Reconciling Non-Gaussian Climate Statistics with Linear Dynamics

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Gaussian statistics are consistent with linear dynamics.

Are non-Gaussian statistics necessarily inconsistent with linear dynamics ?

In particular, are skewed pdfs, implying different behavior of positive and negative anomalies, inconsistent with linear dynamics ?

Thanks also to : Barsugli, Compo, Newman, Penland, and Shin

The Linear Stochastically Forced (LSF) Approximation

$$\frac{dx}{dt} = A\,x + f_{ext} + B\,\eta$$

x = N-component anomaly state vector
η = M-component gaussian noise vector
$f_{ext}(t)$ = N-component external forcing vector
$A(t)$ = N x N matrix
$B(t)$ = N x M matrix

Supporting Evidence

- Linearity of coupled GCM responses to radiative forcings
- Linearity of atmospheric GCM responses to tropical SST forcing
- Linear dynamics of observed seasonal tropical SST anomalies
- Competitiveness of linear seasonal forecast models with global coupled models
- Linear dynamics of observed weekly-averaged circulation anomalies
- Competitiveness of Week 2 and Week 3 linear forecast models with NWP models
- Ability to represent observed second-order synoptic-eddy statistics

Observed and Simulated Spectra of Tropical SST Variability

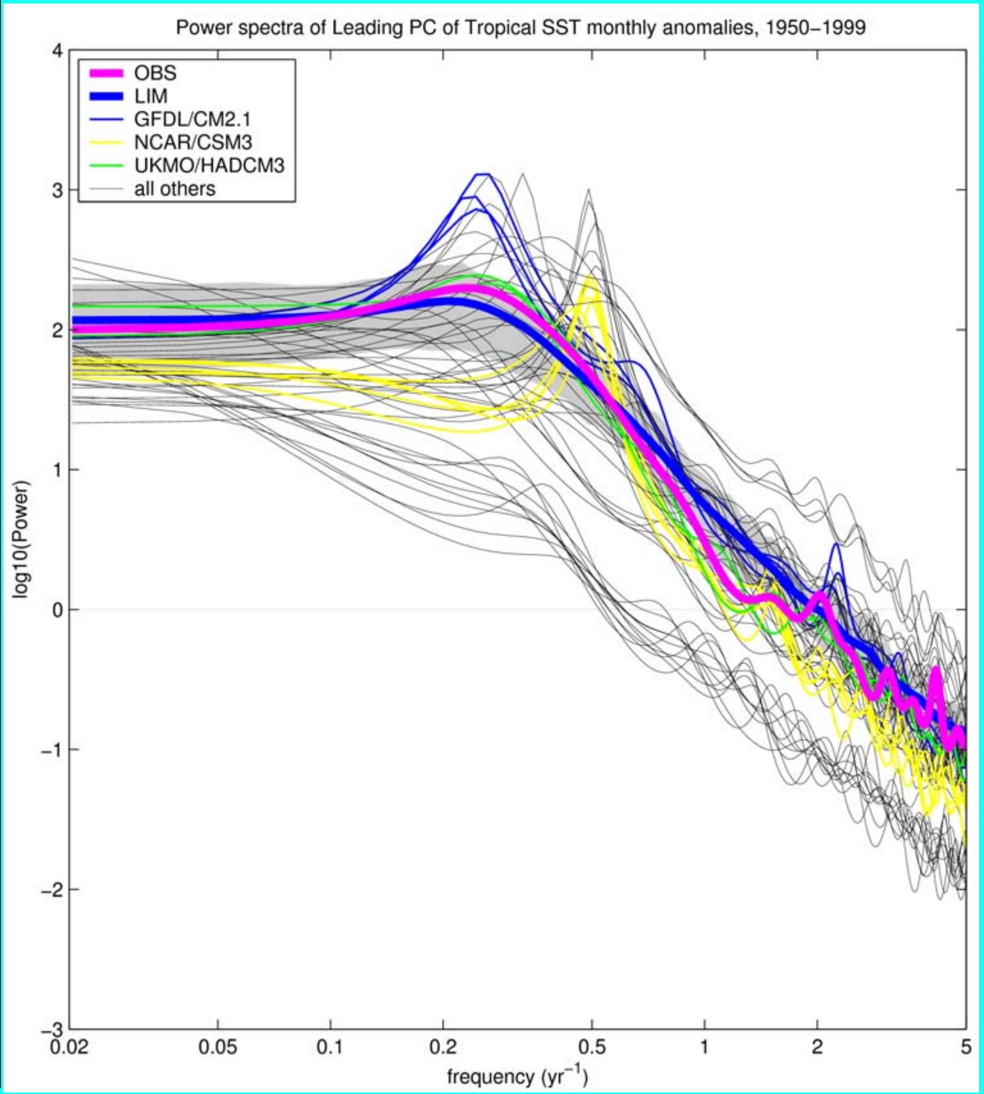
Spectra of the projection of tropical SST anomaly fields on the 1st EOF of observed monthly SST variability in 1950-1999.

Observations (Purple)

IPCC AR4 coupled GCMs
(20^{th}-century (20c3m) runs)
(thin black, yellow, blue, and green)

A linear inverse model (LIM) constructed from 1-week lag covariances of weekly-averaged tropical data in 1982-2005
(Thick Blue)

Gray Shading :
95% confidence interval from the LIM, based on 100 model runs with different realizations of the stochastic forcing.



From Newman, Sardeshmukh and Penland (2008)

Seasonal Predictions of Ocean Temperatures in the Eastern Tropical Pacific : Comparison of linear empirical and nonlinear GCM forecast skill

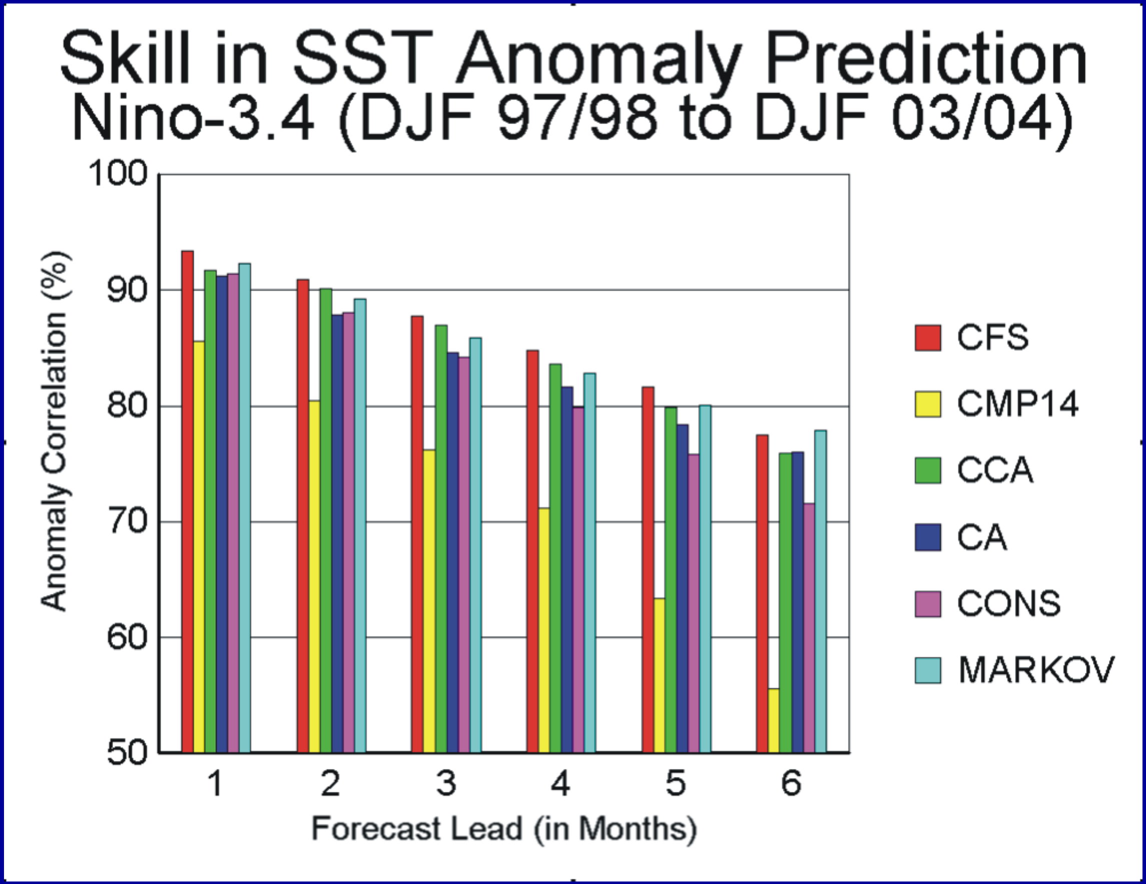
(Courtesy : NCEP)

