On the Euclidean, Affine & Projective invariants of the supercircle, and their associated cocycles

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1. Introduction

2. Contact structure of the supercircle $S^1|1$

3. Euclidean, affine and projective invariants of $S^1|1$
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Find super geometric versions of the Euclidean, affine, and projective invariants of $S^1$. Super cross-ratio?

What are then the 1-cocycles associated with these super extensions of $\text{Diff}^+(S^1)$? Super Schwarzian derivative?

How can one relate these new (super) geometric objects?

Towards a classification of the geometries of the supercircle?
Find super geometric versions of the Euclidean, affine, and projective invariants of $S^1$. Super cross-ratio?

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Towards a classification of the geometries of the supercircle?
Diagrammatic representation: Geometries of $S^{1\mid 1}$

Groups $\xleftarrow{p|q\text{-transitivity}}$ Invariants

Cocycles $\xleftarrow{\text{kernels}}$ Cartan
Bibliographical landmarks

1. Super cross-ratio (even and odd)
   - Aoki ['88] *Super-Riemann surfaces*
   - Nelson ['88] *Superstrings*
   - Uehara & Yasui ['90] *WP on Super-Teichmueller*
   - Manin ['91] *NCG*
   - Giddings ['92] *Punctured Super-Riemann surfaces*
   - ... 

2. Super-Schwarzian
   - Friedan ['86] (N=1) *CFT*
   - Radul ['86] (N=1,2,3) *Super-Bott cocycle*
   - Cohn ['87](N=2) *Super-Riemann surfaces*
   - Gieres & Theisen ['93] *Superconformally covariant operators*
   - ...
Invariants of $E(1) \subset \text{Aff}_+(1) \subset \text{PSL}(2, \mathbb{R}) \subset \text{Diff}_+(S^1)$

- Euclidean invariant (translations): $\text{distance}^1$

  $$[x_1, x_2] = x_2 - x_1$$

- Affine invariant (homotheties, translations): $\text{distance ratio}$

  $$[x_1, x_2, x_3] = \frac{[x_1, x_3]}{[x_1, x_2]}$$

- Projective invariant (homographies): $\text{cross-ratio}$

  $$[x_1, x_2, x_3, x_4] = \frac{[x_1, x_3][x_2, x_4]}{[x_2, x_3][x_1, x_4]}$$

$^1$One deals effectively with $\mathbb{R}P^1$
The associated \( \text{Diff}_+(S^1) \) 1-cocycles

- **Euclidean cocycle** \( \mathcal{E} : \text{Diff}_+(S^1) \to C^\infty(S^1) \):
  \[
  \mathcal{E}(\varphi) = \log(\varphi')
  \]

- **Affine cocycle** \( \mathcal{A} : \text{Diff}_+(S^1) \to \Omega^1(S^1) \):
  \[
  \mathcal{A}(\varphi) = d\mathcal{E}(\varphi)
  \]

- **Projective cocycle**: Schwarzian derivative \( \mathcal{S} : \text{Diff}_+(S^1) \to \mathcal{Q}(S^1) \):
  \[
  \mathcal{S}(\varphi) = \left( \frac{\varphi'''}{\varphi'} - \frac{3}{2} \left( \frac{\varphi''}{\varphi'} \right)^2 \right) dx^2
  = dx \otimes L_{\partial_x} \mathcal{A}(\varphi) - \frac{1}{2} \mathcal{A}(\varphi)^2
  \]
The associated $\text{Diff}_+(S^1)$ 1-cocycles (cont’d)

The Schwarzian $S(\varphi)$ measures, at each point $x$, the shift between a diffeomorphism $\varphi \in \text{Diff}(S^1)$ and its approximating homography, $h \in \text{PGL}(2, \mathbb{R})$,\(^2\)

$$S(\varphi)(x) = (\hat{h}^{-1} \circ \varphi)'''(x)$$

- It is a $\text{PSL}(2, \mathbb{R})$-differential invariant for $\text{Diff}_+(S^1)$: $S(\varphi) = S(\psi)$ iff $\varphi = A \circ \psi$ where $A \in \text{PSL}(2, \mathbb{R})$.

