TUG OF WAR

What is it good for?

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Tug of war

On a graph G: Let X be vertex set of G. Fix a **terminal** subset $Y \subset X$, a **terminal payoff function** $F : Y \to \mathbb{R}$.

Game token is initially placed at a vertex x_0. At kth turn, players toss a coin and the winner is allowed to choose an x_k adjacent to x_{k-1}. Game ends the first time $x_k \in Y$, and player one's payoff is $F(x_k)$.

On a metric space X: Fix a Lipschitz **terminal payoff function** F on a subset Y of a metric space X.

Game token is initially placed at a vertex x_0. At kth turn, players toss a coin and the winner is allowed to choose an x_k with $d(x_{k-1}, x_k) \leq \epsilon$. Game ends the first time $x_k \in Y$, and player one's payoff is $F(x_k)$.

is good for

1. *Infinity Laplacian problem*: existence and uniqueness of solutions to $\Delta_\infty u = g$ on general metric spaces.

2. *Optimal Lipschitz extension problem*: when does a Lipschitz function on a metric space have a unique "tautest" extension?

3. Political and economic modeling?

PART ONE:

THE INFINITY LAPLACIAN

Infinity Laplacian on a graph:

$$\Delta_\infty u(x) = \frac{1}{2}\left(\inf_{y\sim x} u(y) + \sup_{y\sim x} u(y)\right) - u(x)$$

Infinity Laplacian in $\mathbb{R}^n$:

$$\Delta_\infty u(x) = \frac{\sum u_{x_i} u_{x_i x_j} u_{x_j}}{|\nabla u|^2} =$$

"2nd derivative of u in the gradient direction"

Infinity Laplacian: what is it?

On a graph G:

$$\Delta_\infty u(x) = \left(\inf_{y \sim x} u(y) + \sup_{y \sim x} u(y) \right) - 2u(x)$$

In $\mathbb{R}^n$:

$$\Delta_\infty u(x) = \frac{\sum u_{x_i} u_{x_i x_j} u_{x_j}}{|\nabla u|^2} =$$

"2nd derivative of u in the gradient direction"

Convention: $\Delta_\infty u(x)$ undefined in general when $\nabla u(x) = 0$, but if 2nd derivative of u in *every* direction is λ, $\Delta_\infty u(x) = \lambda$.

More definitions...

We say u is **infinity harmonic** if $\Delta_\infty u = 0$ (in the viscosity sense). Infinity harmonic functions are limits of p-harmonic functions (i.e., minimizers of $\int |\nabla u(x)|^p dx$ given boundary data) as $p \to \infty$. The p-harmonic functions solve the **Euler Lagrange equation**

$$\text{div}(|\nabla u|^{p-2}\nabla u) = 0$$

which can be rewritten as:

$$p|\nabla u|^{p-2}\left(p^{-1}\Delta_1 u + (1-p^{-1})\Delta_\infty u\right) = 0,$$

where $\Delta_1 = \Delta - \Delta_\infty$. We write

$$\Delta_p u := p^{-1}\Delta_1 + q^{-1}\Delta_\infty$$

where $p^{-1} + q^{-1} = 1$.

