On the Euclidean, Affine & Projective invariants of the supercircle, and their associated cocycles

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- Find super geometric versions of the Euclidean, affine, and projective invariants of S^1. Super cross-ratio?

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- Towards a classification of the geometries of the supercircle?

Diagrammatic representation : Geometries of $S^{1|1}$

$$
\begin{tikzcd}
\textit{Groups} \arrow[rr, leftrightarrow, "p|q\text{-transitivity}"] & & \textit{Invariants} \arrow[dl, "\text{Cartan}"] \\
& \textit{Cocycles} \arrow[ul, "\text{kernels}"] &
\end{tikzcd}
$$

Bibliographical landmarks

1. Super cross-ratio (even and odd)
   - Aoki ['88] *Super-Riemann surfaces*
   - Nelson ['88] *Superstrings*
   - Uehara & Yasui ['90] *WP on Super-Teichmueller*
   - Manin ['91] *NCG*
   - Giddings ['92] *Punctured Super-Riemann surfaces*
   - ...
2. Super-Schwarzian
   - Friedan ['86] (N=1) *CFT*
   - Radul ['86] (N=1,2,3) *Super-Bott cocycle*
   - Cohn ['87](N=2) *Super-Riemann surfaces*
   - Gieres & Theisen ['93] *Superconformally covariant operators*
   - ...

Invariants of $\mathrm{E}(1) \subset \mathrm{Aff}_+(1) \subset \mathrm{PSL}(2,\mathbb{R}) \subset \mathrm{Diff}_+(\mathrm{S}^1)$

- Euclidean invariant (translations): distance[1]

$$[x_1, x_2] = x_2 - x_1$$

- Affine invariant (homotheties, translations): distance ratio

$$[x_1, x_2, x_3] = \frac{[x_1, x_3]}{[x_1, x_2]}$$

- Projective invariant (homographies): cross-ratio

$$[x_1, x_2, x_3, x_4] = \frac{[x_1, x_3][x_2, x_4]}{[x_2, x_3][x_1, x_4]}$$

[1] One deals effectively with $\mathbb{R}P^1$

The associated $\mathrm{Diff}_+(S^1)$ 1-cocycles

- Euclidean cocycle $\mathcal{E} : \mathrm{Diff}_+(S^1) \to C^\infty(S^1)$:

$$\mathcal{E}(\varphi) = \log(\varphi')$$

- Affine cocycle $\mathcal{A} : \mathrm{Diff}_+(S^1) \to \Omega^1(S^1)$:

$$\mathcal{A}(\varphi) = d\mathcal{E}(\varphi)$$

- Projective cocycle: Schwarzian derivative $\mathcal{S} : \mathrm{Diff}_+(S^1) \to \mathcal{Q}(S^1)$:

$$\begin{aligned} \mathcal{S}(\varphi) &= \left(\frac{\varphi'''}{\varphi'} - \frac{3}{2}\left(\frac{\varphi''}{\varphi'}\right)^2\right) dx^2 \\ &= dx \otimes L_{\partial_x}\mathcal{A}(\varphi) - \frac{1}{2}\mathcal{A}(\varphi)^2 \end{aligned}$$

The associated $\mathrm{Diff}_+(S^1)$ 1-cocycles (cont'd)

The Schwarzian $\mathcal{S}(\varphi)$ measures, at each point x, the shift between a diffeomorphism $\varphi \in \mathrm{Diff}(S^1)$ and its approximating homography, $h \in \mathrm{PGL}(2,\mathbb{R})$,[2]

$$\mathcal{S}(\varphi)(x) = (\widehat{h}^{-1} \circ \varphi)'''(x)$$

- It is a $\mathrm{PSL}(2,\mathbb{R})$-differential invariant for $\mathrm{Diff}_+(S^1)$: $\mathcal{S}(\varphi) = \mathcal{S}(\psi)$ iff $\varphi = A \circ \psi$ where $A \in \mathrm{PSL}(2,\mathbb{R})$.
- It is a non trivial 1-cocycle of $\mathrm{Diff}_+(S^1)$ with values in the module of quadratic differentials $\mathcal{Q}(S^1)$:

$$\mathcal{S}(\varphi \circ \psi) = \psi^* S(\varphi) + \mathcal{S}(\psi)$$

for al $\varphi, \psi \in \mathrm{Diff}_+(S^1)$. It has kernel $\mathrm{PSL}(2\mathbb{R})$.

