An unstable free boundary problem

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July 30, 2009

Background

A semilinear problem

Consider a semilinear equation of the form

$$\Delta u = F(u), \qquad \text{(more generally} \quad F(x,u))$$

where $F(.,t)$ has discontinuity at $t = 0$, i.e. across the zero level sets of u.

What can we say about the optimal regularity of the solution u?

What can we say about the optimal regularity of the set $\partial\{u > 0\}$?

Background

Free boundary points of interest are $\nabla u = 0$, otherwise implicit function theorem gives regularity of the level sets.

Invariant Scaling

Let $u(x) = |\nabla u(x)| = 0$ and set

$$u_r(x) := \frac{u(rx + x^0)}{r^2}.$$

Then

$$\Delta u_r(x) = F(u(rx + x^0)) = F(r^2 u_r)$$

and we retain the problem.

So we need u_r to be uniformly bounded in r?

$C^{1,1}$ regularity

Conditions on F

The condition

$$F_t'(x,t) \geq -C, \qquad \nabla_x F(x,t) \geq -C$$

guarantees a $C^{1,1}$ regularity for u.

Indeed, for $u_e = D_e u$ we get

$$\Delta(u_e)^{\pm}(x) = F_e + F_t'(u_e)^{\pm} \geq -C'$$

and a monotonicity formula of CJK can be applied to obtain bounds on ∇u_e.

Failure of $C^{1,1}$ regularity

Conditions on F

If F' produces a NEGATIVE Dirac type measures, i.e.

$$F'_t \not\geq -C$$

then $C^{1,1}$ regularity may fail.

Good Examples of F: Obstacle problem

Obstacle problem

For the obstacle problem (with smooth obstacle) one has

$$F(x,u) = f(x)\chi_{\{u>0\}}, \qquad u \geq 0 \qquad f > 0$$

$$\Delta u_r(x) = f(rx)\chi_{\{u_r>0\}},$$

and

$$\Delta (u_e)^{\pm} \geq f' \chi_{\{u>0\}} + f(u_e)^{\pm}\delta_{\partial\{u>0\}} \geq -C$$

In this case, not much is left for study!
The case when $f \geq 0$ has zeros is untouched!

Good Examples of F: Two-phase case

Two phase problems

$$F(x,u) = \lambda_+(x)\chi_{\{u>0\}} - \lambda_-(x)\chi_{\{u<0\}},$$

$$\Delta u_r(x) = \lambda_+(rx)\chi_{\{u_r>0\}} - \lambda_-(rx)\chi_{\{u_r<0\}}$$

with $\lambda_+ + \lambda_- \geq 0$ and $C^{0,1}$.
and

$$\Delta(u_e)^{\pm} \geq D_e\lambda_+(x)\chi_{\{u>0\}} - D_e\lambda_-(x)\chi_{\{u<0\}} +$$

$$(\lambda_+ + \lambda_-) \text{ pos. measure } \geq -C$$

Here too, not much is left for study! The case when $\lambda_\pm$ can take zero value or one of them is identically zero are not studied.

Less Good Examples of F: Unstable

Unstable problems

$$F(x,u) = \lambda_+(x)\chi_{\{u>0\}} - \lambda_-(x)\chi_{\{u<0\}},$$

$$\lambda_+(x) + \lambda_-(x) < 0$$

A scaling does not necessarily converge.

Examples of F: Not $C^{1,1}$

A simple case

$$\Delta u = -\chi_{u>0},$$

related to traveling wave solutions in solid combustion with ignition temperature.

J. Andersson, G. Weiss

There exist solutions that are not $C^{1,1}$.

Applications: Unstable case

Composite membrane

Build a body of a prescribed shape out of given materials of varying densities, in such a way that the body has a prescribed mass and with the property that the fundamental frequency of the resulting membrane (with fixed boundary) is lowest possible.

This problem results in an equations of the type above with

$$\lambda_+ + \lambda_- < 0.$$

Ref. S.J. Cox, J.R. Mclaughlin.

Applications: Unstable case

Population dynamics

Other applications are that of Population Dynamics, where the optimal arrangement of favorable and unfavorable regions for species' survival is in consideration. This again is a an eigenvalue problem

$$\Delta u = -\lambda m(x) u$$

with

$$m(x) = m_1 \chi_{u>t} - m_2 \chi_{u<t}$$

($t > 0$, $u > 0$, $\lambda > 0$).

Ref. Y. Lou, E. Yanagida.