Infinity Laplacian and tug of war

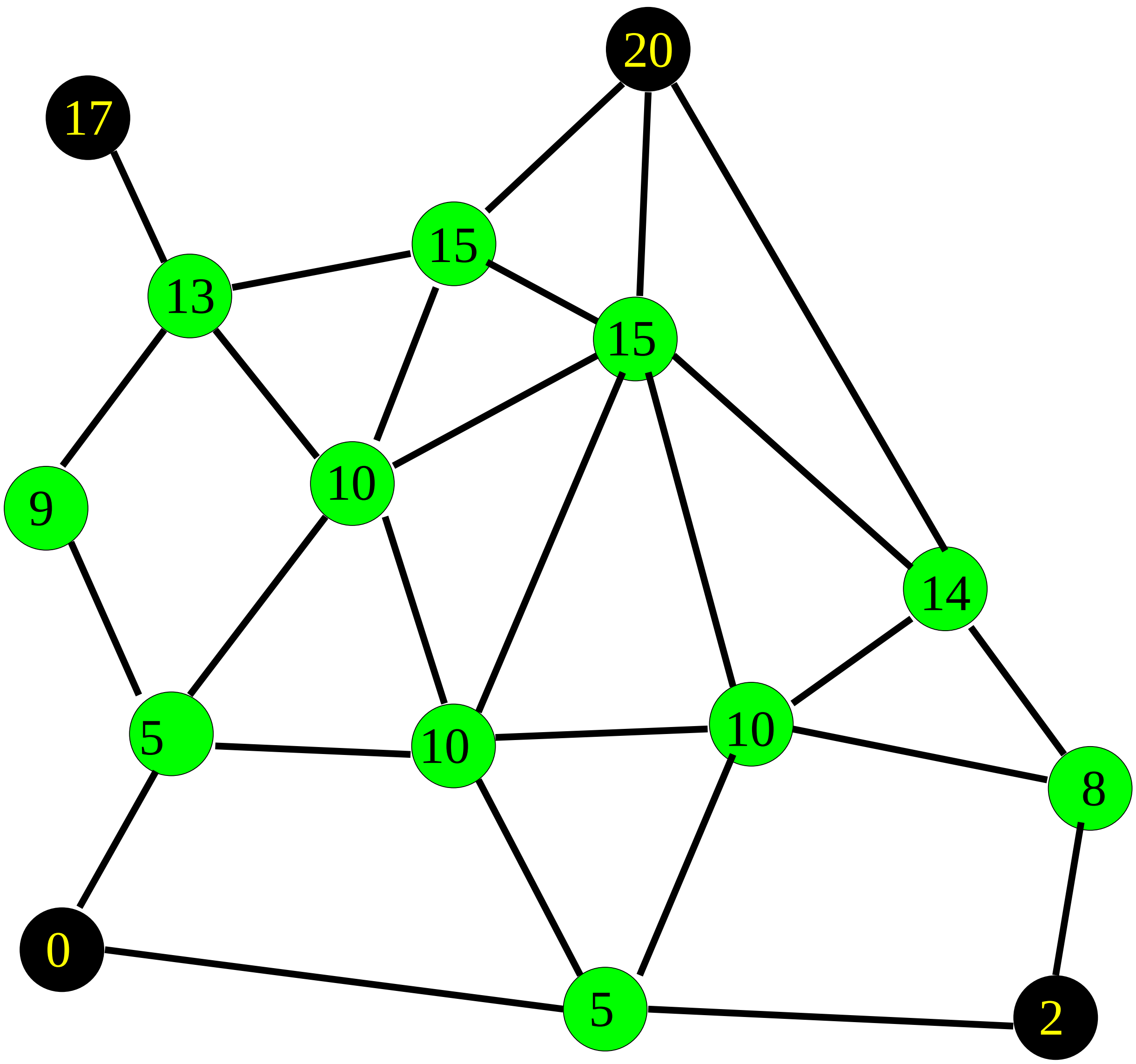
On a graph, when the game starts at v, player one's **value**, denoted $V_1(v)$, is the

supremum, over all player one strategies, of the

infimum, over all player two strategies, of the

expected payoff for player one when the players use those strategies (which we set equal to $-\infty$ if game does not end almost surely).

Define $V_2(v)$ similarly. Say game has a **value function** V if $V_1 = V_2$. The functions V_1 and V_2 are **infinity harmonic**.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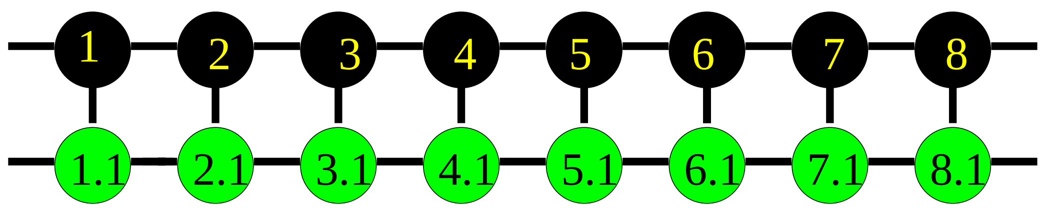
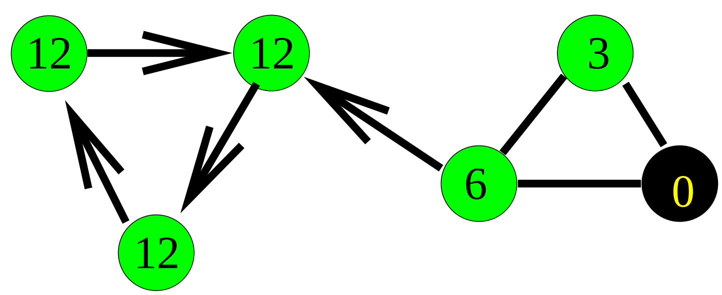
Games without values





Tug of War Application One:

Let u^ϵ be the value function of ϵ tug of war. When the limit $u := \lim_{\epsilon \to 0} u^\epsilon$ exists pointwise, we call u the **continuum value (CV)** of the quadruple (X, Y, F). We define the **continuum value for player one** (or two) analogously. We prove the following:

THEOREM: Suppose X is a length space, $Y \subset X$ is non-empty, $F : Y \to \mathbb{R}$ is Lipschitz and bounded below. Then the continuum value u exists and $|u - u^\epsilon|_\infty = O(\epsilon)$. In a sense we define, this u is the **unique viscosity solution** to $\Delta_\infty u = 0$ with boundary conditions F.