[2] s.t. $\widehat{h}^{-1} \circ \varphi$ has the 2-jet of Id at x

The supercircle $S^{1|1}$

The supercircle $S^{1|1}$: the circle S^1, endowed with (a sheaf of associative and commutative $\mathbb{Z}/(2\mathbb{Z})$-graded algebras, the sections of which are) the superfunctions $C^\infty(S^{1|1}) = C^\infty(S^1)[\xi]$ where $\xi^2 = 0$.

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- If (x, ξ) are local coordinates of (affine) superdomain, every superfunction writes

$$f(x,\xi) = f_0(x) + \xi f_1(x), \qquad \text{where} \qquad f_0, f_1 \in C^\infty(S^1)$$

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- Group of diffeomorphisms: $\mathrm{Diff}(S^{1|1}) = \mathrm{Aut}(C^\infty(S^{1|1}))$. For practical purposes: a diffeomorphism is a pair $\Phi = (\varphi, \psi)$ of superfunctions s.t. $(\varphi(x,\xi), \psi(x,\xi))$ are new coordinates on $S^{1|1}$.

Vector fields & 1-forms of the supercircle

Vector fields $\mathrm{Vect}(S^{1|1}) = \mathrm{SuperDer}(C^\infty(S^{1|1}))$. Every Vector field is written locally as

$$X = f(x,\xi)\partial_x + g(x,\xi)\partial_\xi \qquad \text{where} \qquad f,g \in C^\infty(S^{1|1})$$

NB: $\mathrm{Vect}(S^{1|1})$ is a $C^\infty(S^{1|1})_L$-module locally generated by $(\partial_x, \partial_\xi)$ where $p(\partial_x) = 0$, $p(\partial_\xi) = 1$. It is also a Lie superalgebra with Lie bracket $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$.

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Differential 1-forms $\Omega^1(S^{1|1}) = C^\infty(S^{1|1})_R$-module locally generated by dual basis $(dx, d\xi)$ where $p(dx) = 0$, $p(d\xi) = 1$. The module $\Omega^\star(S^{1|1})$ of differential forms is bigraded (cohomology degree $|\cdot|$, parity p); our choice of *Sign Rule*:

$$\alpha \wedge \beta = (-1)^{|\alpha||\beta| + p(\alpha)p(\beta)} \beta \wedge \alpha$$

Contact structure on supercircle $S^{1|1}$

It is given by direction of contact 1-form

$$\boxed{\alpha = dx + \xi d\xi}$$

We have

$$d\alpha = \beta \wedge \beta \qquad \text{where} \qquad \beta = d\xi$$

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- Contactomorphisms: the automorphisms of $(\mathrm{S}^{1|1}, [\alpha])$:

$$K(1) = \{\Phi \in \mathrm{Diff}(\mathrm{S}^{1|1}) \,|\, \Phi^*\alpha = E_\Phi\, \alpha\}$$

One shows that $\Phi = (\varphi, \psi) \in K(1) \Leftrightarrow \boxed{D\varphi = \psi D\psi}$ and

$$E_\Phi = (D\psi)^2$$

Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of λ-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with the (anti)action ($\Phi \mapsto \Phi_\lambda$) defined by $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$. (One writes *symbolically* $F \in \mathcal{F}_\lambda$ as $F = f\alpha^\lambda$.)

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- The $C^\infty(S^{1|1})_R$-module $\mathcal{Q}(S^{1|1})$ of quadratic differentials is generated by $\alpha^2 = \alpha \otimes \alpha$ et $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.