- It is a non trivial 1-cocycle of $\text{Diff}_+(S^1)$ with values in the module of quadratic differentials $Q(S^1)$:

$$S(\varphi \circ \psi) = \psi^* S(\varphi) + S(\psi)$$

for all $\varphi, \psi \in \text{Diff}_+(S^1)$. It has kernel $\text{PSL}(2\mathbb{R})$.

\(^2\)s.t. $\hat{h}^{-1} \circ \varphi$ has the 2-jet of $\text{Id}$ at $x$
The supercircle $S^{1|1}$

The supercircle $S^{1|1}$: the circle $S^1$, endowed with (a sheaf of associative and commutative $\mathbb{Z}/(2\mathbb{Z})$-graded algebras, the sections of which are) the superfunctions $C^\infty(S^{1|1}) = C^\infty(S^1)[\xi]$ where $\xi^2 = 0$. 
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- If $(x, \xi)$ are local coordinates of (affine) superdomain, every superfunction writes

$$f(x, \xi) = f_0(x) + \xi f_1(x), \quad \text{where} \quad f_0, f_1 \in C^\infty(S^1)$$
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- Projection: $\pi : C^\infty (S^{1|1}) \to C^\infty (S^1)$ where $\ker(\pi)$: ideal generated by nilpotent elements.
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- Group of diffeomorphisms: $\text{Diff}(S^{1|1}) = \text{Aut}(C^\infty(S^{1|1}))$. For practical purposes: a diffeomorphism is a pair $\Phi = (\varphi, \psi)$ of superfunctions s.t. $(\varphi(x, \xi), \psi(x, \xi))$ are new coordinates on $S^{1|1}$. 
Vector fields & 1-forms of the supercircle

**Vector fields** $\text{Vect}(S^{1|1}) = \text{SuperDer}(C^\infty(S^{1|1}))$. Every Vector field is written locally as

$$X = f(x, \xi) \partial_x + g(x, \xi) \partial_\xi$$

where $f, g \in C^\infty(S^{1|1})$

NB: $\text{Vect}(S^{1|1})$ is a $C^\infty(S^{1|1})_L$-module locally generated by $(\partial_x, \partial_\xi)$ where $p(\partial_x) = 0$, $p(\partial_\xi) = 1$. It is also a **Lie superalgebra** with Lie bracket $[X, Y] = XY - (-1)^{p(X)p(Y)} YX$. 
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**Differential 1-forms** \( \Omega^1(S^{1|1}) = C^\infty(S^{1|1})_R \)-module locally generated by dual basis \((dx, d\xi)\) where \( p(dx) = 0, \ p(d\xi) = 1 \). The module \( \Omega^*(S^{1|1}) \) of differential forms is bigraded (cohomology degree \(| \cdot |\), parity \(p\)); our choice of **Sign Rule**:

\[
\alpha \wedge \beta = (-1)^{|\alpha||\beta| + p(\alpha)p(\beta)} \beta \wedge \alpha
\]
Contact structure on supercircle $S^{1|1}$

It is given by direction of contact 1-form

\[ \alpha = dx + \xi d\xi \]

We have

\[ d\alpha = \beta \wedge \beta \quad \text{where} \quad \beta = d\xi \]
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- SUSY contact distribution, \text{ker}(\alpha), generated by “covariant derivative”

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- Contactomorphisms: the automorphisms of $(S^{1|1}, [\alpha])$:

$$K(1) = \{ \Phi \in \text{Diff}(S^{1|1}) \mid \Phi^* \alpha = E_\Phi \alpha \}$$

