Periodic Homogenization For Elliptic Nonlocal Equations
(PIMS Workshop on Analysis of nonlinear PDEs and free boundary problems)

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The Set-Up

Family of Oscillatory Nonlocal Equations:

$$\begin{cases} F(u^\varepsilon, \frac{x}{\varepsilon}) = 0 & \text{in } D \\ u^\varepsilon = g & \text{on } \mathbb{R}^n \setminus D \end{cases}$$

Translation Invariant Limit Nonlocal Equations

$$\begin{cases} \bar{F}(\bar{u}, x) = 0 & \text{in } D \\ \bar{u} = g & \text{on } \mathbb{R}^n \setminus D. \end{cases}$$

GOAL

Prove there is a unique nonlocal operator $\bar{F}$ so that u^ε will be very close to $\bar{u}$ as $\varepsilon \longrightarrow 0$. (Homogenization takes place.)

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$$F(u, \frac{x}{\varepsilon}) =$$

$$\inf_\alpha \sup_\beta \left\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K^{\alpha\beta}(\frac{x}{\varepsilon}, y) \mathrm{dy} \right\}.$$

(Think of a more familiar **2nd order** equation:

$$F(D^2 u, \frac{x}{\varepsilon}) = \inf_\alpha \sup_\beta \left\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + a_{ij}^{\alpha\beta}(\frac{x}{\varepsilon}) u_{x_i x_j}(x) \right\} \Bigg)$$

The Set-Up

Periodic Nonlocal Operator G

for all $z \in \mathbb{Z}^n$

$$G(u, x+z) = G(u(\cdot + z), x)$$

Our F will be periodic when $f^{\alpha\beta}$ and $K^{\alpha\beta}$ are periodic in x.

Translation Invariant Nonlocal Operator G

G is translation invariant if for any $y \in \mathbb{R}^n$,

$$G(u, x+y) = G(u(\cdot + y), x).$$

Main Theorem

Theorem (S. '08; Homogenization of Nonlocal Equations)

If F is periodic and uniformly elliptic, plus technical assumptions, then there exists a **translation invariant** *elliptic nonlocal operator $\bar{F}$ with the same ellipticity as F, such that $u^\varepsilon \to \bar{u}$ locally uniformly and $\bar{u}$ is the unique solution of*

$$\begin{cases} \bar{F}(\bar{u}, x) = 0 & \text{in } D \\ \bar{u} = g & \text{on } \mathbb{R}^n \setminus D. \end{cases}$$

Interpretations and Applications

- Linear Case– Determine effective dynamics of Lévy Process in inhomogeneous media

$$f(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K(\frac{x}{\varepsilon}, y)\mathrm{d}y$$

- Optimal Control Case– Determine an effective optimal cost of control of Lévy Processes in inhmogeneous media

$$\inf_{\alpha} \left\{ f^{\alpha}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K^{\alpha}(\frac{x}{\varepsilon}, y)\mathrm{d}y \right\}$$

- Two Player Game Case– Determine an effective value of a two player game of a Lévy Process in inhomogeneous media

$$\inf_{\alpha} \sup_{\beta} \left\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x))K^{\alpha\beta}(\frac{x}{\varepsilon}, y)\mathrm{d}y \right\}$$

The Set-Up– Assumptions on F

"Ellipticity"

$$\frac{\lambda}{|y|^{n+\sigma}} \leq K^{\alpha\beta}(x,y) \leq \frac{\Lambda}{|y|^{n+\sigma}}$$

Scaling

$$K^{\alpha\beta}(x,\lambda y) = \lambda^{-n-\sigma} K^{\alpha\beta}(x,y).$$

Symmetry

$$K^{\alpha\beta}(x,-y) = K^{\alpha\beta}(x,y)$$

Recent Background– Nonlocal Elliptic Equations

Existence/Uniqueness (Barles-Chasseigne-Imbert)

Given basic assumptions on $K^{\alpha\beta}$ and $f^{\alpha\beta}$, there exist unique solutions to the Dirichlet Problems $F(u^\varepsilon, x/\varepsilon) = 0$, $\bar{F}(\bar{u}, x) = 0$.

Regularity (Silvestre, Caffarelli-Silvestre)

u^ε are Hölder continuous, depending only on λ, Λ, $\|f^{\alpha\beta}\|_\infty$, dimension, and g. (In particular, continuous uniformly in ε.)

