PUZZLES

of the

p-LAPLACIAN

Yuval Peres, Oded Schramm, Scott Sheffield, and David Wilson

References:

Tug of war and the infinity Laplacian, Peres, Schramm, S., Wilson, arXiv.

Tug of war with noise: a game theoretic view of the p-Laplacian, Peres, S., arXiv.

Random turn hex and other selection games, Peres, Schramm, S., Wilson, Amer. Math Monthly, arXiv.

Nonlinear potential theory of degenerate elliptic equations, Heinonen, Kilpeläinen, Martio, book.

Infinity Laplacian on a graph:

$$\Delta_\infty u(x) = \frac{1}{2}\left(\inf_{y\sim x} u(y) + \sup_{y\sim x} u(y)\right) - u(x)$$

Infinity Laplacian in $\mathbb{R}^n$:

$$\Delta_\infty u(x) = \frac{\sum u_{x_i} u_{x_i x_j} u_{x_j}}{|\nabla u|^2} =$$

"2nd derivative of u in the gradient direction"

More definitions...

We say u is **infinity harmonic** if $\Delta_\infty u = 0$ (in the viscosity sense). Infinity harmonic functions are limits of p-harmonic functions (i.e., minimizers of $\int |\nabla u(x)|^p dx$ given boundary data) as $p \to \infty$. The p-harmonic functions solve the **Euler Lagrange equation**

$$\mathrm{div}(|\nabla u|^{p-2}\nabla u) = 0$$

which can be rewritten as:

$$p|\nabla u|^{p-2}\left(p^{-1}\Delta_1 u + (1-p^{-1})\Delta_\infty u\right) = 0,$$

where $\Delta_1 = \Delta - \Delta_\infty$. We write

$$\Delta_p u := p^{-1}\Delta_1 + q^{-1}\Delta_\infty$$

where $p^{-1} + q^{-1} = 1$.

Tug of war with noise

Variant of ϵ-step tug of war in a bounded domain of $\mathbb{R}^d$:

whenever player moves by the vector v, a random "noise vector" is added in a direction orthogonal to v.

THEOREM: As $\epsilon \to 0$, the value function for noisy tug of war converges to the p-harmonic extension of the boundary values (provided domain and boundary data are sufficiently regular). (Peres, S.)

Questions we can answer

1. Is there a natural probabilistic interpretation of what a p-harmonic extension is when $p \neq 2$? Does the answer also make sense for discontinuous boundary data?

2. Is the notion of ∞-harmonic in some sense well defined on general metric spaces (not just subsets of $\mathbb{R}^n$)?

3. (Manfredi) Suppose that U is the unit disc in $\mathbb{R}^2$ and that boundary conditions are 1 on an arc of length a and 0 elsewhere. Does the value of the infinity harmonic extension at 0 have a power law decay as $a \to 0$? Generalization: what if we replace the arc with an a-neighborhood of a Cantor set?

Some other known answers

1. Given continuous boundary conditions on the boundary of a smooth closed domain, there exists a unique p-harmonic extension for each $p \in (1, \infty]$.

2. When $1 < p < \infty$, a p-harmonic function u is everywhere differentiable. It is real analytic except at those points where ∇u vanishes.

3. An ∞-harmonic function is continuous but may not be C^2. For example, in two dimensions, $|x|^{4/3} - |y|^{4/3}$ is ∞-harmonic. On two dimensional domains, an ∞-harmonic function is necessarily C^1 (Savin).

Questions we can't yet answer (but can at least rephrase in terms of game theory)

1. **DIFFERENTIABILITY**: Are ∞-harmonic functions everywhere differentiable in dimensions greater than two? (See the partial results in this direction by Crandall and Evans.)

2. **STRICT MONOTONICITY IN BOUNDARY DATA**: (Manfredi) Suppose $p \in (1, \infty)$ and functions u_1 and u_2 are p-harmonic on an open set U and satisfy $u_1 \geq u_2$ on U and $u_1 = u_2$ on an open subset of U. Does it follow that $u_1 = u_2$ throughout U?

3. **UNIQUE CONTINUATION:** (Manfredi) Suppose $p \in (1, \infty)$ and u_1 and u_2 are p-harmonic on U. If $u_1 = u_2$ on an open subset of U, does it follow that $u_1 = u_2$ throughout u?

4. **BOUNDARY DATA:** For what boundary data is the p-harmonic extension uniquely defined?

Discrete value existence theorem

THEOREM: The payoff function F, defined on a subset Y of the vertices of an **undirected graph** is bounded between two constants, A and B, then there is a function u which is:

1. The value of the game.

2. The unique bounded infinity harmonic function with the given boundary values.

3. The unique bounded AM extension of F.

Three Steps of the Proof:

1. **Existence** of a bounded infinity harmonic function u.

2. **Use u-based strategy** to show it is AM.

3. **Payoff of u achievable for either player**, i.e., given any bounded infinity harmonic u, $V_1 \geq u$ and $V_2 \geq -u$.

From this, we conclude that the value function $V = V_1 = -V_2$ exists, and it is the unique bounded infinity harmonic function.