Systems: An example

Stable (+), and Unstable (-) cases

If we consider systems, then the following problem can be considered a generalization of the the obstacle type problems

$$\Delta u_i = \frac{\pm u_i}{|\mathbf{u}|}, \quad i = 1, 2, ...$$

and it introduces some challenge. $+$ gives the stable case, and $-$ the unstable one.

Systems: An example

Unstable case

The real and imaginary parts of the function

$$S(z) = z^2 \log |z|$$

satisfies the unstable equation (up to a multiplicative constant) and they have a singularity at the origin. Hence optimal $C^{1,1}$ regularity is lost!

Main tools in regularity theory

Quadratic growth

Standard in such problems is that u is $C^{1,1}$ and we have the possibility of scaling the solution

$$u_r(x) := \frac{u(rx + x^0)}{r^2},$$

in order to analyze local properties of the solution and the free boundary.

Observe that u_r is then a solution in $B_{1/r}$ and as $r \to 0$ then $u_r \to u_0$ in the entire space $\mathbb{R}^n$, and we have

$$\Delta u_0 = \lambda_+(0)\chi_{\{u_0>0\}} - \lambda_-(0)\chi_{\{u_0<0\}}, \qquad \text{in } \mathbb{R}^n .$$

Main tools in regularity theory

Non-degeneracy

In blowing up a solution, one needs to show that the limit solution is not identically zero (i.e it doesn't flatten out). Therefore one needs the so-called non-degeneracy:

$$\sup_{B_r(x^0)} u \geq cr^2, \qquad \inf_{B_r(x^0)} u \leq -cr^2.$$

Unstable case: Troubles

Failure!

Both Quadratic growth property and non-degeneracy may fail in the unstable case.

So how do we study the regularity of the free boundary?

Unstable case: Troubleshooting

Consider a solution to the equation

$$\Delta u = -\chi_{\{u>0\}} \text{ in } B_1,$$

and define

$$S(u) = \{X;\ u(X) = \nabla u(X) = 0, u \notin C^{1,1}(B_r(X)) \quad \forall r > 0\}.$$

Can $S(u)$ be embedded in a smooth lower dimensional manifold?

Unstable case: 2-space dimension

Andersson, Sh., Weiss (http://arxiv.org/abs/0905.2811)

In $\mathbb{R}^2$, for x^0 a singular point the following holds
(i) there exists a polynomial $p^{x^0,u} = p$ such that

$$\left\| \frac{u(x^0+sx)}{\sup_{B_s(x^0)}|u|} - p \right\|_{C^{1,\beta}(B_1)} \leq C_{\alpha,\beta,n,M} \left(\frac{\delta}{1+\delta \log(r/s)} \right)^{\alpha}$$

(ii) the set $\{u=0\} \cap B_r(x^0)$ consists of two C^1 curves intersecting each other at right angles at x^0

Here $1/\delta = \sup_{B_r}|u|/r^2$, is large enough, and $s < r$, $0 < \beta, \alpha < 1/2$ are arbitrary, M is supnorm of u.

Unstable case: 3-space dimension

Singular points $S = S(u)$ are points where u is not $C^{1,1}$.

Homogenous scaling

For singular points X^0 we have

$$\Delta\left(\frac{u(r_jX + X^0)}{\sup_{B_{r_j}(X^0)}|u|}\right) = -\frac{r^2}{\sup_{B_{r_j}(X^0)}|u|}\chi_{\{u(r_jX+X^0)>0\}} \quad \to \quad 0$$

Hence

$$\lim_{r_j\to 0}\frac{u(r_jX + X^0)}{\sup_{B_{r_j}(X^0)}|u|} = p(X)$$

where p is a second order homogeneous harmonic polynomial.

Unstable case: 3-space dimension

Questions

- Does p depend on the sequence $r_j \to 0$?
- Are there some further restrictions on p?

Unstable case: 3-space dimension

Limiting harmonic polynomials in $\mathbb{R}^3$

If for some sequence r_j

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{\sup_{B_{r_j}} |u|} = p$$

then the limit exists for all sequences r_j and it is p with

$$p = x^2 - z^2 \qquad \text{or} \qquad p = \pm\left((x^2 + y^2)/2 - z^2\right)$$

after suitable rotation.