Simple linear empirical models are apparently just as good at predicting ENSO

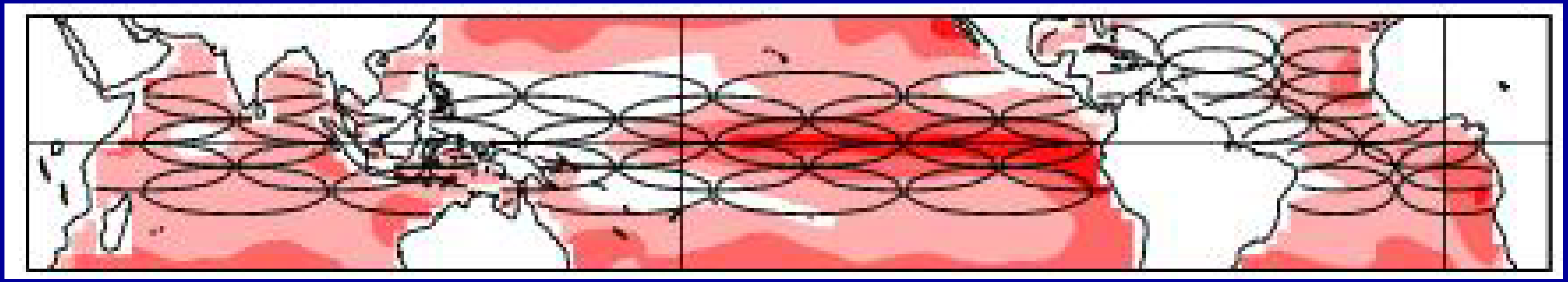
as

"state of the art" coupled

GCMs

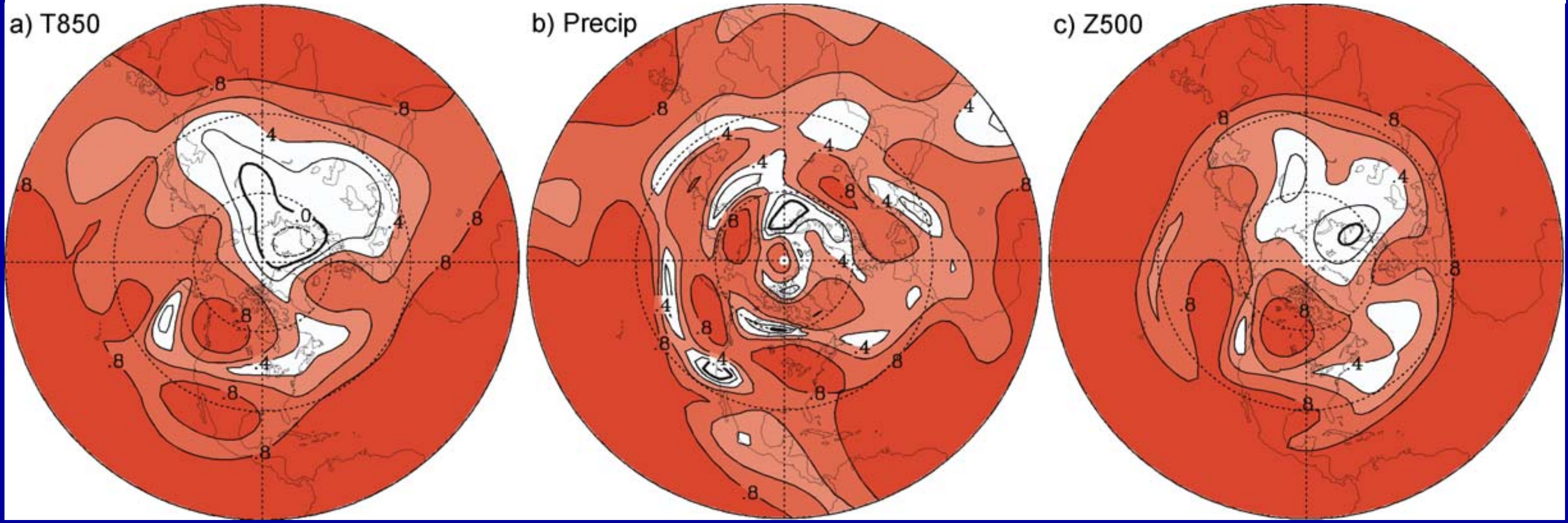


DOMINANCE and LINEARITY of Tropical SST influences on global climate variability



BASIC POINT: The <u>nonlinear</u> NCAR/CCM3 atmospheric GCM's responses to prescribed <u>global</u> SST changes over the last 50 years are well -approximated by <u>linear</u> responses to just the <u>Tropical</u> SST changes, obtained by linearly combining the GCM's responses to SSTs in the 43 localized areas shown above.

Local correlation of annual mean "GOGA" and "Linear TOGA" responses



Sardeshmukh, Barsugli and Shin 2008

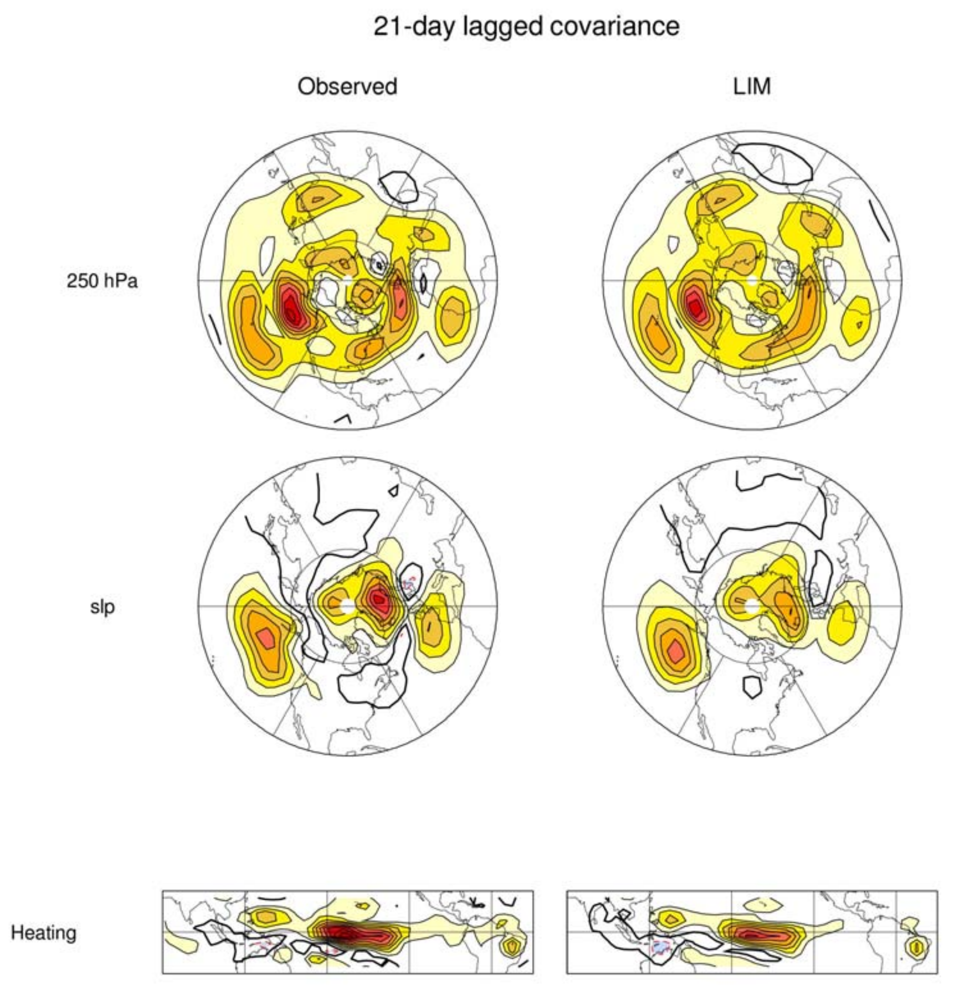
Decay of lag-covariances of weekly anomalies is consistent with linear dynamics

Is $\boxed{C(\tau) \ = \ e^{M\tau} \ C(0)}$?

M is first estimated using the observed $C(\tau = 5$ days$)$ and $C(0)$ in this equation, and then used to "predict" $C(\tau = 21$ days$)$

The components of the anomaly state vector x include the 7-day running mean PCs of 250 and 750 mb streamfunction, SLP, tropical diabatic heating and stratospheric height anomalies.