One shows that $\Phi = (\varphi, \psi) \in K(1) \iff D\varphi = \psi D\psi$ and

$$E_\Phi = (D\psi)^2$$
Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of $\lambda$-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with the (anti)action ($\Phi \mapsto \Phi_\lambda$) defined by $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$. (One writes symbolically $F \in \mathcal{F}_\lambda$ as $F = f \alpha^\lambda$.)
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- The $C^\infty(S^{1|1})_\mathbb{R}$-module $\Omega^1(S^{1|1})$ of 1-forms is generated by $\alpha$ et $\beta$.
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- The $C^\infty(S^{1|1})_R$-module $\Omega^1(S^{1|1})$ of 1-forms is generated by $\alpha$ et $\beta$.
- The $C^\infty(S^{1|1})_R$-module $Q(S^{1|1})$ of quadratic differentials is generated by $\alpha^2 = \alpha \otimes \alpha$ et $\alpha \beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$. 
Densities, 1-forms & quadratic differentials

Let $\mathcal{F}_\lambda(S^{1|1})$ be the $K(1)$-module of $\lambda$-densities ($\lambda \in \mathbb{C}$): $C^\infty(S^{1|1})$ endowed with the (anti)action ($\Phi \mapsto \Phi_\lambda$) defined by $\Phi_\lambda f = (E_\Phi)^\lambda \Phi^* f$.

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- The $C^\infty(S^{1|1})_R$-module $Q(S^{1|1})$ of quadratic differentials is generated by $\alpha^2 = \alpha \otimes \alpha$ et $\alpha\beta = \frac{1}{2}(\alpha \otimes \beta + \beta \otimes \alpha)$.

Proposition

Both $\Omega^1(S^{1|1})$ and $Q(S^{1|1})$ are $K(1)$-modules ; they admit the decomposition into $K(1)$-submodules:

\[ \Omega^1(S^{1|1}) \cong \mathcal{F}_{\frac{1}{2}} \oplus \mathcal{F}_1, \quad Q(S^{1|1}) \cong \mathcal{F}_{\frac{3}{2}} \oplus \mathcal{F}_2 \]

The projections $\Omega^1(S^{1|1}) \to \mathcal{F}_{\frac{1}{2}}$ (resp. $Q(S^{1|1}) \to \mathcal{F}_{\frac{3}{2}}$) are given by $\alpha^\frac{1}{2} \langle D, \cdot \rangle$, and the corresponding sections by $\alpha^\frac{1}{2} L_D$ (resp. $\frac{2}{3} \alpha^\frac{1}{2} L_D$).

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\[a\] Cf. the decomposition: $\text{Vect}(S^{1|1}) \cong \mathcal{F}_{-1} \oplus \mathcal{F}_{-\frac{1}{2}}$ [Gargoubi-Mellouli-Ovsienko].
The Orthosymplectic group, its Euclidean and affine subgroups

It is the subgroup $\text{SpO}(2|1) \subset \text{GL}(2|1)$ of symplectomorphisms of $\mathbb{R}^{2|1}$, with symplectic form $d\varpi$ where $\varpi = \frac{1}{2}(pdq - qdp + \theta d\theta)$; its elements are of the form

$$h = \begin{pmatrix} a & b & \gamma \\ c & d & \delta \\ \alpha & \beta & e \end{pmatrix}$$

where

$$ad - bc - \alpha\beta = 1, \ e^2 + 2\gamma\delta = 1, \ \alpha e - a\delta + c\gamma = 0, \ \beta e - b\delta + d\gamma = 0.$$ 

The group $\text{SpO}(2|1)$ also preserving the 1-forme $\varpi = \frac{1}{2}p^2\alpha$ (where $p \neq 0$), it acts by contactomorphisms via the projective action of $S^{1|1}$:

$$\hat{h}(x, \xi) = \left( \frac{ax + b + \gamma\xi}{cx + d + \delta\xi}, \frac{\alpha x + \beta + e\xi}{cx + d + \delta\xi} \right)$$

The Berezinian is $\text{Ber}(h) = e + \alpha\beta e^{-1}$ and $\text{SpO}_+(2|1) = \text{Ber}^{-1}(1)$ is a super-extension of $\text{Sp}(2, \mathbb{R}) = \text{SL}(2, \mathbb{R})$. 

