Moduli spaces of stable sheaves 
& the Brauer class

Huybrechts- Lehn " The geometry 
of moduli spaces 
of sheaves"

X proj variety / field k = k
Fix embedding X \hookrightarrow \mathbb{P}^n \rightarrow \mathcal{O}_X(1)
goal: (quasi-) proj. variety param.
vector bundles / coherent sheaves on X.

1st issue: infinitely many connected components \implies not finite type
ex: line bundles on \mathbb{P}^1
each (0\alpha) gives different connected component, \alpha \in \mathbb{Z}

solution: fix some numerical invariants
e.g. Chern classes, or...

**Def:** For $F \in \text{Coh}(X)$, the **Hilbert polynomial** of $F$ is

$$P_F(t) := \chi(F(t))$$

$F(t) = F \otimes \mathcal{O}_X(t)$

depends on choice!

If $t >> 0$, $h^i(F(t)) = 0 \ \forall \ i > 0$

$$P_F(t) = h^0(F(t))$$

& is a polynomial in $t$

- compute use HRR
- $\text{deg } P_F = \dim \text{Supp } F$
- leading coeff. > 0
- constant in flat families

2nd issue: space param all sheaves w/ fixed Hilb. poly. is typically not separated.
ex: on \( P' \), \( O(1) \oplus O(1), \; O \oplus O(2), \; O(-1) \oplus O(3) \)

have Hilb poly \( 2t + 4 \)

Can form family \( \mathcal{F} \) over \( A' \) w/\n\[
\mathcal{F}|_{P' \times \{ z \}} = \begin{cases} 
O(1) \oplus O(1) & z \neq 0 \\
O \oplus O(2) & z = 0
\end{cases}
\]

\( \rightarrow \) map \( A' \to M \)
\( z \to 0 \mapsto (O(1) \oplus O(1)) \to \text{is not separated.} \)

\( z = 0 \mapsto (O \oplus O(2)) \)

**Solution**: add a stability condition

**Def**: \( F \) is pure if \( \forall 0 \neq E \leq F, \dim \text{Supp } E = \dim \text{Supp } F \)

(\( X \) integral, \( \text{rk } F > 0 \) \( \Rightarrow \) pure = torsion-free)

**Def**: The reduced Hilbert polynomial of \( F \) is
\[ p_F(t) = \frac{P_F(t)}{\text{leading coeff. of } P_F} \quad \text{(monic)} \]
Def: F is stable (resp. semi-stable) if \( \forall E \in F,\ P_E(t) < P_F(t) \) \( \forall t \gg 0 \).

(resp. \( \leq \))

Ex: \( F = 0 \oplus 0(2) \)
\[ P_F(t) = 2t + 4 \]
\[ P_E(t) = t + 2 \]
\[ P_E(t) = t + 3 = P_F(t) \]

\( \therefore F \) unstable

But \( 0(1) \oplus 0(1) \) is semi-stable

\[ P_{0(1)}(t) = t + (d+1) \leq t + 2 \]

In fact, its polystable = sum of stable sheaves

Thm: There exists a quasi-proj. variety param. stable sheaves w/ any given Hilb. poly., & a proj. variety param. polystable sheaves.
Note:
- If $k 
eq k_{0}$, change "stable" to "geometrically stable"
- can also work over $\text{Spec} \mathbb{Z}$, $\text{Spec} \mathbb{Z}_{p}$

Examples:

1. A connected component of $\text{Pic} X$

Prop: if $X$ is geometrically integral, all rank-1 torsion-free sheaves are stable wrt any embedding $X \hookrightarrow \mathbb{P}^{n}$.

PF: \[ 0 \to E \to F \to F/E \to 0 \]
\[ \text{rk} E = 1, \text{rk}(F/E) = 0 \]
\[ O \to E \to F \to F/E \to 0 \]
\[ P_{E}(t) = a t^{n} + \cdots, a > 0, \ n = \dim X \]
\[ P_{F/E}(t) = b t^{m} + \cdots, b > 0, \ m = \dim \text{Supp}(F/E) \leq n \]
\[ P_{E}(t) = P_{F}(t) - P_{F/E}(t) \]

divide by $a$ to get $P_{E}(t) < P_{F}(t)$

\[ \square \]
Gives natural compactification of Pic \( X \) - component, & being line bundle is open condition

Thm: \( X \) smooth \( \Rightarrow \) its also a closed condition

\( \exists \) \text{Hilb}^n \text{X} \text{ param ideal sheaves of 0-dim' length n subsch. of X}

\( \exists X = \text{intersection of 2 quadrics} \)

\[ \mathbb{P}^5 \setminus \mathbb{C} \]

\[ = \{ f = g = 0 \} \]

\( \rightarrow \) pencil of quadrics

\[ Q(a:b) = \{ af + bg = 0 \} \quad [a:b] \in \mathbb{P}^1 \]

If \( f, g \) generic, \( X \) smooth

\( \Rightarrow Q(a:b) \) smooth, except when

\[ \det \left( a M f + b M g \right) = 0, \quad \text{i.e.} \]

\[ \text{symm. matrix of } df \]
except at 6 pts of $\mathbb{P}^1$.