From Newman and Sardeshmukh (2008)



An attractive feature of the LSF Approximation

$$\frac{dx}{dt} = A\,x + f_{ext} + B\,\eta$$

Equations for the first two moments

(Applicable to both Marginal and Conditional Moments)

$<x>$ = ensemble mean anomaly

C = covariance of departures from ensemble mean

$$\frac{d}{dt}<x> = A<x> + f_{ext}$$

$$\frac{d}{dt}C = A\,C + C\,A^T + B\,B^T$$

If A(t), B(t) , and f_{ext}(t) are constant, then

*First two **Marginal** moments* →

$$<x> = -A^{-1} f_{ext}$$

$$\frac{dC}{dt} = 0 = A\,C + C\,A^T + B\,B^T$$

*First two **Conditional** moments*
Ensemble mean forecast
Ensemble spread →

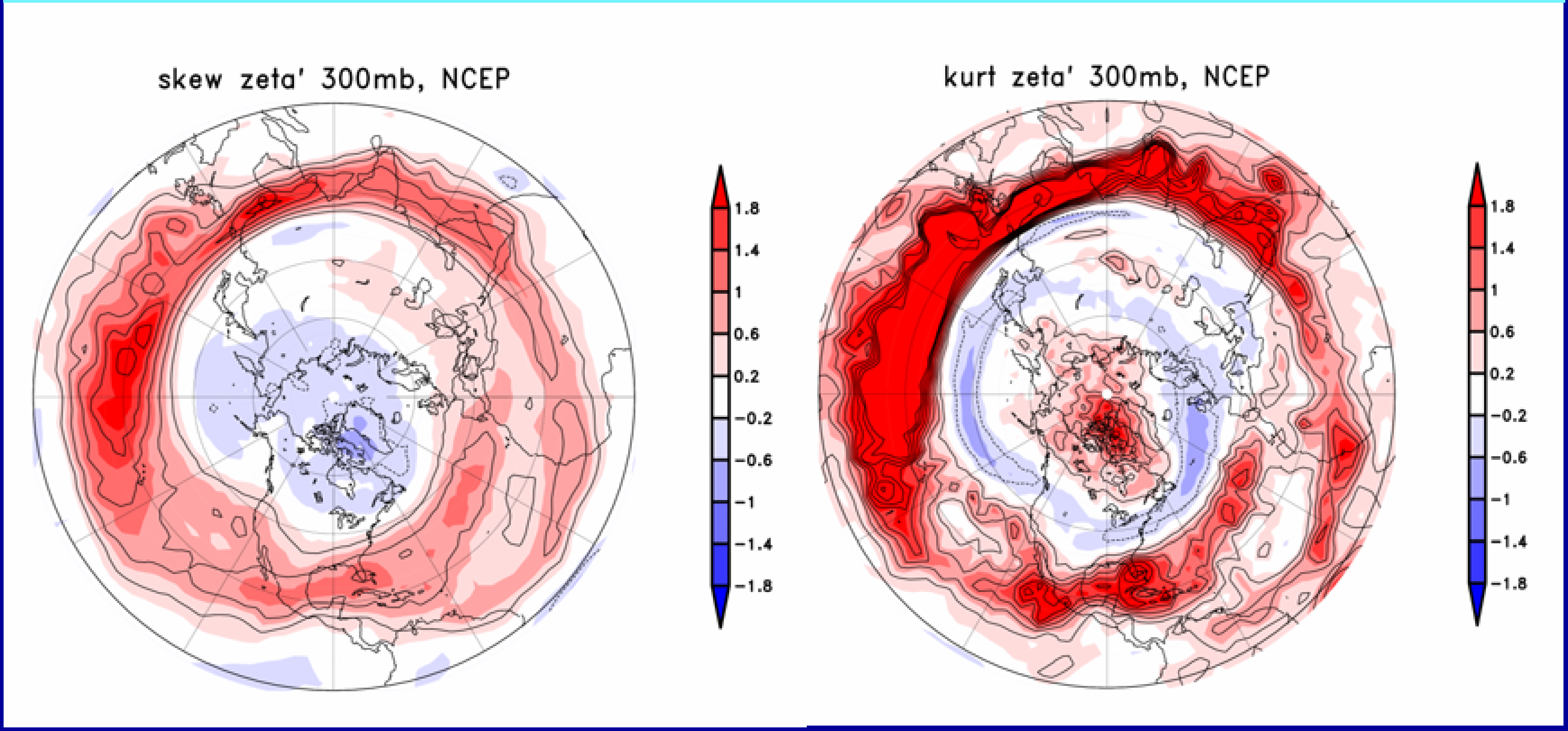
$$\hat{x}'(t) \equiv < x'(t) \mid x'(0) > = e^{At} x'(0)$$

$$\hat{C}(t) \equiv < (\hat{x}' - x')\,(\hat{x}' - x')^T > = C - e^{At} C e^{A^T t}$$

If *x* is Gaussian, then these moment equations **COMPLETELY** characterize system variability ***and*** predictability

But . . . atmospheric circulation statistics are not Gaussian . . .

Observed Skew *S* and (excess) Kurtosis *K* of daily 300 mb Vorticity (DJF)

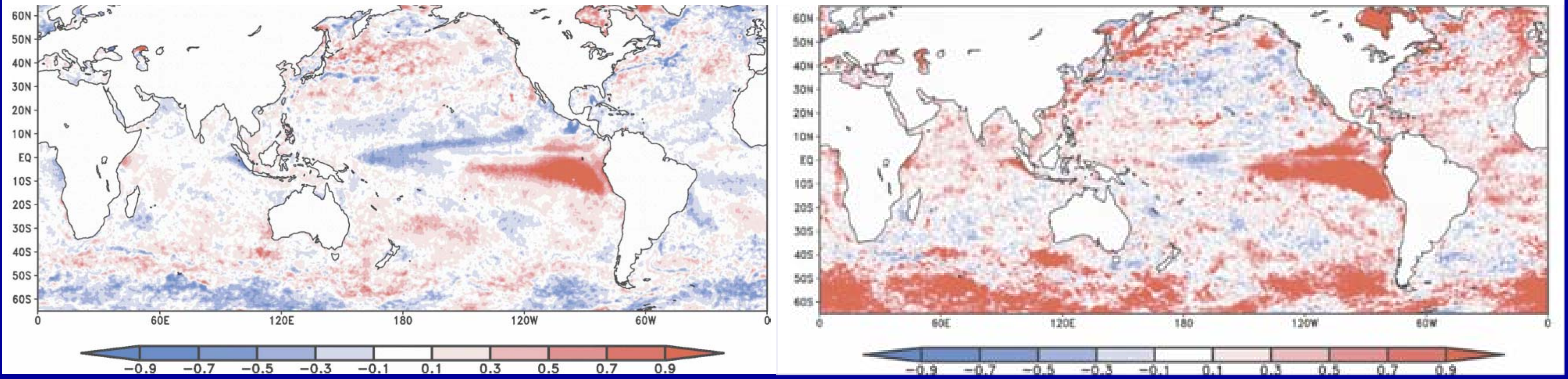


From Sardeshmukh and Sura 2008

Sea Surface Temperature statistics are also not Gaussian . . .

Observed Skew *S* and (excess) Kurtosis *K* of daily SSTs (DJF)

Skew **Kurtosis**



From Sura and Sardeshmukh 2008

Modified LSF Dynamics

$$\left.\begin{array}{ll} \textit{Model 1:} & \boxed{\dfrac{dx}{dt} = Ax + f_{ext} + B\eta} \\ \textit{Model 2:} & \boxed{\dfrac{dx}{dt} = Ax + f_{ext} + B\eta + (Ex)\xi} \\ \textit{Model 3:} & \boxed{\dfrac{dx}{dt} = Ax + f_{ext} + B\eta + (Ex + g)\xi - \dfrac{1}{2}Eg} \end{array}\right\}$$

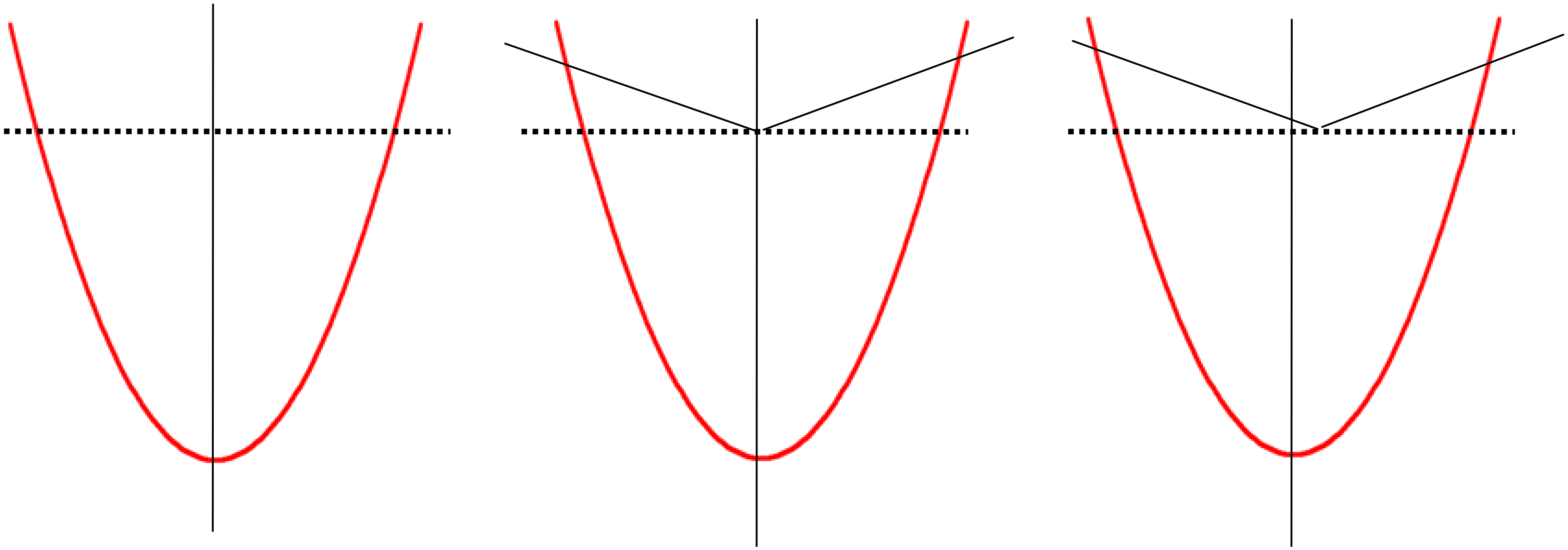
For simplicity consider a scalar ξ here

$A(t), B(t), E(t)$ are matrices; $g(t)$, $f_{ext}(t)$, η are vectors

Moment Equations :