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Proposition

Both $\Omega^1(S^{1|1})$ and $\mathcal{Q}(S^{1|1})$ are $K(1)$-modules ; they admit the decomposition into $K(1)$-submodules:[a]

$$\Omega^1(S^{1|1}) \cong \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_1, \qquad \mathcal{Q}(S^{1|1}) \cong \mathcal{F}_{\frac{3}{2}} \oplus \mathcal{F}_2$$

The projections $\Omega^1(S^{1|1}) \to \mathcal{F}_{\frac{1}{2}}$ (resp. $\mathcal{Q}(S^{1|1}) \to \mathcal{F}_{\frac{3}{2}}$) are given by $\alpha^{\frac{1}{2}}\langle D, \cdot\rangle$, and the corresponding sections by $\alpha^{\frac{1}{2}} L_D$ (resp. $\frac{2}{3}\alpha^{\frac{1}{2}} L_D$).

[a] Cf. the decomposition: $\mathit{Vect}(S^{1|1}) \cong \mathcal{F}_{-1} \oplus \mathcal{F}_{-\frac{1}{2}}$ [Gargoubi-Mellouli-Ovsienko].

The Orthosymplectic group, its Euclidean and affine subgroups

It is the subgroup $\mathrm{SpO}(2|1) \subset \mathrm{GL}(2|1)$ of symplectomorphisms of $\mathbb{R}^{2|1}$, with symplectic form $d\varpi$ where $\varpi = \frac{1}{2}(pdq - qdp + \theta d\theta)$; its elements are of the form

$$h = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$$

where
$ad - bc - \alpha\beta = 1, e^2 + 2\gamma\delta = 1, \alpha e - a\delta + c\gamma = 0, \beta e - b\delta + d\gamma = 0.$
The group $\mathrm{SpO}(2|1)$ also preserving the 1-forme $\varpi = \frac{1}{2}p^2\alpha$ (where $p \neq 0$), it acts by contactomorphisms via the projective action of $\mathrm{S}^{1|1}$:

$$\widehat{h}(x,\xi) = \left(\frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi}\right)$$

The Berezinian is $\mathrm{Ber}(h) = e + \alpha\beta e^{-1}$ and $\mathrm{SpO}_+(2|1) = \mathrm{Ber}^{-1}(1)$ is a super-extension of $\mathrm{Sp}(2,\mathbb{R}) = \mathrm{SL}(2,\mathbb{R})$.

Le orthosymplectic group (cont'd)

One has the local factorization

$$\mathrm{SpO}_+(2|1) \ni h = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{c} & 1 & \tilde{\delta} \\ \tilde{\delta} & 0 & 1 \end{pmatrix} \underbrace{\begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \overbrace{\begin{pmatrix} \epsilon & \tilde{b} & -\tilde{\beta} \\ 0 & \epsilon & 0 \\ 0 & \epsilon\tilde{\beta} & 1 \end{pmatrix}}^{\mathrm{E}(1|1)}}_{\mathrm{Aff}(1|1)}$$

where $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}$, with $\epsilon^2 = 1$, $\tilde{a} > 0$.

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where $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}$, with $\epsilon^2 = 1$, $\tilde{a} > 0$.

- The subgroup $\mathrm{E}(1|1)$ is the group of those $\Phi \in \mathrm{Diff}(\mathrm{S}^{1|1})$ s.t. $\Phi^*\alpha = \alpha$ and $\Phi^*\beta = \epsilon\beta$.

- One also has $\mathrm{Aff}(1|1) = \{\Phi \in K(1) \,|\, \Phi^*\beta = F_\Phi \beta\}$.

Notion of $p|q$-transitivity

Extension of the notion of n-transitivity to supergroup actions.

Consider $E = E_0 \times E_1$ and p_0 & p_1 its canonical projections.
Two n-uplets $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of E are said $p|q$-equivalent, $s \stackrel{p|q}{=} t$, where $n = \max(p, q)$, if $p_0(s_i) = p_0(t_i)$ $\forall i = 1, \ldots, p$ and $p_1(s_i) = p_1(t_i)$ $\forall i = 1, \ldots, q$.[3]

[3] The $n|n$-equivalence is an ... equality!