Le orthosymplectic group (cont’d)

One has the local factorization

\[ \text{SpO}_+(2|1) \ni h = \begin{pmatrix} 1 & 0 & 0 \\ \tilde{c} & 1 & \delta \\ \delta & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \tilde{b} & -\tilde{\beta} \\ 0 & \epsilon & 0 \\ 0 & \epsilon \tilde{\beta} & 1 \end{pmatrix} \]

where \((\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}\), with \(\epsilon^2 = 1, \tilde{a} > 0\).
Le orthosymplectic group (cont’d)

One has the local factorization

$$\text{SpO}_+(2|1) \ni h = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \tilde{c} & 1 & \tilde{\alpha} \\ \tilde{\delta} & 0 & 1 \end{pmatrix} \begin{pmatrix} \tilde{a} & 0 & 0 \\ 0 & \tilde{a}^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \epsilon & \tilde{b} & -\tilde{\beta} \\ 0 & \epsilon & 0 \\ 0 & \epsilon \tilde{\beta} & 1 \end{pmatrix}$$

where \((\tilde{a}, \tilde{b}, \tilde{c}, \tilde{\beta}, \tilde{\delta}) \in \mathbb{R}^{3|2}\), with \(\epsilon^2 = 1\), \(\tilde{a} > 0\).

- The subgroup \(E(1|1)\) is the group of those \(\Phi \in \text{Diff}(S^{1|1})\) s.t. \(\Phi^* \alpha = \alpha\) and \(\Phi^* \beta = \epsilon \beta\).

- One also has \(\text{Aff}(1|1) = \{ \Phi \in K(1) | \Phi^* \beta = F_\Phi \beta \} \).
Notion of $p|q$-transitivity

Extension of the notion of $n$-transitivity to supergroup actions.

Consider $E = E_0 \times E_1$ and $p_0$ & $p_1$ its canonical projections.
Two $n$-uplets $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of $E$ are said $p|q$-equivalent, $s \equiv_{p|q} t$, where $n = \max(p, q)$, if $p_0(s_i) = p_0(t_i) \ \forall i = 1, \ldots, p$ and $p_1(s_i) = p_1(t_i) \ \forall i = 1, \ldots, q$.  

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$^3$The $n|n$-equivalence is an ... equality!
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The action of group $G$ on $E$ is said (simply) $p|q$-transitive if for all $s = (s_1, \ldots, s_n)$ and $t = (t_1, \ldots, t_n)$ of distinct points of $E$, there exists a (unique) $g \in G$ s.t. $\hat{g}(t) \overset{p|q}{\equiv} s$.

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Example: The $\text{PGL}(2, \mathbb{R})$-action on $S^1$ is simply 3-transitive.

\[^3\text{The } n|n\text{-equivalence is an } \ldots \text{equality!}\]
Construction of the invariants

**Theorem**

Let the group $G$ act simply $p|q$-transitively on $E = E_0 \times E_1$, and $m$ be an $n$-uplet, $n = \max(p, q)$, of distinct points of $E$. The $n + 1$ point-function $I[m]$ of $E$ with values in $E$ defined by

$$I[m](t_1, \ldots, t_{n+1}) = \hat{h}(t_{n+1})$$

where $\hat{h}(t) \overset{p|q}{=} m$, and $t = (t_1, \ldots, t_n) \in E^n \setminus \Gamma$ enjoys the properties:

1. $I[m]$ is $G$-invariant
2. If $\Phi \in E!$ preserves $I[m]$, then $\Phi = \hat{g}$ where $g \in G$