$$\boxed{\begin{array}{l} \dfrac{d}{dt} < x > \;=\; M < x > + f_{ext} \qquad \text{where} \quad M = (A + \dfrac{1}{2}E^2) \\ \dfrac{d}{dt} C \;=\; M\,C + C\,M^T + B\,B^T + E\,\{\,C + <x><x>^T\,\}\,E^T + g\,g^T \end{array}}$$

A simple view of how additive and linear multiplicative noise can generate skewed PDFs even in a deterministically linear system



Additive noise only
Gaussian
No skew

Additive and uncorrelated Multiplicative noise
Symmetric non-Gaussian

Additive and correlated Multiplicative noise
Asymmetric non-Gaussian

A simple rationale for Correlated Additive and Multiplicative (CAM) noise

In a quadratically nonlinear system with "slow" and "fast" components ***x*** and ***y***, the anomalous nonlinear tendency has terms of the form :

$$
\begin{aligned}
(xy)' &= x'\,\bar{y} + \bar{x}\,y' + x'y' - \overline{x'y'} \\
&= \bar{y}\,x' + (\bar{x} + x')y' - \overline{x'y'}
\end{aligned}
$$

CAM noise **mean Noise Induced Drift**

Note that it is the ***STOCHASTICITY*** of *y'* that enables the mean drift to be parameterized in terms of the noise amplitude parameters

$$
\begin{aligned}
(vT)' &= T'\,\bar{v} + v'\,\bar{T} + v'T' - \overline{v'T'} \\
&= \bar{v}\,T' + (\bar{T} + T')v' - \overline{v'T'}
\end{aligned}
$$

Rationalizing linear anomaly dynamics with correlated additive and linear multiplicative stochastic noise

$$\frac{dX_i}{dt} = L_{ij} X_j + N_{ijk} X_j X_k + F_i$$

Einstein Summation Convention

$$\frac{dX_i'}{dt} = [L_{ij} + (N_{ijk} + N_{ikj}) \bar{X}_k]\, X_j' \;+\; N_{ijk} (X_j' X_k' - \overline{X_j' X_k'}) \;+\; F_i'$$

Let $X' = \begin{bmatrix} x' \\ \eta' \end{bmatrix}$ and $\bar{X} = \begin{bmatrix} \bar{x} \\ \bar{\eta} \end{bmatrix}$

$$\frac{dx_i'}{dt} = [L_{ij} + (N_{ijm} + N_{imj}) \bar{\eta}_m]\, x_j'$$

Linear terms ($= A_{ij} x_j'$)

$$+ \; [(N_{ijm} + N_{imj}) x_j' + \{L_{im} + (N_{ijm} + N_{imj}) \bar{x}_j\}]\; \eta_m'$$

Correlated additive and multiplicative noise

$$- \; (N_{ijm} + N_{imj})\; \overline{x_j' \eta_m'}$$

Mean noise-induced drift

$$+ \; N_{imn} (\eta_m' \eta_n' - \overline{\eta_m' \eta_n'})$$

Other additive noise ($= B_{ik} \xi_k$)

$$+ \; N_{ijk} \; (x_j' x_k' - \overline{x_j' x_k'})$$

Hard nonlinearity

$$+ \; f_i'$$

External forcing

Neglecting the hard nonlinearity, and using the FPE to derive the noise-induced drift, we obtain

$$\boxed{\begin{aligned} \frac{dx_i'}{dt} &= A_{ij}\, x_j' \;+\; (E_{ijm} x_j' + L_{im} + E_{ijm} \bar{x}_j)\; \eta_m' \;-\; \frac{1}{2} E_{ijm} (L_{jm} + E_{jkm} \bar{x}_k) \;+\; B_{ik} \xi_k \;+\; f_i' \\ &= A_{ij} x_j' \;+\; (E_{ijm} x_j' + \; G_{im} \;)\; \eta_m' \;-\; \frac{1}{2} E_{ijm} G_{jm} \;+\; B_{ik} \xi_k \;+\; f_i' \end{aligned}}$$

where $E_{ijm} = (N_{ijm} + N_{imj})$, and $G_{im} = L_{im} + (N_{ijm} + N_{imj}) \bar{x}_j = L_{im} + E_{ijm} \bar{x}_j$

A 1-D system with Correlated Additive and Multiplicative ("CAM") noise

Stochastic Differential Equation :

$$\frac{dx}{dt} \cong Ax + (Ex + g)\eta + B\xi - \frac{1}{2}Eg$$

Fokker-Planck Equation :

$$Mxp = \frac{1}{2}\frac{d}{dx}\left[\left(E^2x^2 + 2Egx + g^2 + B^2\right)p\right]$$

Moments :

$$<x> \;=\; 0$$

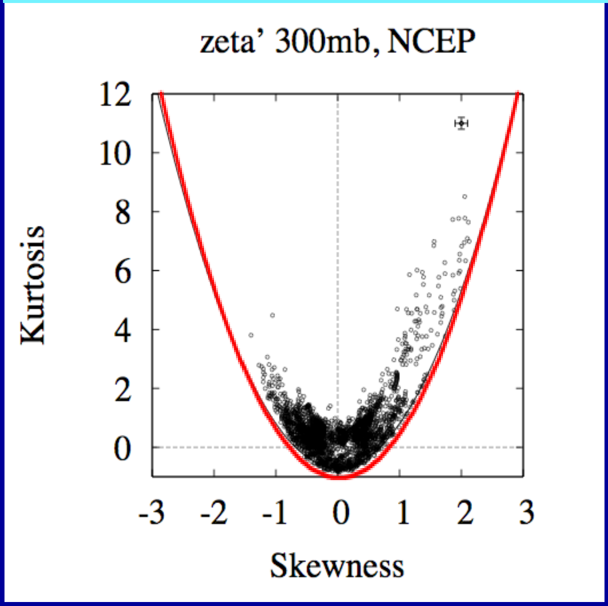
$$<x^n> \;=\; -\left(\frac{n-1}{2}\right)\left[2Eg <x^{n-1}> + (g^2 + B^2) <x^{n-2}>\right] \Big/ \left[M + \left(\frac{n-1}{2}\right)E^2\right]$$

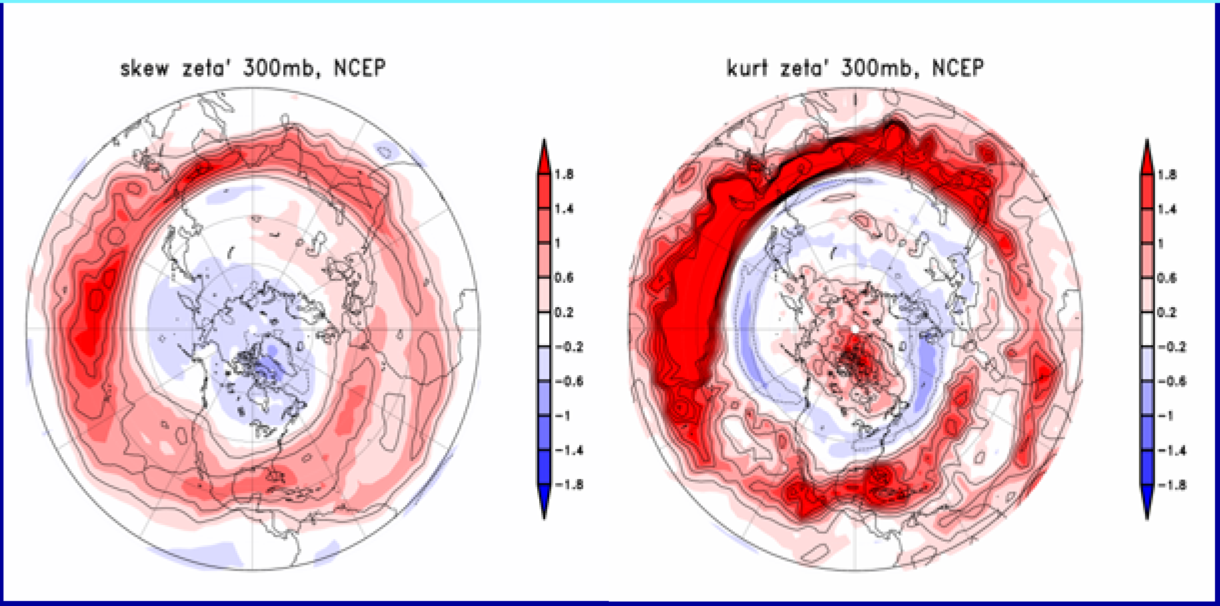
A simple relationship between Skew and Kurtosis :