Nonlocal Ellipticity (Caffarelli-Silvestre)

If u and v are $C^{1,1}$ at a point, x, then

$$M^-(u-v)(x) \leq F(u,x) - F(v,x) \leq M^+(u-v)(x).$$

$$M^-u(x) = \inf_{\alpha\beta}\left\{L^{\alpha\beta}u(x)\right\} \text{ and } M^+u(x) = \sup_{\alpha\beta}\left\{L^{\alpha\beta}u(x)\right\}.$$

Recent Background– 2nd Order Homogenization

The "Corrector" Equation (Caffarelli-Souganidis-Wang)

For each matrix, Q, fixed, $\bar{F}(Q)$ is the **unique** constant such that the solutions, v^ε, of

$$\begin{cases} F(Q + D^2 v^\varepsilon, \frac{x}{\varepsilon}) & = \bar{F}(Q) \text{ in } B_1 \\ v^\varepsilon(x) & = 0 \text{ on } \partial B_1, \end{cases}$$

satisfy the **decay property** as $\varepsilon \to 0$, $\|v^\varepsilon\|_\infty \to 0$.

This generalizes the notion of the

True Corrector Equation (Lions-Papanicolaou-Varadhan, 1st order HJE)

$\bar{F}(Q)$ is the unique constant such that there is a global periodic solution of

$$F(Q + D^2 v, y) = \bar{F}(Q) \text{ in } \mathbb{R}^n.$$

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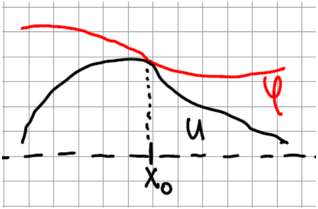
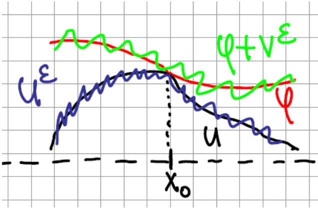
Perturbed Test Function Method

Need to Determine Effective operator $\Rightarrow$ All information is in original operator $F(\cdot, x/\varepsilon) = 0$



Can we perturb ϕ to $\phi + v^\varepsilon$ to COMPARE WITH u^ε???



$\bar{F}(\phi, x_0) \geq 0 \quad \Rightarrow \quad F(\phi + v^\varepsilon, x/\varepsilon) = \bar{F} \geq 0$

To go BACK from comparison of $\phi + v^\varepsilon$ and u^ε TO comparison of ϕ and u NEED

$$|v^\varepsilon| \to 0 \quad \text{as} \quad \varepsilon \to 0$$

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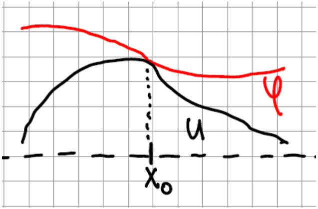

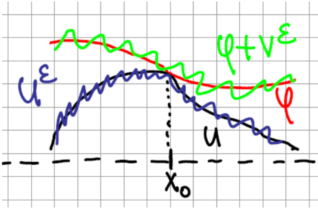

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Strategy

- Most of the arguments for 2nd order homogenization are based on COMPARISON + REGULARITY
- Nonlocal equations have good COMPARISON + REGULARITY properties

$\Longrightarrow$ We should try to modify techniques of the 2nd order setting to the nonlocal setting

Difficulties Taking Ideas to Nonlocal Setting

- The space of test functions is much larger! $C_b^2(\mathbb{R}^n)$ versus $\mathcal{S}^n$
- Test function space is not invariant under the scaling of the operators $u \mapsto \varepsilon^\sigma u(\cdot/\varepsilon)$
- $\bar{F}(\phi, \cdot)$ is a function, not a constant
- What should be the "corrector" equation? We can't just "freeze" the hessian, $D^2\phi(x_0)$, at a point x_0

Scaling Test Functions?

Bad Test Function Scaling, But Good F Scaling

$$L^{\alpha\beta}[\varepsilon^{\sigma} u(\frac{\cdot}{\varepsilon})](x) = L^{\alpha\beta}[u](\frac{x}{\varepsilon})$$

$$L^{\alpha\beta}u(x) = \int_{\mathbb{R}^n} (u(x+y) + u(x-y) - 2u(x)) K^{\alpha\beta}(\frac{x}{\varepsilon}, y) \mathrm{d}y$$

Put The Test Function Inside

$$\begin{cases} F(\phi + v^{\varepsilon}, \frac{x}{\varepsilon}) & = \mu \text{ in } B_1 \\ v^{\varepsilon}(x) & = 0 \text{ on } \mathbb{R}^n \setminus B_1, \end{cases}$$