If $n = p > q$, the $n$-point functions with values in $E_1$

$$J_{[m],j}(t) = p_1(\hat{h}(t_j)) \quad (j = q + 1, \ldots, n)$$

are $G$-invariant. The $l[m]$ and $J_{[m],j}$ generate all invariants with $n + 1$ and $n$ points.
Theorem I

- **Euclidean invariant:** $l_e(t_1, t_2) = ([t_1, t_2], \{t_1, t_2\})$ with
  
  $[t_1, t_2] = x_2 - x_1 - \xi_2 \xi_1, \quad \{t_1, t_2\} = \xi_2 - \xi_1$

- **Affine invariant,** $l_a(t_1, t_2, t_3) = ([t_1, t_2, t_3], \{t_1, t_2, t_3\})$, where, if $x_1 < x_2,$

  $[t_1, t_2, t_3] = \frac{[t_1, t_3]}{[t_1, t_2]}, \quad \{t_1, t_2, t_3\} = [t_1, t_2, t_3]^\frac{1}{2} \frac{\{t_1, t_3\}}{[t_1, t_3]^\frac{1}{2}}$

- **Projective invariant,** $l_p(t_1, t_2, t_3, t_4) = ([t_1, t_2, t_3, t_4], \pm\{t_1, t_2, t_3, t_4\})$, i.e., super cross-ratio, where, if $\text{ord}(t_1, t_2, t_3) = 1,$

  $[t_1, t_2, t_3, t_4] = \frac{[t_1, t_3][t_2, t_4]}{[t_2, t_3][t_1, t_4]},$

  $\{t_1, t_2, t_3, t_4\} = [t_1, t_2, t_3, t_4]^\frac{1}{2} \frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^\frac{1}{2}}$
Theorem I (cont’d)

- If a bijection $\Phi$ de $S^{1|1}$ preserves $l_e$, or $l_a$, or $l_p$, then $\Phi = \hat{h}$ for $h$ in $E_+(1|1)$, or $\text{Aff}_+(1|1)$, or $\text{SpO}_+(2|1)$, respectively.

- If a contactomorphism $\Phi \in K(1)$ preserves the even part of $l_e$, or $l_a$, or $l_p$, then $\Phi = \hat{h}$ for $h$ in $E(1|1)$, or $\text{Aff}(1|1)$, or $\text{SpO}_+(2|1)$, respectively.
Sketch of proof: One shows that the $\text{SpO}_+(2|1)$-action is $3|2$-transitive. The projective invariant $l_p$ stems from the general theorem via the triple

$$p = ((\infty, 0), (0, 0), (1, \zeta))$$

Now for every oriented triple of “points” $t = (t_1, t_2, t_3)$ there exist two elements $h_\pm \in \text{SpO}_+(2|1)$ linking $t$ to the class $p = [p]$; $\Rightarrow$ the odd invariant is defined up to an overall sign.

In fact: $\hat{h}_\pm(t_1) = (\infty, 0), \hat{h}_\pm(t_2) = (0, 0), \hat{h}_\pm(t_3) = (1, \pm \zeta_3)$ where

$$\zeta_3 = \frac{\{t_2, t_4\}[t_1, t_2] - \{t_1, t_2\}[t_2, t_4]}{([t_1, t_2][t_2, t_4][t_1, t_4])^{1/2}}$$

is uniquely determined by $t_1, t_2, t_3$. Hence

$$l_p(t_1, t_2, t_3, t_4) = \hat{h}_\pm(t_4)$$

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4 More precisely that of the group generated by $\text{SpO}_+(2|1)$ and $(x, \xi) \mapsto (-x, \xi)$. 4
The Cartan formula

Consider $\Phi \in \text{Diff}(S^1)$, the flow $\phi_\varepsilon = \text{Id} + \varepsilon X + O(\varepsilon^2)$ of a vector field $X$, and 4 points $t_1, t_2 = \phi_\varepsilon(t_1), t_3 = \phi_2\varepsilon(t_1), t_4 = \phi_3\varepsilon(t_1)$.