What is a viscosity solution?

As a degenerate elliptic operator, Δ_∞ has a monotonicity property: if $u_1, u_2 \in C_2$, $u_1(x) = u_2(x)$, and $u_1 \geq u_2$ in neighborhood of x, then $\Delta_\infty u_1 \geq \Delta_\infty u_2$.

We say u is a **viscosity solution** (definition due to Crandall and Lions, 1983) to $\Delta_\infty u = g$ in an open set U if it is continuous and there exists **no** $x \in U$ and C^2 function ϕ for which $\phi(x) = u(x)$ *AND EITHER* $u \geq \phi$ in neighborhood of x and $\Delta_\infty \phi(x) > g(x)$ *OR* $u \leq \phi$ in neighborhood of x and $\Delta_\infty \phi(x) < g(x)$.

EQUIVALENT: replace "C^2 function" with "quadratic function" or with "a quadratic function of path-distance from a fixed point z." **The latter makes sense for any metric space.** If $g = 0$ then affine functions of distance (cones) suffice.

PART TWO:

OPTIMAL LIPSCHITZ EXTENSIONS

Optimal ("tautest") Lipschitz Extension Problem

Given a Lipschitz function u, defined on a subset Y of a metric space X with metric δ, what is the **"tautest"** Lipschitz extension of u to all of X?

If "tautest" means merely "minimizing Lipschitz norm"

$$\mathrm{Lip}(u, X) := \sup_{x,y \in X} \frac{|u(x) - u(y)|}{\delta(x, y)},$$

then (noting that $\mathrm{Lip}(u, X) \geq \mathrm{Lip}(u, Y)$) the **McShane-Whitney extensions** (1934) would are the largest and smallest minimizers:

$$\inf_{y \in Y} (u(y) + \mathrm{Lip}(u, Y)|x - y|)$$

$$\sup_{y \in Y} (u(y) - \mathrm{Lip}(u, Y)|x - y|)$$

A notion of "tautest"

Say $u : X \to \mathbb{R}$ is an **absolutely minimizing (AM)** extension of its values on Y if $\text{Lip}(u, U) = \text{Lip}(u, \partial U)$ for all open $U \subset X \backslash Y$.

Theorem [Aronsson, Jensen]: When X is a bounded, closed subset of $\mathbb{R}^n$, $Y = \partial X$, and F is a Lipschitz function on Y, there exists a **unique** AM extension u of F to X. This is also the unique continuous extension of F that is **infinity harmonic** (i.e., a viscosity solution to $\Delta_\infty = 0$) on $X \backslash Y$.

AM/Infinity Harmonic equivalence: We show that on any rectifiable-path-connected metric space, a bounded function is infinity harmonic if and only if it is AM (w.r.t. path-distance metric).

AMLE Model

Caselles, Masnou, Morel, and Sbert. **Image Interpolation** Seminaire de l'Ecole Polytechnique, Palaiseau, Paris, 1998.

Figure 8:

Above Original image where occlusions are in white.

Infinity harmonic/AM history

EXISTENCE PROOF: G. Aronsson (1967) proves existence when X is bounded subset of $\mathbb{R}^n$, also shows that all C^2 solutions are **infinity harmonic**.

EXISTENCE EXTENSION: Juutinen (2002) extends existence to case that X is a **separable length space**.

UNIQUENESS PROOF 1: Jensen (1993), also shows that u is AM iff if infinity harmonic.

UNIQUENESS PROOF 2: Barles and Busca (2001)

UNIQUENESS PROOF 3: Crandall, Aronsson, and Juutinen (2004): generalizes X to uniformly convex norms on $\mathbb{R}^n$.

We prove existence *and* **uniqueness for** *all* **length spaces using** *Tug of War*.

Tug of War Application Two:

THEOREM Let X be a length space, $Y \subset X$, and $F : Y \to \mathbb{R}$. If F is Lipschitz and positive, then the continuum value described above is an AM extension of F. If F is Lipschitz and bounded, then it is the *unique* AM extension of F.

COUNTEREXAMPLES: AM extension need not be unique when F is merely bounded below, even though the continuum value exists uniquely in that setting.