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The action of group G on E is said (simply) **$p|q$-transitive** if for all $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of E, there exists a (unique) $g \in G$ s.t. $\hat{g}(t) \overset{p|q}{=} s$.

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Example: The $\mathrm{PGL}(2, \mathbb{R})$-action on S^1 is simply 3-transitive.

[3] The $n|n$-equivalence is an . . . equality!

Construction of the invariants

Theorem

Let the group G act simply $p|q$-transitively on $E = E_0 \times E_1$, and m be an n-uplet, $n = \max(p, q)$, of distinct points of E. The $n+1$ point-function $I_{[m]}$ of E with values in E defined by

$$I_{[m]}(t_1, \ldots, t_{n+1}) = \widehat{h}(t_{n+1})$$

where $\widehat{h}(t) \stackrel{p|q}{=} m$, and $t = (t_1, \ldots, t_n) \in E^n \setminus \Gamma$ enjoys the properties:

1. $I_{[m]}$ is G-invariant
2. If $\Phi \in E!$ preserves $I_{[m]}$, then $\Phi = \widehat{g}$ where $g \in G$

If $n = p > q$, the n-point functions with values in E_1

$$J_{[m],j}(t) = p_1(\widehat{h}(t_j)) \qquad (j = q+1, \ldots, n)$$

are G-invariant. the $I_{[m]}$ and $J_{[m],j}$ generate all invariants with $n+1$ and n points.

Theorem I

- Euclidean invariant: $I_e(t_1, t_2) = ([t_1, t_2], \{t_1, t_2\})$ with

$$[t_1, t_2] = x_2 - x_1 - \xi_2\xi_1, \qquad \{t_1, t_2\} = \xi_2 - \xi_1$$

- Affine invariant, $I_a(t_1, t_2, t_3) = ([t_1, t_2, t_3], \{t_1, t_2, t_3\})$, where, if $x_1 < x_2$,

$$[t_1, t_2, t_3] = \frac{[t_1, t_3]}{[t_1, t_2]}, \qquad \{t_1, t_2, t_3\} = [t_1, t_2, t_3]^{\frac{1}{2}} \frac{\{t_1, t_3\}}{[t_1, t_3]^{\frac{1}{2}}}$$

- Projective invariant, $I_p(t_1, t_2, t_3, t_4) = ([t_1, t_2, t_3, t_4], \pm\{t_1, t_2, t_3, t_4\})$, i.e., super cross-ratio, where, if $\mathrm{ord}(t_1, t_2, t_3) = 1$,

$$[t_1, t_2, t_3, t_4] = \frac{[t_1, t_3][t_2, t_4]}{[t_2, t_3][t_1, t_4]},$$

$$\{t_1, t_2, t_3, t_4\} = [t_1, t_2, t_3, t_4]^{\frac{1}{2}} \frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^{\frac{1}{2}}}$$

Theorem I (cont'd)

- If a bijection Φ de $\mathrm{S}^{1|1}$ preserves I_e, or I_a, or I_p, then $\Phi = \widehat{h}$ for h in $\mathrm{E}_+(1|1)$, or $\mathrm{Aff}_+(1|1)$, or $\mathrm{SpO}_+(2|1)$, respectively.
- If a contactomorphism $\Phi \in K(1)$ preserves the even part of I_e, or I_a, or I_p, then $\Phi = \widehat{h}$ for h in $\mathrm{E}(1|1)$, or $\mathrm{Aff}(1|1)$, or $\mathrm{SpO}_+(2|1)$, respectively.

Sketch of proof: One shows that the $\mathrm{SpO}_+(2|1)$-action is 3|2-transitive.[4] The projective invariant I_{p} stems from the general theorem via the triple

$$p = ((\infty, 0), (0, 0), (1, \zeta))$$

Now for every oriented triple of "points" $t = (t_1, t_2, t_3)$ there exist two elements $h_\pm \in \mathrm{SpO}_+(2|1)$ linking t to the class $\mathrm{p} = [p]$; $\Rightarrow$ the odd invariant is defined up to an overall sign.
In fact: $\widehat{h}_\pm(t_1) = (\infty, 0), \widehat{h}_\pm(t_2) = (0, 0), \widehat{h}_\pm(t_3) = (1, \pm\zeta_3)$ where

$$\zeta_3 = \frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^{\frac{1}{2}}}$$

is uniquely determined by t_1, t_2, t_3. Hence

$$I_{\mathrm{p}}(t_1, t_2, t_3, t_4) = \widehat{h}_\pm(t_4)$$

[4] More precisely that of the group generated by $\mathrm{SpO}_+(2|1)$ and $(x, \xi) \mapsto (-x, \xi)$.