The Schwarzian derivative of $\Phi$ is defined, via the cross-ratio, as the quadratic differential $S(\Phi) \in \mathcal{Q}(S^1)$ appearing in

$$\frac{\Phi^*[t_1, t_2, t_3, t_4]}{[t_1, t_2, t_3, t_4]} - 1 = \langle \varepsilon X \otimes \varepsilon X, S(\Phi) \rangle + O(\varepsilon^3)$$

This formula (and its avatar for $A(\Phi)$) admits a prolongation to the case of the supercircle; it leads to the following result:
Theorem II

The even Euclidean, affine and projective invariants \( \Rightarrow \) three 1-cocycles of \( K(1) \), with kernel \( \text{E}(1|1) \), \( \text{Aff}(1|1) \) et \( \text{SpO}_+(2|1) \) resp.:

- the Euclidean cocycle \( \mathcal{E} : K(1) \rightarrow \mathcal{F}_0(S^{1|1}) \):
  \[
  \mathcal{E}(\Phi) = \log E_\Phi = \log (D\psi)^2
  \]

- the affine cocycle \( \mathcal{A} : K(1) \rightarrow \Omega^1(S^{1|1}) \):
  \[
  \mathcal{A}(\Phi) = d\mathcal{E}(\Phi)
  \]

- the projective cocycle (superSchwarzian) \( S : K(1) \rightarrow \mathcal{Q}(S^{1|1}) \):
  \[
  S(\Phi) = \frac{2}{3} \alpha \frac{1}{2} L_D S(\Phi)
  \]

where

\[
S(\Phi) = \frac{1}{4} \left( \frac{D^3 E_\Phi}{E_\Phi} - \frac{3}{2} \frac{DE_\Phi D^2 E_\Phi}{E_\Phi^2} \right) \alpha^{3/2}
\]
Theorem II (cont’d)

Using the projections of \( Q(S^{1|1}) \) to summands of densities, one obtains two new affine and projective 1-cocycles:

- the projection of the affine cocycle \( A : K(1) \to \mathcal{F}_{\frac{1}{2}}(S^{1|1}) \):
  \[
  A(\Phi) = \alpha^{\frac{1}{2}} \langle D, A(\Phi) \rangle = \frac{DE_\Phi}{E_\Phi} \alpha^{\frac{1}{2}}
  \]

- the projection of the Schwarzian cocycle \( S : K(1) \to \mathcal{F}_{3/2}(S^{1|1}) \):
  \[
  S(\Phi) = \alpha^{\frac{1}{2}} \langle D, S(\Phi) \rangle = \frac{1}{4} \left( \frac{D^3 E_\Phi}{E_\Phi} - \frac{3}{2} \frac{DE_\Phi D^2 E_\Phi}{E_\Phi^2} \right) \alpha^{\frac{3}{2}}
  \]

This expression is due to Radul.
Alternative formulae for the superSchwarzian

Let us point out the following expression

$$S(\Phi) = \frac{1}{6} \alpha^2 \tilde{S}(\Phi) + \frac{1}{2} \alpha \beta \tilde{S}(\Phi)$$

where

$$\tilde{S}(\Phi) = S(\Phi)$$

which duly returns, via projection $\pi$:

$$C_\infty(S_1|1) \rightarrow C_\infty(S_1),$$

the classical Schwarzian derivative (in contrast to $S(\Phi)$).

Notice the alternative formulae

$$S(\Phi) = \frac{1}{4} \alpha \frac{1}{2} \left< \frac{D}{\alpha^2 L D} A(\Phi) - \frac{1}{2} A(\Phi) \right> = -\frac{1}{2} E^\frac{1}{2} \Phi D^3 (E - \frac{1}{2} \Phi) \alpha^3$$

yielding the Schwarzian [Radul] in terms of the affine cocycle $A$ and the multiplier $E$.