Remembering that Skew $S = \dfrac{<x^3>}{\sigma^3}$ and Kurtosis $K = \dfrac{<x^4>}{\sigma^4} - 3$, we have

$$K = \frac{3}{2}\left[\frac{M + E^2}{M + (3/2)E^2}\right]S^2 + 3\left[\frac{M + (1/2)E^2}{M + (3/2)E^2} - 1\right] \geq \frac{3}{2}S^2$$

Observed Skew *S* and (excess) Kurtosis *K* of daily 300 mb Vorticity (DJF)

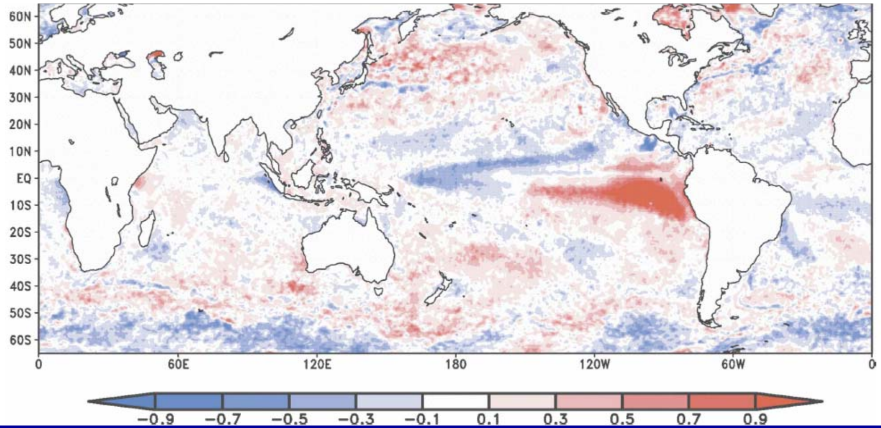




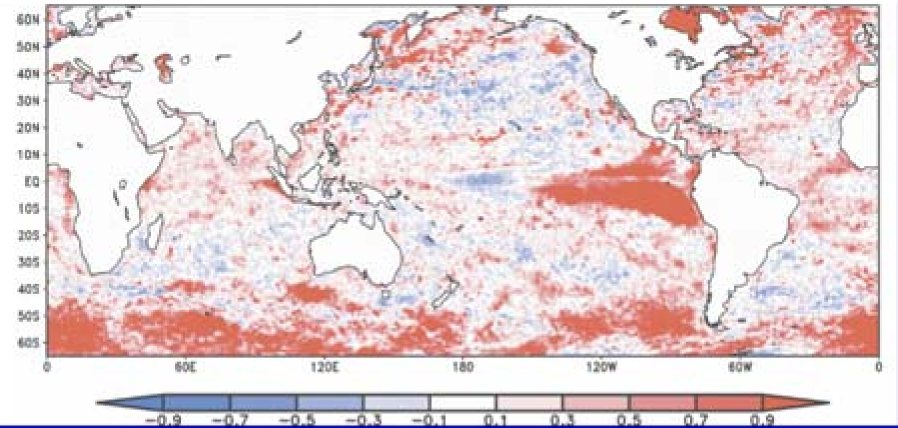
Note the quadratic relationship between *K* and *S* : $\boxed{K \geq 3/2\ S^2}$

Observed Skew S and (excess) Kurtosis K of daily SSTs (DJF)

Skew



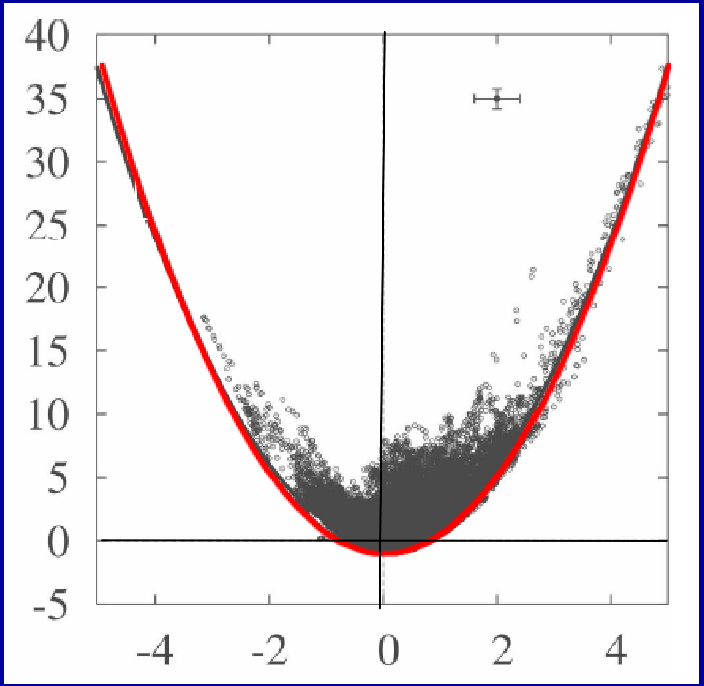
Kurtosis



Note the quadratic relationship

between K and S : $\boxed{K \geq 3/2\ S^2}$

From Sura and Sardeshmukh 2008



Understanding the patterns of Skewness and Kurtosis

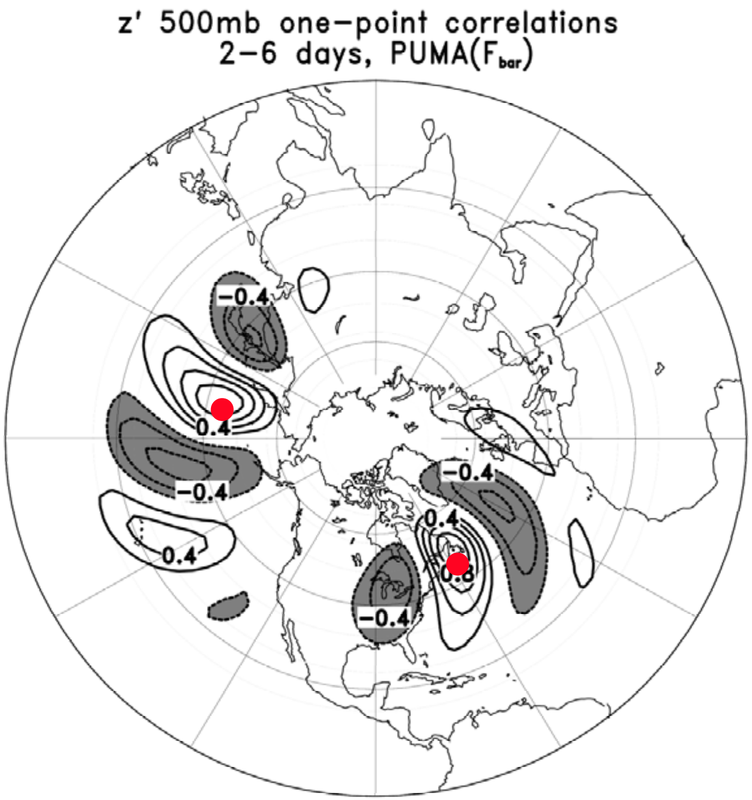
Are diabatic or adiabatic stochastic transients more important ?

To clarify this, we examined the circulation statistics in a 1200 winter simulation generated with a T42 5-level dry adiabatic GCM ("PUMA") with the observed time-mean diabatic forcing specified as a **fixed** forcing.

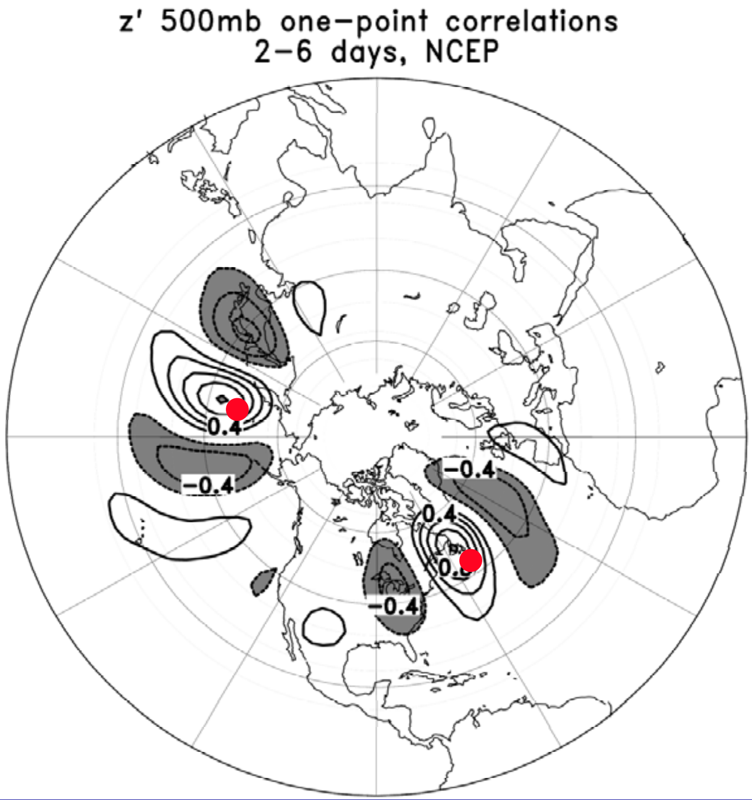
There is thus NO transient diabatic forcing in these runs.