The Cartan formula

Consider $\Phi \in \mathrm{Diff}(S^1)$, the flow $\phi_\varepsilon = \mathrm{Id} + \varepsilon X + O(\varepsilon^2)$ of a vector field X, and 4 points t_1, $t_2 = \phi_\varepsilon(t_1)$, $t_3 = \phi_{2\varepsilon}(t_1)$, $t_4 = \phi_{3\varepsilon}(t_1)$.

The Schwarzian derivative of Φ is defined, via the cross-ratio, as the quadratic differential $\mathcal{S}(\Phi) \in \mathcal{Q}(S^1)$ appearing in

$$\frac{\Phi^*[t_1, t_2, t_3, t_4]}{[t_1, t_2, t_3, t_4]} - 1 = \langle \varepsilon X \otimes \varepsilon X, \mathcal{S}(\Phi) \rangle + O(\varepsilon^3)$$

This formula (and its avatar for $\mathcal{A}(\Phi)$) admits a prolongation to the case of the supercircle; it leads to the following result:

Theorem II

The even Euclidean, affine and projective invariants $\Rightarrow$ three 1-cocycles of $K(1)$, with kernel $\mathrm{E}(1|1)$, $\mathrm{Aff}(1|1)$ et $\mathrm{SpO}_+(2|1)$ resp.:

- the Euclidean cocycle $\mathcal{E} : K(1) \to \mathcal{F}_0(\mathrm{S}^{1|1})$:

$$\mathcal{E}(\Phi) = \log E_\Phi = \log(D\psi)^2$$

- the affine cocycle $\mathcal{A} : K(1) \to \Omega^1(\mathrm{S}^{1|1})$:

$$\mathcal{A}(\Phi) = d\mathcal{E}(\Phi)$$

- the projective cocycle (superSchwarzian) $\mathcal{S} : K(1) \to \mathcal{Q}(\mathrm{S}^{1|1})$:

$$\boxed{\mathcal{S}(\Phi) = \frac{2}{3}\,\alpha^{\frac{1}{2}} L_D\,\mathrm{S}(\Phi)}$$

where

$$\mathrm{S}(\Phi) = \frac{1}{4}\left(\frac{D^3 E_\Phi}{E_\Phi} - \frac{3}{2}\frac{DE_\Phi\, D^2 E_\Phi}{E_\Phi^2}\right)\alpha^{3/2}$$

Theorem II (cont'd)

Using the projections of $\mathcal{Q}(\mathrm{S}^{1|1})$ to summands of densities, one obtains two new affine and projective 1-cocycles:

- the projection of the affine cocycle $\mathrm{A} : K(1) \to \mathcal{F}_{\frac{1}{2}}(\mathrm{S}^{1|1})$:

$$\mathrm{A}(\Phi) = \alpha^{\frac{1}{2}} \langle D, \mathcal{A}(\Phi) \rangle = \frac{DE_\Phi}{E_\Phi}\, \alpha^{\frac{1}{2}}$$

- the projection of the Schwarzian cocycle $\mathrm{S} : K(1) \to \mathcal{F}_{3/2}(\mathrm{S}^{1|1})$:

$$\mathrm{S}(\Phi) = \alpha^{\frac{1}{2}} \langle D, \mathcal{S}(\Phi) \rangle = \frac{1}{4} \left(\frac{D^3 E_\Phi}{E_\Phi} - \frac{3}{2} \frac{DE_\Phi\, D^2 E_\Phi}{E_\Phi^2} \right) \alpha^{3/2}$$

This expression is due to Radul.