Christian DUVAL CPT & UM (Aix-Marseille II)
On the three geometries of the supercircle
Vancouver, 29-07-2009
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Alternative formulae for the superSchwarzian

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Notice the alternative formulae

\[ S(\Phi) = \frac{1}{4} \alpha^{1/2} \langle D, (\alpha^{1/2} L_D)^2 A(\Phi) - \frac{1}{2} A(\Phi)^2 \rangle \]

\[ = -\frac{1}{2} E_{\Phi}^{1/2} D^3 (E_{\Phi}^{-1/2}) \alpha^{3/2} \]

yielding the Schwarzian [Radul] in terms of the affine cocycle \( A \) & the multiplier \( E \).
Determination of $H^1(K(1), \mathcal{M})$ for $\mathcal{M} = \mathcal{F}_\lambda, \Omega^1, Q$

The 1-cocycles of $k(1)$ (Lie superalgebra hamiltonian vector fields of $(S^{1|1}, \alpha)$) associated with $\mathcal{E}, \mathcal{A}$ et $S$ are trivially the $c_i : k(1) \to \mathcal{F}_{i/2}$:

$$c_i(X_f) = (D^{i+2} f) \alpha^{i/2} \quad (i = 0, 1, 3)$$

These are the 3 out of 4 generators of $H^1(k(1), \mathcal{F}_\lambda)$ [Agrebaoui & Ben Fraj], the only ones which integrate as (non trivial) $K(1)$-cocycles.

**Theorem**

The cohomology spaces

$$H^1(K(1), \mathcal{F}_\lambda(S^{1|1})) = \begin{cases} \mathbb{R} & \text{si } \lambda = 0, \frac{1}{2}, \frac{3}{2} \\ \{0\} & \text{sinon} \end{cases}$$

are resp. generated by $\mathcal{E}, \mathcal{A}$ et $S$. The cohomology spaces

$$H^1(K(1), \Omega^1(S^{1|1})) = \mathbb{R} \quad \text{et} \quad H^1(K(1), Q(S^{1|1})) = \mathbb{R}$$

are resp. generated by $\mathcal{A}$ et $S$. 
Case of the supercircle $S^{1|N}$

For the supercircle $S^{1|N}$, endowed with the contact 1-form

$$\alpha = dx + \sum_{i,j=1}^{N} \delta_{ij} \xi^i d\xi^j$$

the invariants of $E_+(1|N)$, $A_+(1|N)$ and $SpO_+(2|N)$ retain the same form as for $N = 1$. However, the odd invariant $(l_p)_1$ is no longer determined up to a sign (mod $O(1)$), but up to the action of $O(N)$. 

Remark: In the Cartan formula $\Rightarrow$ 1-cocycles of $K(N)$, $\Phi^* [t_1, t_2]$ is no more proportional to $[t_1, t_2]$, up to $O(\varepsilon^3)$, for $N \geq 3$. The Schwarzian, $S(\Phi)$, is no longer given by the Cartan formula if $N \geq 3$. But . . .
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For the supercircle $S^{1|N}$, endowed with the contact 1-form

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the invariants of $E_+(1|N)$, $A_+(1|N)$ and $SpO_+(2|N)$ retain the same form as for $N = 1$. However, the odd invariant $(I_p)_1$ is no longer determined up to a sign (mod $O(1)$), but up to the action of $O(N)$.

**Remark:** In the Cartan formula $[\Rightarrow 1$-cocycles of $K(N)]$, $\Phi^*[t_1, t_2]$ is no more proportional to $[t_1, t_2]$, up to $O(\varepsilon^3)$, for $N \geq 3$. The Schwarzian, $S(\Phi)$, is no longer given by the Cartan formula if $N \geq 3$. 
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**Remark**: In the Cartan formula $\Rightarrow$ 1-cocycles of $K(N)$, $\Phi^*[t_1, t_2]$ is *no more* proportional to $[t_1, t_2]$, up to $O(\varepsilon^3)$, for $N \geq 3$. The Schwarzian, $S(\Phi)$, is no longer given by the Cartan formula if $N \geq 3$.