1-point anomaly correlations of synoptic (2 to 6 day period) variations with respect to base points in the Pacific and Atlantic sectors

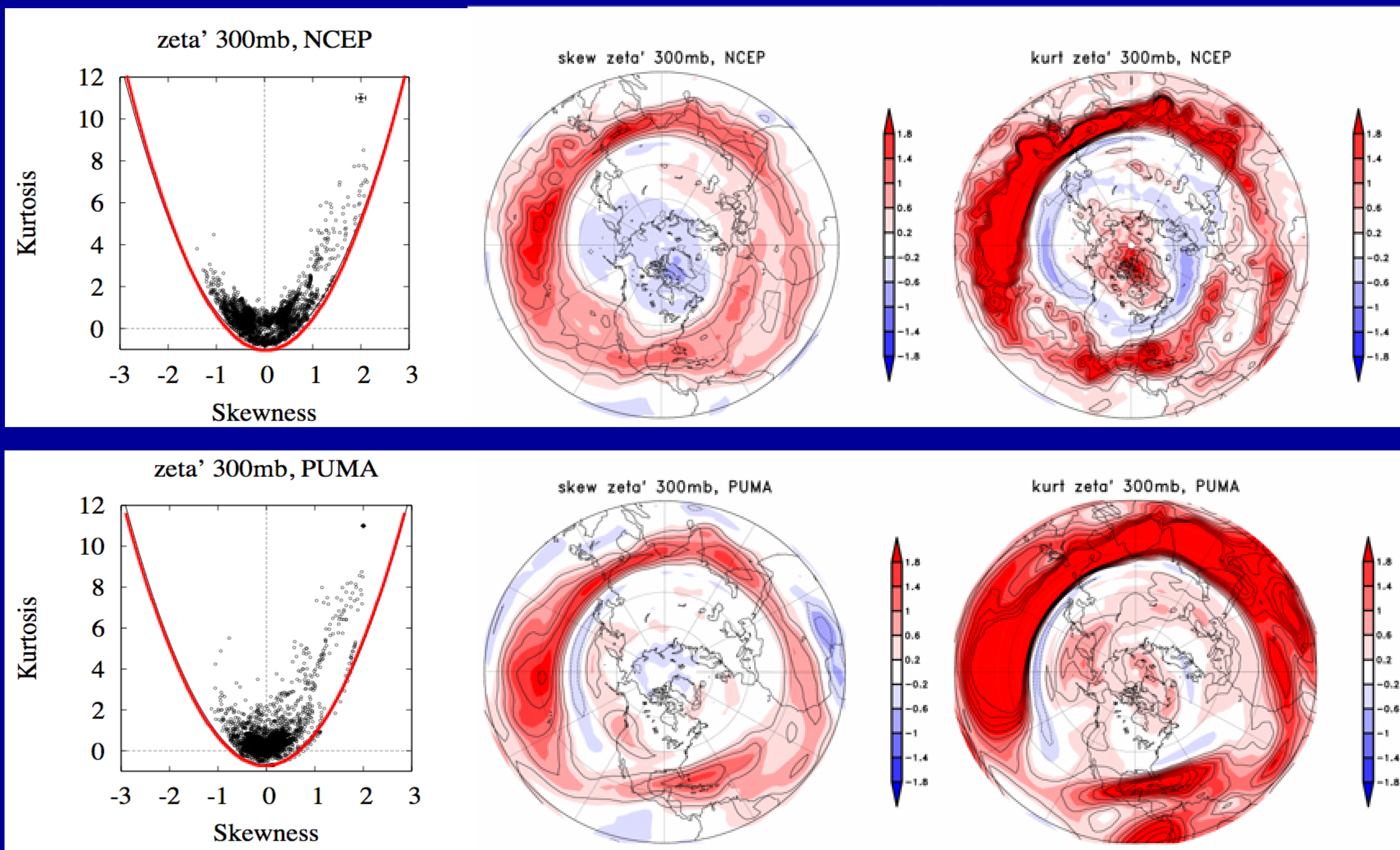
Simulated



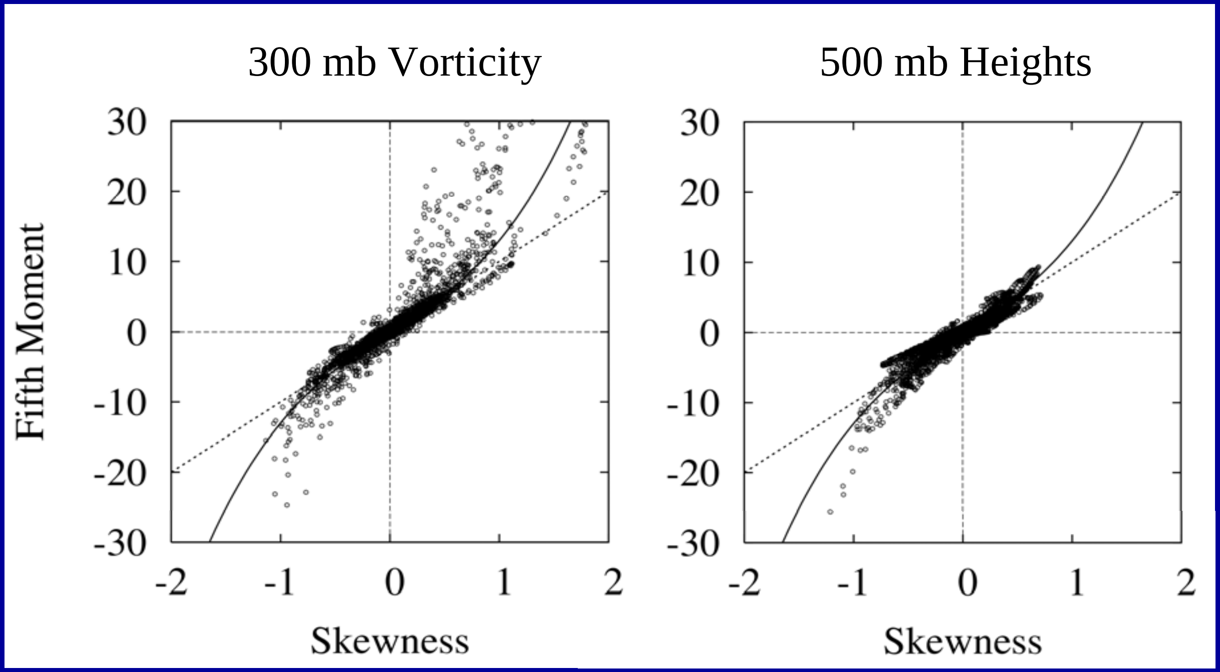
Observed



Observed (NCEP, Top) and Simulated (PUMA, Bottom) *S* and *K* of 300 mb Vorticity



Scatter plots of **Fifth Moments versus Skew** in the dry adiabatic GCM



The 1-d model predicts $\mu_5 \equiv \dfrac{< x^5 >}{\sigma^5} \quad \begin{matrix} > & \mathbf{10}\boldsymbol{s} + \mathbf{3}\boldsymbol{S}^3 \text{ for } \boldsymbol{S} > \mathbf{0} \\ < & \mathbf{10}\boldsymbol{s} + \mathbf{3}\boldsymbol{S}^3 \text{ for } \boldsymbol{S} < \mathbf{0} \end{matrix}$!!