Each smooth $Q_{[a:b]} \cong Gr(2,4)$

$$0 \to S' \to \mathcal{O}^4 \to Q \to 0$$

tautological bundle

quotient bundle

Consider $S'|_X$, $Q|_X$ - rk 2 stable sheaves on $X$

moduli space param. these rk 2 stable sheaves $\cong$ double cover of $\mathbb{P}^1$ branched over those 6 pts.

The Brauer class:

$M =$ mod. sp. of stable sheaves w/ fixed Hilb. poly

goal: when does $\mathcal{F}$ a universal sheaf on $X \times M$, i.e. $U$ s.t.

$U|_{X \times \{F\}} \cong \mathcal{F}$?
If $U$ exists, then $M$ fine, i.e. represents functor:

For family $\mathcal{F}$ of sheaves on $X \times T$,

exists map $f : T \to M$ s.t.

\[(1 \times f)^* U \otimes \pi_2^* L \cong \mathcal{F} \]

A universal sheaf always exists locally (analytic/étale), but $\exists$ a Brauer class that can obstruct it globally:

1st, replace $F \in M \rightsquigarrow F(n)$ for $n \gg 0$

(boundedness $\Rightarrow \exists \ n$ that works $\forall F \in M$)

So assume $F$ is globally generated & $h^i(F) = 0 \ \forall \ i > 0$. 

Let \( m = h^0(F) = \chi(F) \)

\( U \) univ. sheaf - only well-defined up to \( \otimes \) line bundle

\[
\begin{array}{c}
X \times M' \xrightarrow{\pi_M'} M' \xrightarrow{\text{étale}} M
\end{array}
\]

\[ E := \pi_{M'}^* U \] \( \text{rk} \) \( m \) vector bundle on \( M' \)

\[ P' := \mathcal{P} \mathcal{E} \]

\[ \rho : P' \to M' \]

\( U \) vs \( U \otimes \pi_{M'}^* L \) also universal

\( E \) vs \( E \otimes L \)

\( \mathcal{O}_{P'E}(1) \) vs \( \mathcal{O}_{P' \otimes L(1)}(1) \equiv \mathcal{O}_{P'E}(1) \otimes L^* \)

on \( X \times P' \), \((1 \times \rho)^* U 
\otimes \mathcal{O}_{P'E}(1) \) is well-defined

\[ \Rightarrow \text{get } \mathbb{P}^{m-1}-\text{bundle } \pi : P \to M, \]

\[ P_{(F)} = \mathbb{P} H^0(F), \text{ w/ univ. sh. } \widetilde{U} \]

on \( X \times P \).
If \( E \) relative \( \mathcal{O}(1) \) for \( \pi \), then
\[
\tilde{U} \otimes \pi_2^*(\mathcal{O}(-1))
\]
descends to a univ.
sheaf on \( X \times M \).

Recall: relative \( \mathcal{O}(1) \) exists
\[
\iff P = \mathbb{P}(\text{v.b.})
\]
\[
\iff \text{Brauer class vanishes}
\]

In general, no relative \( \mathcal{O}(1) \).

Call the Brauer class the obstruction to the existence of
a univ. sheaf on \( X \times M \).

When does \( E \) relative \( \mathcal{O}(1) \)?

As above, \( \pi_2^* \tilde{U} \) \( \text{rk} \) \( m \) v.b. \( m \)
P, & restriction to any \( \mathbb{P}^{m-1} \)-fiber
is \( \mathcal{O}(1)^m \)
\[ \Rightarrow \det (\pi_2^* \tilde{\mathcal{A}}) \text{ is a relative } O(\log) \]

If \( E \) v.b. on \( X \) w/ \( \chi(E \otimes F) = k \)

\( \Rightarrow \) \( \det R\pi_2^* (\tilde{\mathcal{A}} \otimes \pi_1^* E) \) is a relative \( O(k) \)

If \( F \in M \)

\( \Rightarrow \det R\pi_2^* (\tilde{\mathcal{A}} \otimes \pi_1^* E) \) is a relative \( O(k) \)

If \( F \in \mathcal{L} \) w/ \( \gcd (\chi(E \otimes F)) = 1 \),

then \( \mathcal{L} \) relative \( O(1) \), & hence universal sheaf on \( X \times M \).