Alternative formulae for the superSchwarzian

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- Let us point out the following expression

$$\mathcal{S}(\Phi) = \frac{1}{6}\alpha^2 D\widetilde{S}(\Phi) + \frac{1}{2}\alpha\beta\widetilde{S}(\Phi) \qquad \text{where} \qquad \widetilde{S}(\Phi) = \mathrm{S}(\Phi)\alpha^{-3/2}$$

which duly returns, via projection $\pi : C^\infty(S^{1|1}) \to C^\infty(S^1)$, the classical Schwarzian derivative (in contrast to $\mathrm{S}(\Phi)$ (!)).

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- Notice the alternative formulae

$$\begin{aligned}\mathrm{S}(\Phi) &= \frac{1}{4}\alpha^{\frac{1}{2}}\langle D, (\alpha^{\frac{1}{2}} L_D)^2 \mathcal{A}(\Phi) - \frac{1}{2}\mathcal{A}(\Phi)^2\rangle \\ &= -\frac{1}{2} E_\Phi^{\frac{1}{2}} D^3 (E_\Phi^{-\frac{1}{2}})\,\alpha^{3/2}\end{aligned}$$

yielding the Schwarzian [Radul] in terms of the affine cocycle $\mathcal{A}$ & the multiplier E.

Determination of $H^1(K(1),\mathcal{M})$ for $\mathcal{M}=\mathcal{F}_\lambda,\Omega^1,\mathcal{Q}$

The 1-cocycles of $k(1)$ (Lie superalgebra hamiltonian vector fields of $(\mathrm{S}^{1|1},\alpha)$) associated with $\mathcal{E}$, A et S are trivially the $c_i : k(1)\to\mathcal{F}_{i/2}$:

$$c_i(X_f) = (D^{i+2}f)\,\alpha^{i/2} \qquad (i=0,1,3)$$

These are the 3 out of 4 generators of $H^1(k(1),\mathcal{F}_\lambda)$ [Agrebaoui & Ben Fraj], the only ones which integrate as (non trivial) $K(1)$-cocycles.

Theorem

The cohomology spaces

$$H^1(K(1),\mathcal{F}_\lambda(\mathrm{S}^{1|1})) = \begin{cases} \mathbb{R} & \text{si } \lambda = 0,\ \frac{1}{2},\ \frac{3}{2} \\ \{0\} & \text{sinon} \end{cases}$$

are resp. generated by $\mathcal{E}$, A et S. The cohomology spaces

$$H^1(K(1),\Omega^1(\mathrm{S}^{1|1}))=\mathbb{R} \qquad \text{et} \qquad H^1(K(1),\mathcal{Q}(\mathrm{S}^{1|1}))=\mathbb{R}$$

are resp. generated by $\mathcal{A}$ et $\mathcal{S}$.

Case of the supercircle $S^{1|N}$

For the supercircle $\mathrm{S}^{1|N}$, endowed with the contact 1-form

$$\alpha = dx + \sum_{i,j=1}^{N} \delta_{ij}\, \xi^i d\xi^j$$

the invariants of $\mathrm{E}_+(1|N)$, $\mathrm{A}_+(1|N)$ and $\mathrm{SpO}_+(2|N)$ retain th same form as for $N = 1$. However, the odd invariant $(I_\mathrm{p})_1$ is no longer determined up to a sign (mod $\mathrm{O}(1)$), but up to the action of $\mathrm{O}(N)$.

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Remark: In the Cartan formula [$\Rightarrow$ 1-cocycles of $K(N)$], $\Phi^*[t_1, t_2]$ is *no more* proportional to $[t_1, t_2]$, up to $O(\varepsilon^3)$, for $N \geq 3$. The Schwarzian, $\mathcal{S}(\Phi)$, is no longer given by the Cartan formula if $N \geq 3$.

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But . . .