A linear 1-D system with non-Gaussian statistics, forced by "CAM" noise

$$\frac{dx}{dt} = Ax + b\eta_1 + (Ex + g)\eta_2 - \frac{1}{2}Eg \qquad \textbf{SDE}$$

$$[\, Mx \,]\, p = \frac{1}{2}\,\frac{d}{dx}\,[\, E^2x^2 + 2Egx + (g^2 + b^2)\, p \,] \qquad \textbf{FPE}$$

$$p(x) = \frac{1}{N}\left[(Ex+g)^2 + b^2\right]^{\frac{1}{\alpha}-1} \exp\left[-\frac{2g}{\alpha b}\,\arctan\left(\frac{Ex+g}{b}\right)\right] \qquad \textbf{PDF}$$

Such a system satisfies $K > (3/2)S^2$ and its PDF has power-law tails

$M = A + 0.5\,E^2$

$\alpha = E^2 / M$

Both < 0

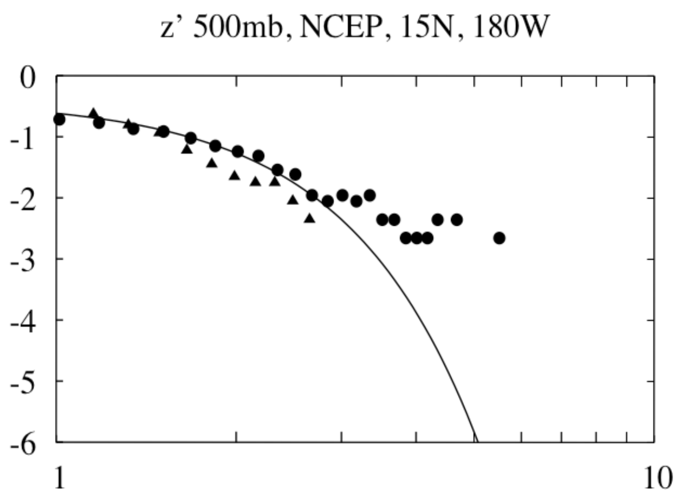
Observed and Simulated pdfs in the North Pacific

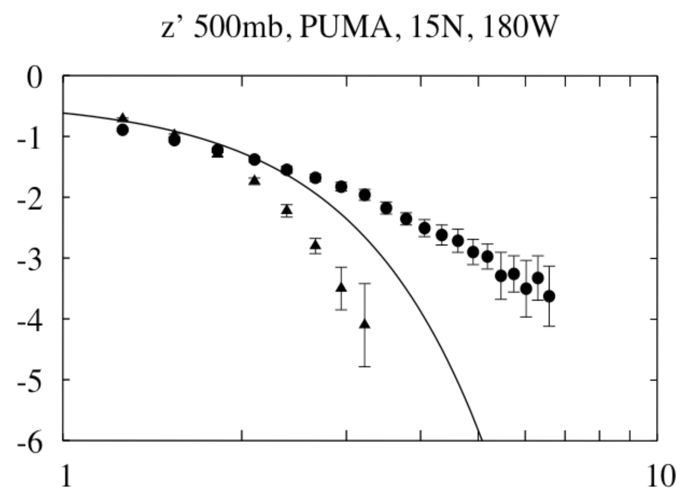
(On a log-log plot, and with the negative half folded over into the positive half)

Observed (NCEP Reanalysis)

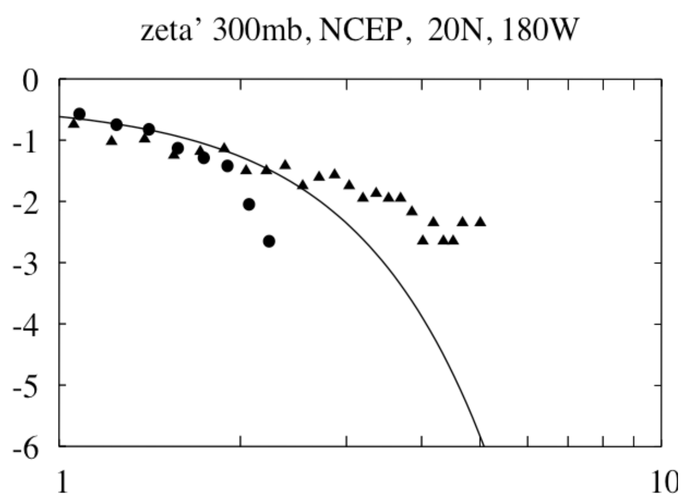
Simulated by a dry adiabatic GCM with fixed forcing

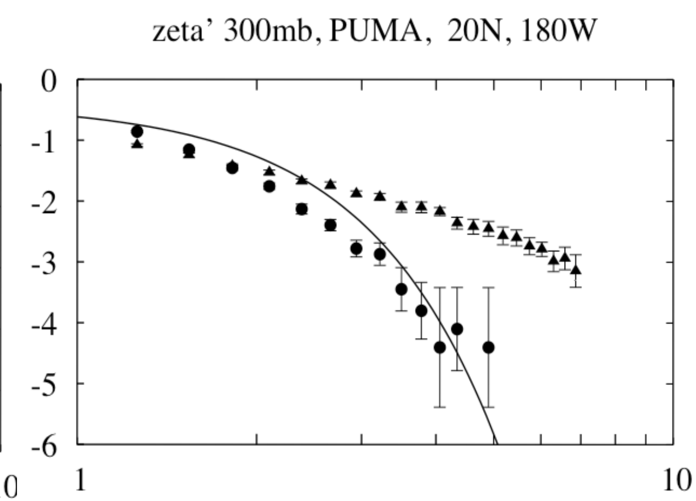
500 mb Height





300 mb Vorticity





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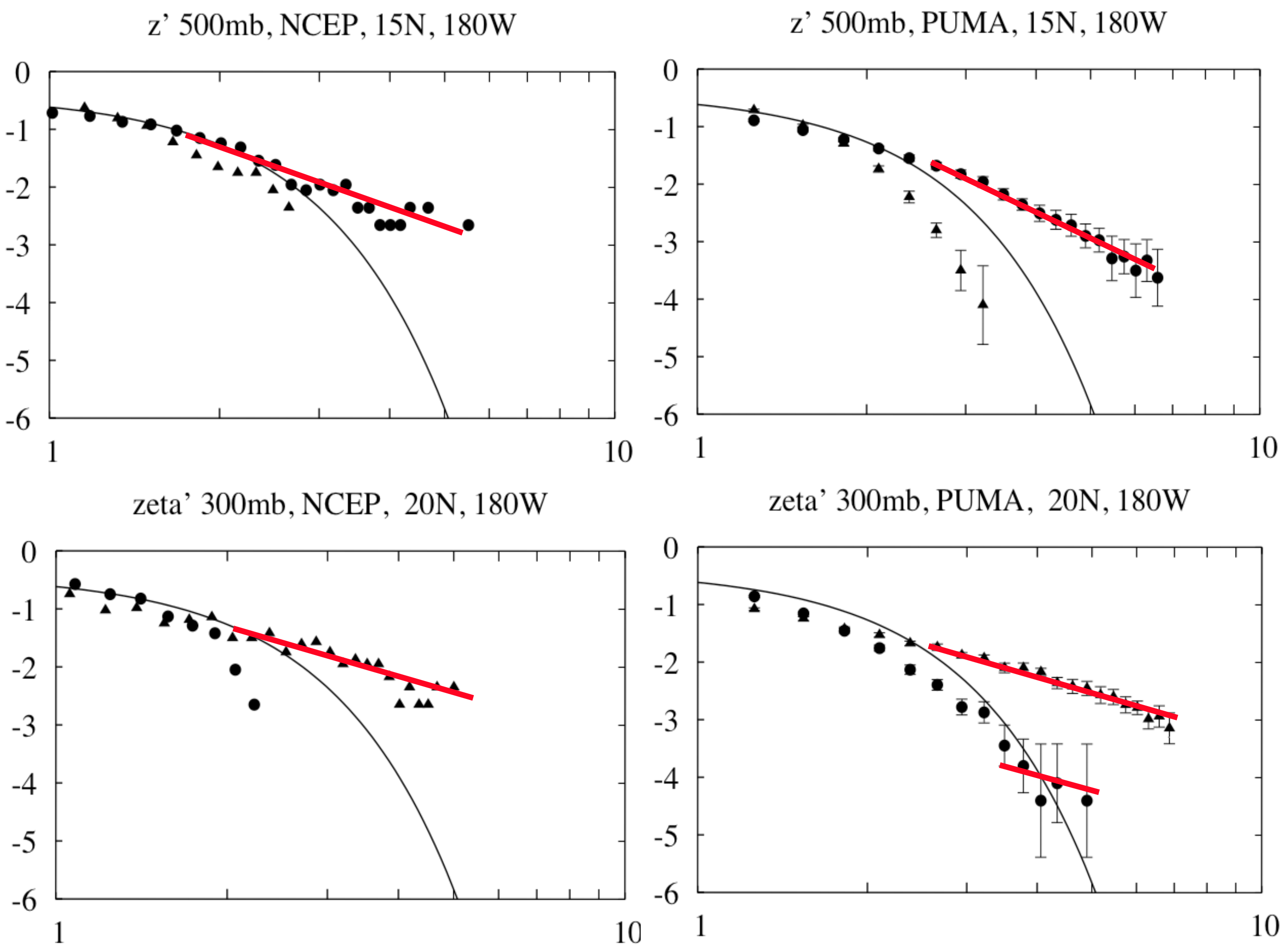
500 mb Height