But …
Theorem

One deduces, from the even cross-ratio \((I_p)_0\) and the Cartan formula, the projective 1-cocycle \(S : K(2) \to \mathcal{Q}(S^{1|2})\)

\[
S = \frac{1}{6} \alpha^2 \left( D_1 D_2 S_{12} + \frac{1}{2} S_{12}^2 \right) + \frac{1}{2} \alpha (\beta^1 D_2 + \beta^2 D_1) S_{12} + \beta^1 \beta^2 S_{12}
\]

with \(S_{12} = 2 S \alpha^{-1}\) where

\[
S(\Phi) = \left( \frac{D_2 D_1 E_\Phi}{E_\Phi} - \frac{3}{2} \frac{D_2 E_\Phi D_1 E_\Phi}{E_\Phi^2} \right) \alpha
\]

The projection of quadratic differentials to 1-densities of \(S^{1|2}\) returns the above Schwarzian derivative \(S : K(2) \to \mathcal{F}_1(S^{1|2})\).

The kernels of these cocycles coincide and are isomorphic to \(PC(2|2)\).
Perspectives

- Classification of the geometries of \((S^{1|2}, [\alpha])\) — see [Ben Fraj & Salem]

- Construction of the Bott cocycle of \(K(1)\) and \(K(2)\) via the cup product of \(\mathcal{E}\) and \(\mathcal{A}\).

- Detailed study of the Möbius supercircle \(S^{1|1}\).

- Superization of the Lorentzian hyperboloid of one sheet \(\mathcal{H}^{1,1} \subset \mathfrak{sl}(2, \mathbb{R})\) whose conformal geometry is holographically related to the projective geometry of conformal infinity, \(S^1\) — [Kostant-Sternberg, Guieu-ChD].

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5 Its superfunctions are defined as the smooth superfunctions of \(\mathbb{R}^{1|1}\) invariant under \((x, \xi) \mapsto (x + 2\pi, -\xi)\).
Reference

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Intermezzo: Supermanifolds

- A supermanifold of dim $n|N$ is a pair $\mathcal{M} = (M, \mathcal{O}_M)$ with $M$ a $n$-dimensional smooth manifold and $\mathcal{O}_M$ a sheaf of commutative superalgebras, locally isomorphic to a superdomain of dim $n|N$.

- A superdomain is a triple $\mathcal{U} = (U, \mathcal{A}(U), \text{ev})$ where $U \subset \mathbb{R}^n$ open, $\mathcal{A}(U) = C^\infty(U) \otimes \Lambda(\xi_1, \ldots, \xi_N)$, and for all $x \in U$, evaluation map $\text{ev}_x : \mathcal{A}(U) \to \mathbb{R}$ defined by $\text{ev}_x(f \otimes 1) = f(x)$ & $\text{ev}_x(1 \otimes \xi_i) = 0$ is a morphism of superalgebras.

- A morphism between superdomains $\Phi : \mathcal{U}_1 \to \mathcal{U}_2$ is a pair $(\phi, \chi)$ where $\phi \in C^\infty(U_1, U_2)$ and $\chi : \mathcal{A}(U_2) \to \mathcal{A}(U_1)$ is a morphism of superalgebras s.t. $\text{ev}_x(\chi(f)) = \text{ev}_{\phi(x)}(f)$.

- A morphism of supermanifolds $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ is a pair $(\phi, \chi)$ where $\phi \in C^\infty(M_1, M_2)$ and $\chi : \mathcal{O}_{M_2} \to \phi_*\mathcal{O}_{M_1}$ is a morphism of sheaves, inducing locally a morphism of superdomains.

- A diffeomorphism of supermanifolds is a morphism $\Phi : \mathcal{M}_1 \to \mathcal{M}_2$ s.t. $\Phi^{-1} : \mathcal{M}_2 \to \mathcal{M}_1$ exists and is a morphism.