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"Corrector" Equation

equation for $\phi + v^\varepsilon$

$$F(\phi + v^\varepsilon, \frac{x}{\varepsilon}) =$$

$$\inf_\alpha \sup_\beta \Big\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + \int_{\mathbb{R}^n} (\phi(x+y) + \phi(x-y) - 2\phi(x)) K^{\alpha\beta}(\frac{x}{\varepsilon}, y) \mathrm{dy}$$

$$+ \int_{\mathbb{R}^n} (v^\varepsilon(x+y) + v^\varepsilon(x-y) - 2v^\varepsilon(x)) K^{\alpha\beta}(\frac{x}{\varepsilon}, y) \mathrm{dy} \Big\}$$

"frozen" operator on ϕ at x_0

$$[L^{\alpha\beta}\phi(x_0)](x) = \int_{\mathbb{R}^n} (\phi(x_0+z) + \phi(x_0-z) - 2\phi(x_0)) K^{\alpha\beta}(x, z) \mathrm{dz}$$

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“Corrector” Equation

Analogy to 2nd order equation

$$a_{ij}(\frac{x}{\varepsilon})(\phi + v)_{x_i x_j}(x) = a_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_j}(x) + a_{ij}(\frac{x}{\varepsilon})v_{x_i x_j}(x)$$

and $a_{ij}(\frac{x}{\varepsilon})\phi_{x_i x_j}(x)$ is uniformly continuous in x.

Free and frozen variables, x and x_0

Uniform continuity (Caffarelli-Silvestre)

$[L^{\alpha\beta}\phi(x_0)](x)$ is **uniformly continuous in $\mathbf{x_0}$**, independent of x and $\alpha\beta$

"Corrector" Equation

NEW OPERATOR F_{ϕ,x_0}

$$F_{\phi,x_0}(v^\varepsilon,\frac{x}{\varepsilon}) = \inf_\alpha \sup_\beta \Big\{ f^{\alpha\beta}(\frac{x}{\varepsilon}) + [L^{\alpha\beta}\phi(x_0)](\frac{x}{\varepsilon})$$

$$+ \int_{\mathbb{R}^n} (v^\varepsilon(x+y) + v^\varepsilon(x-y) - 2v^\varepsilon(x))K^{\alpha\beta}(\frac{x}{\varepsilon}, y)\mathrm{dy}\Big\}$$

New "Corrector" Equation

$$\begin{cases} F_{\phi,x_0}(v^\varepsilon,\frac{x}{\varepsilon}) = \bar{F}(\phi,x_0) & \text{in } B_1(x_0) \\ v^\varepsilon = 0 & \text{on } \mathbb{R}^n \setminus B_1(x_0). \end{cases}$$

“Corrector” Equation

Proposition (S. ’08; “Corrector” Equation)

There exists a **unique** *choice for the value of* $\bar{F}(\phi, x_0)$ *such that the solutions of the “corrector” equation also satisfy*

$$\lim_{\varepsilon \to 0} \max_{B_1(x_0)} |v^{\varepsilon}| = 0.$$

(via the perturbed test function method, this proposition is equivalent to homogenization)

Finding $\bar{F}$... Variational Problem

(Caffarelli-Sougandis-Wang... *In spirit)

Consider a generic choice of a Right Hand Side, l is fixed

$$\begin{cases} F_{\phi,x_0}(v_l^\varepsilon, \frac{x}{\varepsilon}) = l & \text{in } B_1(x_0) \\ v_l^\varepsilon = 0 & \text{on } \mathbb{R}^n \setminus B_1(x_0). \end{cases}$$

How does the choice of l affect the decay of v_l^ε?

decay property

$\lim_{\varepsilon \to 0} \max_{B_1(x_0)} |v^\varepsilon| = 0 \iff (v_l^\varepsilon)^* = (v_l^\varepsilon)_* = 0$

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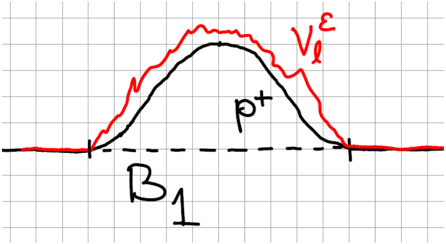
Variational Problem