Theorem

One deduces, from the even cross-ratio $(I_p)_0$ and the Cartan formula, the projective 1-cocycle $\mathcal{S} : K(2) \to \mathcal{Q}(\mathrm{S}^{1|2})$

$$\mathcal{S} = \frac{1}{6}\alpha^2 \left(D_1 D_2 \mathrm{S}_{12} + \frac{1}{2}\mathrm{S}_{12}^2 \right) + \frac{1}{2}\alpha(\beta^1 D_2 + \beta^2 D_1)\mathrm{S}_{12} + \beta^1 \beta^2\, \mathrm{S}_{12}$$

with $\mathrm{S}_{12} = 2\,\mathrm{S}\,\alpha^{-1}$ where

$$\mathrm{S}(\Phi) = \left(\frac{D_2 D_1 E_\Phi}{E_\Phi} - \frac{3}{2}\frac{D_2 E_\Phi D_1 E_\Phi}{E_\Phi^2} \right)\alpha$$

The projection of quadratic differentials to 1-densities of $\mathrm{S}^{1|2}$ returns the above Schwarzian derivative $\mathrm{S} : K(2) \to \mathcal{F}_1(\mathrm{S}^{1|2})$.

The kernels of these cocycles coincide and are isomorphic to $\mathrm{PC}(2|2)$.

Perspectives

- Classification of the geometries of $(\mathrm{S}^{1|2}, [\alpha])$ — see [Ben Fraj & Salem]
- Construction of the Bott cocycle of $K(1)$ and $K(2)$ via the cup product of $\mathcal{E}$ and $\mathcal{A}$.
- Detailed study of the Möbius supercircle $\mathrm{S}_+^{1|1}$.[5]
- Superization of the Lorentzian hyperboloid of one sheet $\mathcal{H}^{1,1} \subset \mathrm{sl}(2, \mathbb{R})$ whose conformal geometry is holographically related to the projective geometry of conformal infinity, S^1 — [Kostant-Sternberg, Guieu-ChD].

[5] Its superfunctions are defined as the smooth superfunctions of $\mathbb{R}^{1|1}$ invariant under $(x, \xi) \mapsto (x + 2\pi, -\xi)$.

Reference

J.-P. Michel & ChD, On the projective geometry of the supercircle: a unified construction of the super cross-ratio and Schwarzian derivative,
`http://xxx.lanl.gov/abs/0710.1544v2`
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Intermezzo: Supermanifolds

- A supermanifold of dim $n|N$ is a pair $\mathcal{M} = (M, \mathcal{O}_M)$ with M a n-dimensional smooth manifold and $\mathcal{O}_M$ a sheaf of commutative superalgebras, locally isomorphic to a superdomain of dim $n|N$.
- A superdomain is a triple $\mathcal{U} = (U, \mathcal{A}(U), \mathrm{ev})$ where $U \subset \mathbb{R}^n$ open, $\mathcal{A}(U) = C^\infty(U) \otimes \Lambda(\xi_1, \ldots, \xi_N)$, and for all $x \in U$, evaluation map $\mathrm{ev}_x : \mathcal{A}(U) \to \mathbb{R}$ defined by $\mathrm{ev}_x(f \otimes 1) = f(x)$ & $\mathrm{ev}_x(1 \otimes \xi_i) = 0$ is a morphism of superalgebras.
- A morphism between superdomains $\Phi : \mathcal{U}_1 \to \mathcal{U}_2$ is a pair (ϕ, χ) where $\phi \in C^\infty(U_1, U_2)$ and $\chi : \mathcal{A}(U_2) \to \mathcal{A}(U_1)$ is a morphism of superalgebras s.t. $\mathrm{ev}_x(\chi(f)) = \mathrm{ev}_{\phi(x)}(f)$.
- A morphism of supermanifolds $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a pair (ϕ, χ) where $\phi \in C^\infty(M_1, M_2)$ and $\chi : \mathcal{O}_{M_2} \to \phi_* \mathcal{O}_{M_1}$ is a morphism of sheaves, inducing locally a morphism of superdomains.
- A diffemorphism of supermanifolds is a morphism $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ s.t. $\Phi^{-1} : \mathcal{M}_2 \to \mathcal{M}_1$ exists and is a morphism.