Unstable case: 3-space dimension

Limiting harmonic polynomials in $\mathbb{R}^3$

Observe that this is a very strong statement, since there are a range of other possible quadratic harmonic polynomials, that we exclude

$$ax^2 + by^2 - z^2, \qquad a + b = 1.$$

Unstable case: 3-space dimension

Theorem; Andersson, Sh., Weiss

In $\mathbb{R}^3$, the singular set S is divided into two parts

$$S_1 = \{x \in S;\ \lim_{r \to 0} u(x + r\cdot)/\sup_{B_r} |u| = \pm(x^2 + y^2 - 2z^2)\}$$

and

$$S_2 = \{x \in S;\ \lim_{r \to 0} u(x + r\cdot)/\sup_{B_r} |u| = xy\}$$

where S_1 consists only of isolated points and S_2 is contained in a C^1 manifold.

Unstable case: Ideas of the proof

A non-standard blow-up

The above blow-up does not provide us with enough information about the solution, since the non-linearity of $\chi_{\{u>0\}}$ disappears in the limit.

Instead we suggest a blow-up of the following kind

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{r_j^2} - \Pi(u, r_j, X^0),$$

where $\Pi(u, r_j, X^0)$ is the projection of $u(r_j X + X^0)/r_j^2$ in B_1 into the homogeneous harmonic second order polynomials.

Unstable case: Ideas of the proof

Preserving Non-linearity in the limit

One can show that

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{r_j^2} - \Pi(u, r_j, X^0) = Z_p$$

where

$$\Delta Z_p = -\chi_{\{p>0\}}.$$

The proof is either indriect, using compactness, or direct using the fact that the second derivatives of u are in BMO, so a harmonic analysis approach will work.

Unstable case: Ideas of the proof

Ideas

One notices further that

$$\lim_{r_j \to 0} \frac{u(r_j X + X^0)}{\sup_{B_{r_j}(X^0)} |u|} = \lim_{r_j \to 0} \frac{\Pi(u, r_j, X^0)}{\sup_{B_1} |\Pi(u, r_j, X^0)|},$$

which follows from $u(r_j X + X^0) = (\Pi(u, r_j, X^0) + Z_p) r_j^2$ and that $\sup_{B_1} |u(rx)| = \sup_{B_1} \Pi(u(rx))$, along with the fact that the $r^2 / \sup_{B_{r_j}} |u|$ tend to zero.

So if we want to prove uniqueness of p it is enough to control how $\Pi(u, r, X^0)$ changes in r.

Unstable case: Ideas of the proof

Ideas

Specifically we would want to estimate

$$\left|\frac{\Pi(u, r, X^0)}{\sup_{B_1} |\Pi(u, r, X^0)|} - \frac{\Pi(u, r/2, X^0)}{\sup_{B_1} |\Pi(u, r/2, X^0)|}\right|,$$

that is how much the orientation of u changes when we pass from the ball $B_r(X^0)$ to $B_{r/2}(X^0)$.

Unstable case: Ideas of the proof

Ideas

To do this we notice that if

$$\Pi(u,r)/\tau_r \approx p(X)$$

for some polynomial p where $\tau_r = \sup_{B_1} |\Pi(u,r)|$ then

$$\Delta u \approx -\chi_{\{p>0\}} \qquad \text{in } B_r$$

and thus

$$u \approx r^2(\tau_r p + Z_p) \qquad \text{in } B_r.$$

Observer that τ_r tends to infinity with r tending to zero.

Unstable case: Ideas of the proof

Ideas

The main point here is that if $\Pi(u,r)/\tau_r \approx p(X)$ then

$$\Pi(u,r/2) \approx \Pi(\tau_r p + Z_p, 1/2) \approx \Pi(\tau_r p, 1/2) + \Pi(Z_p, 1/2)$$

$$= \tau_r p + \Pi(Z_p, 1/2) = \Pi(u,r) + \Pi(Z_p, 1/2)$$

so

$$|\Pi(u,r) - \Pi(u,r/2)| \approx |\Pi(Z_p, 1/2)|.$$

And therefore to estimate how much $\Pi(u,r)$ changes when we change r we need to be able to control $\Pi(Z_p, \cdot)$.

Unstable case: Ideas of the proof

Explicit computation of Z_p

Next one explicitly calculates $Z_p = q(X) \ln |X| + |X|^2 \phi(X)$, when

$$p = xy \qquad \text{or} \qquad p = \pm\big((x^2 + y^2)/2 - z^2\big).$$

Using these calculations we can show that at singular points and for small r we have

$$\sup_{B_r} |u(X + X^0)| \geq cr^2 |\ln r|.$$

This is the first step in a series of EXACT estimates to come, and that leads us to the an accurate estimate of the turn of the polynomials (or u_r) in terms of r.

Unstable case: Existing results

Partial results

Monneau-Weiss, Chanillo-Kenig, Chanillo-Kenig-Tou, Andersson-Weiss, Sh., ...