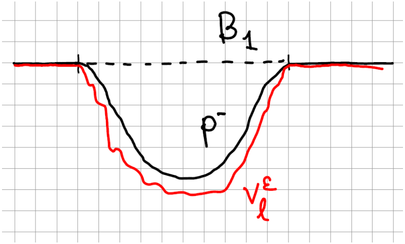
l very negative

$p^+(x) = (1 - |x|^2)^2 \cdot \mathbb{1}_{B_1}$ is a **subsolution** of equation
$\implies (v_l^\varepsilon)_* > 0$ **and we missed the goal.**



$$\left(F_{\phi,x_0}(v_l^\varepsilon, \frac{x}{\varepsilon}) = l\right)$$

Variational Problem



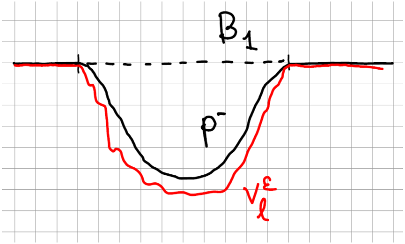
$$\left(F_{\phi,x_0}(v_l^\varepsilon, \frac{x}{\varepsilon}) = l\right)$$

l very positive

$p^-(x) = -(|x|^2 - 1)^2 \cdot \mathbb{1}_{B_1}$ is a **supersolution** of equation
$\implies$ $(v_l^\varepsilon)^* < 0$ **and we missed the goal,** but in the other direction.

Can we choose an l in the middle that is "JUST RIGHT"?

Variational Problem



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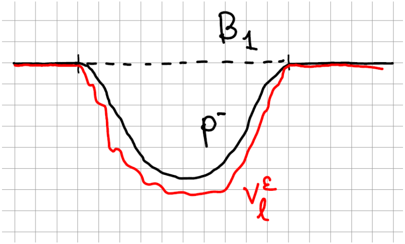

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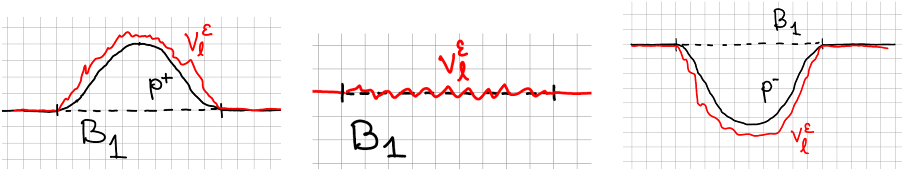

$\left(l << 0\right)$ $\quad$ $l = ?????$ $\quad$ $\left(l >> 0\right)$

Can we choose an l in the middle that is "JUST RIGHT"?

Obstacle Problem

(Caffarelli-Sougandis-Wang) The answer is YES.

Information From Obstacle Problem

The obstacle problem gives relationship between the choice of l and the decay of v_l^ε.

Obstacle Problem

The Solution of The Obstacle Problem In a Set A

$$U_A^l = \inf\left\{u : F_{\phi,x_0}(u,y) \leq l \text{ in } A \text{ and } u \geq 0 \text{ in } \mathbb{R}^n\right\}$$

equation: U_A^l is the least supersolution of $F_{\phi,x_0} = l$ in A
obstacle: U_A^l must be above the obstacle which is 0 in all of $\mathbb{R}^n$

Lemma (Hölder Continuity)

U_A^l is γ-Hölder Continuous depending only on λ, Λ, $\|f^{\alpha\beta}\|_\infty$, ϕ, dimension, and A.

Monotonicity and Periodicity of Obstacle Problem

If $A \subset B$, then $U_A^l \leq U_B^l$. For $z \in \mathbb{Z}^n$, $U_{A+z}^l(x) = U_A^l(x - z)$

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Obstacle Problem

NOTATION

Rescaled Solution

$$u^{\varepsilon,l} = \inf \left\{ u : F_{\phi,x_0}(u, \frac{y}{\varepsilon}) \leq l \text{ in } Q_1 \text{ and } u \geq 0 \text{ in } \mathbb{R}^n \right\}.$$

Solution in Q_1 and Solution in $Q_{1/\varepsilon}$

$$u^{\varepsilon,l}(x) = \varepsilon^{\sigma} U^l_{Q_{1/\varepsilon}}(\frac{x}{\varepsilon})$$

Obstacle Problem

Dichotomy

(i) For all $\varepsilon > 0$, $U^l_{Q_{1/\varepsilon}} = 0$ for at least one point in **every** complete cell of $\mathbb{Z}^n$ contained in $Q_{1/\varepsilon}$.

(ii) There exists some ε_0 and some cell, C_0, of $\mathbb{Z}^n$ such that $U^l_{Q_{1/\varepsilon_0}}(y) > 0$ for all $y \in C_0$.

Lemma (Part (i) of The Dichotomy)

If (i) occurs, then $(v^\varepsilon_l)^* \leq 0$.