PART THREE:

SOME PROOF SKETCHES

Discrete value existence theorem

THEOREM: The payoff function F, defined on a subset Y of the vertices of an **undirected graph** is bounded between two constants, A and B, then there is a function u which is:

1. The value of the game.

2. The unique bounded infinity harmonic function with the given boundary values.

3. The unique bounded AM extension of F.

Three Steps of the Proof:

1. **Existence** of a bounded infinity harmonic function u.

2. **Use u-based strategy** to show it is AM.

3. **Payoff of u achievable for either player**, i.e., given any bounded infinity harmonic u, $V_1 \geq u$ and $V_2 \geq -u$.

From this, we conclude that the value function $V = V_1 = -V_2$ exists, and it is the unique bounded infinity harmonic function.

1. Existence

Define u_n to be the best player one can do in a game modified so that if the boundary is not reached in n steps, player one gets A (the lowest possible value). Observe that $u_0(x) = A$ on non-terminal states and

$$u_n(x) = \frac{1}{2}\left(\sup_{y \sim x} u_{n-1}(y) + \inf_{y \sim x} u_{n-1}(y)\right)$$

The u_n's are increasing and bounded between A and B. By induction, each u_n is infinity **sub-harmonic** and the supremum u is clearly infinity **superharmonic** (otherwise it would get bigger after another step), so u is **infinity harmonic**.

Clearly, $V_1 \geq u$, and since player two can play in such a way that u is a supermartingale, $V_1 \leq u$. Hence $u = V_1$.

2. Increasing increment and AM extensions

Suppose graph is locally finite and u is bounded and infinity harmonic and players play the **natural strategy suggested by u**, i.e., player 1 always moves to where u is maximal, player 2 to where u is minimal.

If both players play this way and x_n is game position after n steps, $u(x_n)$ is a martingale with **non-decreasing increment sizes**, i.e.,

$$|u(x_{n+1}) - u(x_n)| \geq |u(x_n) - u(x_{n-1})|.$$

Thus, for *any* edge $e = (x, y)$ with $u(y) - u(x) = \delta > 0$ and *any* induced subgraph X' of X containing e, there is a path from y to the boundary of $\partial X'$ on which u **increases by at least δ at each step**, and path from x to $\partial X'$ on which u **decreases by at least δ at each step**. Conclusion: Lipschitz norm of u in X is at most the Lipschitz norm of u in $\partial X'$. Thus u **is AM**.

3. Value is achievable:

Suppose graph is locally finite, x_0 is starting point, and there is a $\delta > 0$ and a y neighboring x_0 with $|y - x_0| \geq \delta$. Let $\mathcal{V}_\delta$ be the collection of all vertices on which u differs by δ or more from its neighbors.

STRATEGY: when player two leaves $\mathcal{V}_\delta$, player one can always "backtrack" until returning to $\mathcal{V}_\delta$. Let v_n be the last vertex of $\mathcal{V}_\delta$ visited during the first n moves; let y_n be the number of surplus turns player two has had since the last visit to $\mathcal{V}_\delta$. Then observe:

$$u(v_n) - \delta y_n$$

is a **submartingale** which at each step goes up by at least δ with probability $1/2$. Convergence follows from martingale convergence theorem, and thus the game must end.

Value for continuum game

Tug of war variant: **player-one-ϵ-target tug of war**.

At each step, player one targets a point y up to ϵ units away. Then with probability 1/2, player one reaches y (or hits the boundary at a place within $B_{2\epsilon}(y)$) and with probability 1/2 the game state moves to a point in $\overline{B_{2\epsilon}(y)}$ of the second player's choice. If the game does not terminate in n steps, player one receives A, the lowest possible payoff. Denote by v_ϵ^n the value function for this game.

OBSERVE: $v_\epsilon = \sup v_\epsilon^n$ is smaller than or equal to any function which is bounded below by A and satisfies comparison with distance functions. Define w_ϵ using second player and we have:

Any bounded AM extension u satisfies $v_\epsilon \leq u \leq w_\epsilon$.

Sandwich argument

CLAIM: $|v_\epsilon - u_\epsilon| = O(\epsilon)$ and $|w_\epsilon - u_\epsilon| = O(\epsilon)$ where u_ϵ is value of ordinary ϵ-step tug of war.

The claim implies $w_\epsilon - v_\epsilon = O(\epsilon)$. Since any bounded AM u satisfies $v_\epsilon \leq u \leq w_\epsilon$, letting ϵ go to zero gives uniqueness.

PROOF OF CLAIM: When game position is more than 2ϵ away from the boundary, one way to think of the game is that player one always takes one step, and then with probability $1/2$ player two gets two steps.

Now, suppose every time player two gets one of these two-step strings, player one uses the next step to **backtrack** the latter of player two's moves. Then this reduces the game to ordinary ϵ tug of war, with an error of $O(\epsilon)$ that comes from what happens near the boundary.