300 mb Vorticity



A linear 1-D system with non-Gaussian statistics, forced by "CAM" noise

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$$[\,Mx\,]\,p = \frac{1}{2}\,\frac{d}{dx}\,[\,E^2x^2 + 2Egx + (g^2 + b^2)\,p\,] \qquad \textit{FPE}$$

$$p(x) = \frac{1}{N}\left[(Ex+g)^2 + b^2\right]^{\frac{1}{\alpha}-1} \exp\left[-\frac{2g}{\alpha b}\arctan\left(\frac{Ex+g}{b}\right)\right] \qquad \textit{PDF}$$

Such a system satisfies $K > (3/2)S^2$ and its PDF has power-law tails

$M = A + 0.5\,E^2$

$\alpha = E^2 / M$

Both < 0

A linear 1-D system with non-Gaussian statistics, forced by "CAM" noise

$$\frac{dx}{dt} = Ax + b\eta_1 + (Ex + g)\eta_2 - \frac{1}{2}Eg \qquad SDE$$

$$[\, Mx \,]\, p = \frac{1}{2}\,\frac{d}{dx}\,[\, E^2x^2 \; + \; 2Egx \; + \; (g^2 + b^2)\, p \,] \qquad FPE$$

$$p(x) \;=\; \frac{1}{N}\left[(Ex+g)^2 + b^2\right]^{\frac{1}{\alpha}-1} \exp\left[-\,\frac{2g}{\alpha b}\;\arctan\left(\frac{Ex+g}{b}\right)\right] \qquad PDF$$

Such a system satisfies $K > (3/2)S^2$ and its PDF has power-law tails

$M \;=\; A \;+\; 0.5\; E^2$

$\alpha \;=\; E^2 \,/\, M$

Both $< \;\; 0$

The most general linear 1-D system with non-Gaussian statistics, forced by "radical" noise

$$\frac{dx}{dt} \;=\; Ax \;+\; \sum_m \sqrt{[(E_m x + g_m)^2 + c_m x]}\;\;\eta_m \;-\; \frac{\beta}{2} \;+\; f_{ext} \qquad SDE$$

$$\left[Mx + f_{ext}\right] p \;=\; \frac{1}{2}\,\frac{d}{dx}\,[\;(E^2x^2 + 2\beta x \;+ G^2)\; p\;] \qquad FPE$$

$$p(x) \;=\; \frac{1}{N}\left[E^2x^2 + 2\beta x + G^2\right]^{\frac{1}{\alpha}-1} \exp\left[\;\frac{2}{\gamma}\left(f_{ext} - \frac{\beta}{\alpha}\right)\;\arctan\left(\frac{E^2x+\beta}{\gamma}\right)\right] \qquad PDF$$

Such a system satisfies $K \;\geq\; (3/2)S^2$ and its PDF also has power-law tails

$$\beta \;=\sum_m\left(E_m g_m + \frac{c_m}{2}\right)$$

$$E^2 = \sum_m E_m^2$$

$$G^2 = \sum_m g_m^2$$

Why does a local 1-D system capture the relationships between the higher-order moments of the N-d climate system with obviously important non-local dynamics ?

Mainly because the equations for the higher moments in the N-d system are increasingly dominated by **self-correlation** terms. We call this a principle of **"DIAGONAL DOMINANCE"**

$$\boldsymbol{K} = \frac{3}{2} \boldsymbol{S}^2 + \boldsymbol{r}$$

$$\boldsymbol{r} = 3 \left[\frac{M + (1/2)E^2}{M + (3/2)E^2} - 1 \right] - 3 \left[\frac{M + (1/2)E^2}{M + (3/2)E^2} \right] \varepsilon^{(2)} - \cancel{\frac{3}{2} \left[\frac{M + E^2}{M + (3/2)E^2} \right] S\, \varepsilon^{(3)}} + \cancel{\varepsilon^{(4)}}$$

$> 0 \qquad\qquad < 0 \;\; \text{if} \;\; \varepsilon^{(2)} > 0$

The quantities $\varepsilon^{(n)}$ represent the error made in $< x^n > / \ \sigma^n$ by ignoring the non-local dynamics.

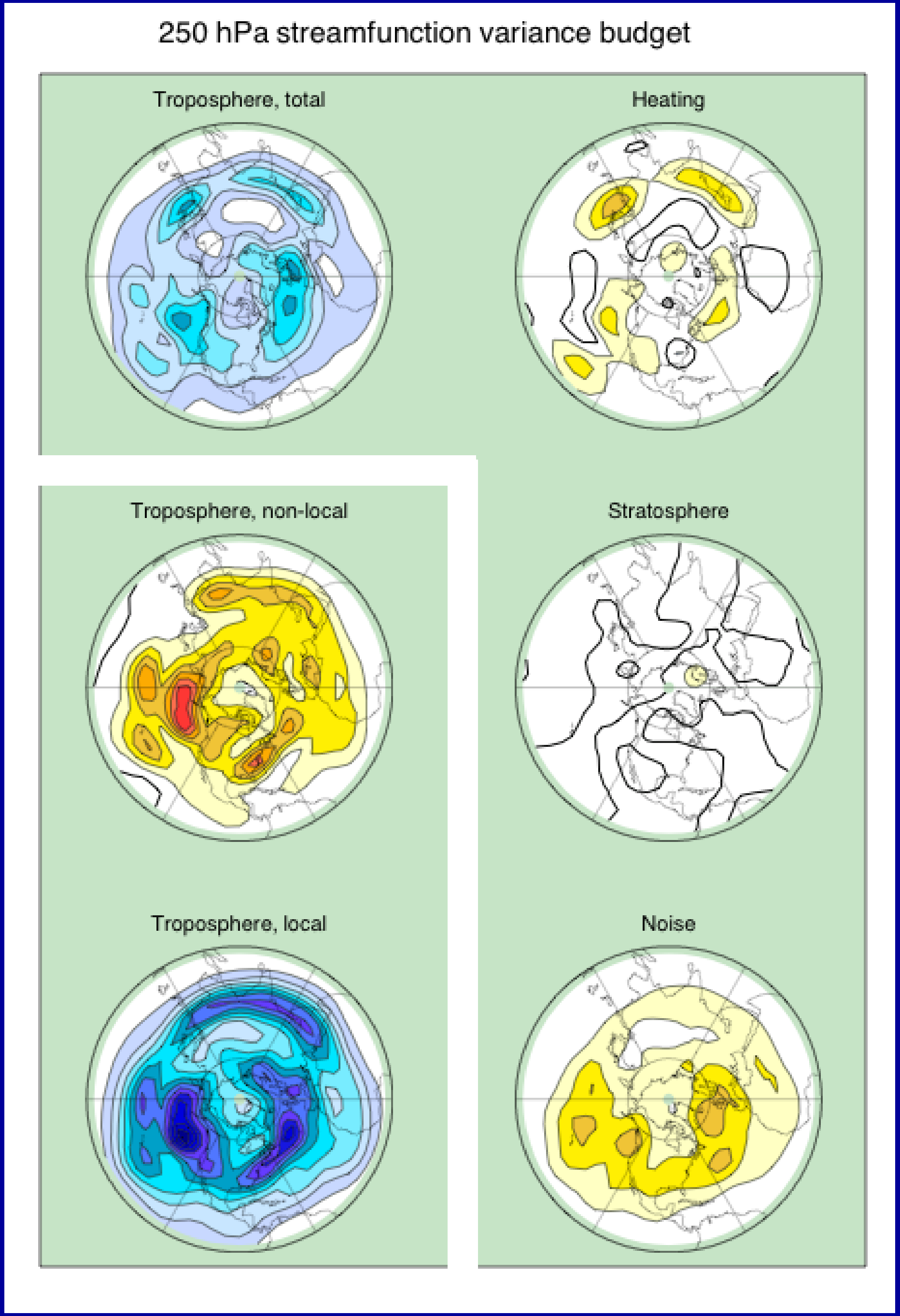
From Diagonal Dominance, we expect that $|\,\varepsilon^{(4)}\,| \;<\; |\,\varepsilon^{(3)}\,| \;<\; |\,\varepsilon^{(2)}\,|$ etc.

Variance Budget of 250 mb Streamfunction in winter

Note the approximate balance between stochastic forcing and local damping.

The non-local interactions <u>increase</u> the variance, everywhere.

Newman and Sardeshmukh (2008)



<u>Summary</u>

1. Strong evidence for "coarse-grained" linear dynamics is provided by
   - (a) the observed decay of correlations with lag
   - (b) the success of linear forecast models, and
   - (c) the approximately linear system response to external forcing.

2. The simplest dynamical model with the above features is a linear model perturbed by **additive** Gaussian stochastic noise. <u>**Such a model, however, cannot generate non-Gaussian statistics**</u>.

3. A linear model with **a mix of multiplicative and additive noises** can generate non-Gaussian statistics; but not odd moments (such as skew) without external forcing; and therefore are not viable models of anomalies with zero mean.

4. **Linear models with correlated multiplicative and additive ("CAM") noise can generate both odd and even moments, and can also explain the remarkable observed quadratic K-S relationship between Kurtosis and Skew, as well as the Power-Law tails of the pdfs.**