Lemma (Part (ii) of The Dichotomy)

If (ii) occurs, then $(v^\varepsilon_l)_* \geq 0$

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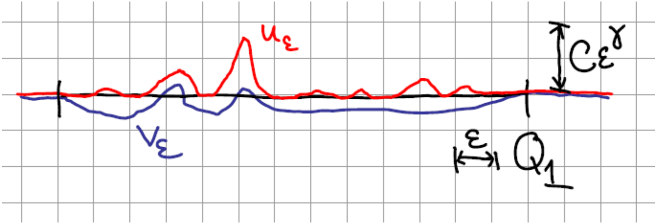
Proof of First Lemma (If (i) occurs, then $(v_l^\varepsilon)^* \leq 0$)...

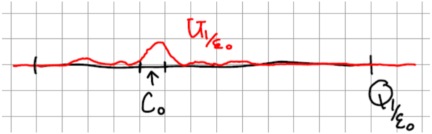
- Rescale back to Q_1.
  Definition of $u^{\varepsilon,l} \implies v_l^\varepsilon \leq u^{\varepsilon,l}$
- (i) $\implies u^{\varepsilon,l} = 0$ at least once in EVERY cell of $\varepsilon\mathbb{Z}^n$. Hölder Continuity $\implies u^{\varepsilon,l} \leq C\varepsilon^\gamma$.

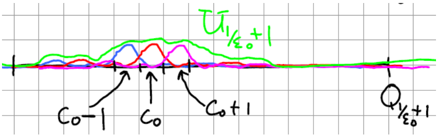


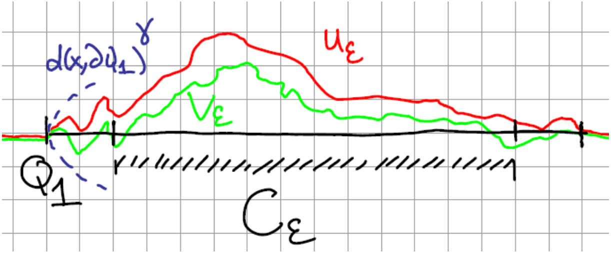
Obstacle Problem

Proof of Second Lemma (If (ii) occurs, then $(v_l^\varepsilon)_* \geq 0$)...

- Given any $\delta > 0$, Periodicity, Monotonicity, and (ii) allow construction of a **connected cube** $C_\varepsilon \subset Q_1$ such that $u^{\varepsilon,l} > 0$ in C_ε **and** $|C_\varepsilon| / |Q_1| \geq 1 - \delta$.







Obstacle Problem

Proof of Second Lemma continued (If (ii) occurs, then $(v_l^\varepsilon)_* \geq 0$)

- Properties of $u^{\varepsilon,l} \implies u^{\varepsilon,l}$ is a **solution** inside C_ε.
- Comparison with v_l^ε and boundary continuity $\implies$ $u^{\varepsilon,l} - v_l^\varepsilon \leq C(\delta^{1/n})^\gamma$.
- Upper limit in ε: $(-v_l^\varepsilon)^* \leq 0$
- Same as $(v_l^\varepsilon)_* \geq 0$

Choice for $\bar{F}$

Choose a special I such that I is ARBITRARILY CLOSE to values that give (i) **and** values that give (ii).

The Good Choice of $\bar{F}$

$$\bar{F}(\phi, x_0) = \sup\left\{ I : (ii) \text{ happens for the family } (U^I_{Q_{1/\varepsilon}})_{\varepsilon>0} \right\}$$

Choice for $\bar{F}$

Choose a special l such that l is ARBITRARILY CLOSE to values that give (i) **and** values that give (ii).

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Needed Properties for $\bar{F}$

Still need to show

Elliptic Nonlocal Equation

- $\bar{F}(u,x)$ is well defined whenever u is bounded and "$C^{1,1}$ at the point, x".
- $\bar{F}(u,\cdot)$ is a continuous function in an open set, Ω, whenever $u \in C^2(\Omega)$.
- Ellipticity holds: If u and v are $C^{1,1}$ at a point, x, then

$$M^-(u-v)(x) \le \bar{F}(u,x) - \bar{F}(v,x) \le M^+(u-v)(x).$$

Comparison

This follows from ellipticity and translation invariance.

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True Corrector Equation

Periodic Corrector

$\bar{F}(\phi, x_0)$ is the unique constant such that the equation,

$$F_{\phi,x_0}(w, y) = \bar{F}(\phi, x_0) \quad \text{in} \quad \mathbb{R}^n$$

admits a global periodic solution, w.

Inf-Sup Formula

Corollary: Inf-Sup formula

$$\bar{F}(\phi, x_0) = \inf_{\{W \text{ periodic}\}} \sup_{y \in \mathbb{R}^n} \left(F_{\phi, x_0}(W, y)\right)$$

Thank You!