1. Value existence

Define u_n to be the best player one can do in a game modified so that if the boundary is not reached in n steps, player one gets A (the lowest possible value). Observe that $u_0(x) = A$ on non-terminal states and

$$u_n(x) = \frac{1}{2}\left(\sup_{y\sim x} u_{n-1}(y) + \inf_{y\sim x} u_{n-1}(y)\right)$$

The u_n's are increasing and bounded between A and B. By induction, each u_n is infinity **subharmonic** and the supremum u is clearly infinity **superharmonic** (otherwise it would get bigger after another step), so u is **infinity harmonic**.

Clearly, $V_1 \geq u$, and since player two can play in such a way that u is a supermartingale, $V_1 \leq u$. Hence $u = V_1$.

2. Increasing increment sizes and extensions

Suppose graph is locally finite and u is bounded and infinity harmonic and players play the **natural strategy suggested by** u, i.e., player 1 always moves to where u is maximal, player 2 to where u is minimal.

If both players play this way and x_n is game position after n steps, $u(x_n)$ is a martingale with **non-decreasing increment sizes**, i.e.,

$$|u(x_{n+1}) - u(x_n)| \geq |u(x_n) - u(x_{n-1})|.$$

Thus, for *any* edge $e = (x, y)$ with $u(y) - u(x) = \delta > 0$ and *any* induced subgraph X' of X containing e, there is a path from y to the boundary of $\partial X'$ on which u **increases by at least** δ **at each step**, and path from x to $\partial X'$ on which u **decreases by at least** δ **at each step**. Conclusion: Lipschitz norm of u in X is at most the Lipschitz norm of u in $\partial X'$. Thus u **is an optimal extension**.

3. Value is achievable:

Suppose graph is locally finite, x_0 is starting point, and there is a $\delta > 0$ and a y neighboring x_0 with $|y - x_0| \geq \delta$. Let $\mathcal{V}_\delta$ be the collection of all vertices on which u differs by δ or more from its neighbors.

STRATEGY: when player two leaves $\mathcal{V}_\delta$, player one can always "backtrack" until returning to $\mathcal{V}_\delta$. Let v_n be the last vertex of $\mathcal{V}_\delta$ visited during the first n moves; let y_n be the number of surplus turns player two has had since the last visit to $\mathcal{V}_\delta$. Then observe:

$$u(v_n) - \delta y_n$$

is a **submartingale** which at each step goes up by at least δ with probability $1/2$. Convergence follows from martingale convergence theorem, and thus the game must end.

Tug of war with running payoffs

If g is fixed, solutions to $\Delta_\infty u = g$ have meaning as the values of games in which player one collects $g(x)$ from player two each time x is visited.

If g is positive some places and negative other places, the game may not have a value. The reason is that it may turn out that neither player has an incentive to end the game—and each player has to "waste" one or more valuable turns in order to force the game to end.

Fixed targets and comparison with cones

Suppose player one begins the game with a "target a single point" strategy. That is, player one picks a fixed point x_0 and a set S of states and at each turn moves in a way that decreases the distance to x_0 by 1—stopping when game position either reaches x_0 or exits S. Playing in this way makes distance to x_0 a supermartingale, and this leads to an inequality. Namely, for any constants $a > 0, b$, *if* $u(x) \geq a\delta(x, x_0) + b$ **on the boundary of $S \backslash \{x\}$**, then $u(x) \geq a\delta(x, x_0) + b$ **throughout S**.

A function satisfying these inequalities and the corresponding inequalities for player two is said to satisfy **comparison with cones**. It is well known and easy to show that **on a length space, satisfying comparison with cones is equivalent to being an optimal Lipschitz extension**.

Value for continuum game

Tug of war variant: **player-one-ϵ-target tug of war**.

At each step, player one targets a point y up to ϵ units away. Then with probability 1/2, player one reaches y (or hits the boundary at a place within $B_{2\epsilon}(y)$) and with probability 1/2 the game state moves to a point in $\overline{B_{2\epsilon}(y)}$ of the second player's choice. If the game does not terminate in n steps, player one receives A, the lowest possible payoff. Denote by v_ϵ^n the value function for this game.

OBSERVE: $v_\epsilon = \sup v_\epsilon^n$ is smaller than or equal to any function which is bounded below by A and satisfies comparison with cones. Define w_ϵ using second player and we have:

Any bounded optimal extension u satisfies $v_\epsilon \leq u \leq w_\epsilon$.

Sandwich argument

CLAIM: $|v_\epsilon - u_\epsilon| = O(\epsilon)$ and $|w_\epsilon - u_\epsilon| = O(\epsilon)$ where u_ϵ is value of ordinary ϵ-step tug of war.

The claim implies $w_\epsilon - v_\epsilon = O(\epsilon)$. Since any optimal u satisfies $v_\epsilon \leq u \leq w_\epsilon$, letting ϵ go to zero gives uniqueness.

PROOF OF CLAIM: When game position is more than 2ϵ away from the boundary, one way to think of the game is that player one always takes one step, and then with probability $1/2$ player two gets two steps.

Now, suppose every time player two gets one of these two-step strings, player one uses the next step to **backtrack** the latter of player two's moves. Then this reduces the game to ordinary ϵ tug of war, with an error of $O(\epsilon)$ that comes from what happens near the boundary.