A Continuous-Time Version of the Principal-Agent Problem.

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October 25, 2007

Abstract

This paper describes a new continuous-time principal-agent model, in which the output is a diffusion process with drift determined by the agent's unobserved effort. The risk-averse agent receives consumption continuously. The optimal contract, based on the agent's continuation value as a state variable, is computed by a new method using a differential equation. During employment the output path stochastically drives the agent's continuation value until it reaches a point that triggers retirement, quitting, replacement or promotion. The paper explores how the dynamics of the agent's wages and effort, as well as the optimal mix of short-term and long-term incentives, depend on the contractual environment.[1]

Keywords: Principal-agent model, continuous time, optimal contract, career path, retirement, promotion

JEL Numbers: C63, D82, E2

[1]I am most thankful to Michael Harrison for his guidance and encouragement to develop a rigorous continuous time model from my early idea and for helping me get valuable feedback, and to Andy Skrzypacz for a detailed review of the paper and for many helpful suggestions. I am also grateful to Darrell Duffie, Yossi Feinberg, Xavier Gabaix, Bengt Holmstrom, Chad Jones, Gustavo Manso, Paul Milgrom, John Roberts, Thomas Sargent, Florian Scheuer, Sergio Turner, Cheng Wang, Ivan Werning and Robert Wilson for valuable feedback, and to Susan Athey for comments on an earlier version.

1 Introduction.

The understanding of dynamic incentives is central in economics. How do companies motivate their workers through piecerates, bonuses, and promotions? How is income inequality connected with productivity, investment and economic growth? How do financial contracts and capital structure give incentives to the managers of a corporation? The methods and results of this paper provide important insights to many such questions.

This paper introduces a continuous-time principal-agent model that focuses on the dynamic properties of optimal incentive provision. We identify factors that make the agent's wages increase or decrease over time. We examine the degree to which current and future outcomes motivate the agent. We provide conditions under which the agent eventually reaches retirement in the optimal contract. We also investigate how the costs of creating incentives and the dynamic properties of the optimal contract depend on the contractual environment: the agent's outside options, the difficulty of replacing the agent, and the opportunities for promotion.

Our new dynamic insights are possible due to the technical advantages of continuous-time methods over the traditional discrete-time ones. Continuous time leads to a much simpler computational procedure to find the optimal contract by solving an ordinary differential equation. This equation highlights the factors that determine optimal consumption and effort. The dynamics of the agent's career path are naturally described by the drift and volatility of the agent's payoffs. The geometry of solutions to the differential equation allows for easy comparisons to see how the agent's wages, effort and incentives depend on the contractual environment. Finally, continuous time highlights many essential features of the optimal contract, including the agent's eventual retirement.

In our benchmark model a risk-averse agent is tied to a risk-neutral principal forever after employment starts. The agent influences output by his continuous unobservable effort input. The principal sees only the output: a Brownian motion with a drift that depends on the agent's effort. The agent dislikes effort and enjoys consumption. We assume that the agent's utility function has the income effect, that is, as the agent's income increases it becomes costlier to compensate him for effort. Also, we assume that the agent's utility of consumption is bounded from below.

At time 0 the principal can commit to any history-dependent contract. Such a contract specifies the agent's consumption at every moment of time contingent on the entire past output path. The agent demands an initial reservation utility from the entire contract in order to begin, and the principal offers a contract only if he can derive a positive profit from it. After we solve our benchmark model, we examine how the optimal contract changes if the agent may quit, be replaced or promoted.

As in related discrete-time models, the optimal contract can be described in terms of the agent's continuation value as a single state variable, which completely determines the agent's effort and consumption. After any history of output the agent's continuation value is the total future expected utility. The agent's value depends on his future wages and effort. While in discrete time the optimal contract is described by cumbersome functions that map current continuation values and output realizations into future continuation values and consumption, continuous time offers more natural descriptors of employment dynamics: the drift and volatility of the agent's continuation value.

The volatility of the agent's continuation value is related to effort. The agent has incentives to put higher effort when his value depends more strongly on output. Thus, higher effort requires a higher volatility of the agent's value. The agent's optimal effort varies with his continuation value. To determine optimal effort, the principal maximizes expected output minus the costs of compensating the agent for effort and the risk required by incentives. If the agent is very patient, so that incentive provision is costless, the optimal effort decreases with the agent's continuation value due to the income effect. Apart from this extreme case, the agent's effort is typically nonmonotonic because of the costs of exposing the agent to risk.

The drift of the agent's value is related to the allocation of payments over time. The agent's value has an upward drift when his wages are backloaded, i.e. his current consumption is small relative to his expected future payoff. A downward drift of the agent's value corresponds to frontloaded payments. The agent's intertemporal consumption is distorted to facilitate the provision of incentives. The drift of the agent's value always points in the direction where it is cheaper to provide the agent with incentives. Unsurprisingly, when the agent gets patient, so that incentive provision is costless, his continuation value does not have any drift.

Over short time intervals, our optimal contract resembles that of Holmstrom and Milgrom (1987) (hereafter HM), who study a simple continuous-time model in which the agent gets paid at the end of a finite time interval. HM show that optimal contracts are linear in aggregate output when the agent has exponential utility with a monetary cost of effort.[2]

[2]Many other continuous-time papers have extended the linearity results of HM. Schattler and Sung (1993) develop a more general mathematical framework for such results, and Sung (1995) allows the agent

These preferences have no income effect. According to Holmstrom and Milgrom (1991), the model of HM is "especially well suited for representing compensation paid over short period." Therefore, it is not surprising that the optimal contract in our model is also approximately linear in incremental output over short time periods.

In the long-run, the optimal contract involves complex nonlinear patterns of the agent's wages and effort. In our benchmark setting, where the contract binds the agent to the principal forever, the agent eventually reaches retirement. After retirement, which occurs when the agent's continuation value reaches a low endpoint or a high endpoint, the agent receives a constant stream of consumption and stops putting effort.

The agent eventually reaches retirement in the optimal contract for two reasons. First, as shown in Spear and Wang (2005), the agent must retire when his continuation payoff becomes very low or very high.[3] For the low retirement point, the assumption that the agent's consumption utility is bounded from below implies that payments to the agent must stop when his value reaches the lower bound. For the high continuation values, retirement becomes optimal due to the income effect. When the agent's consumption is high, it costs too much to compensate him for positive effort. Second, retirement depends on the relative time preferences of the agent and the principal. If the agent had a higher discount rate than the principal, then with time the principal's benefit from output outweighs the cost of the agent's effort. Under these conditions, it is sensible to avoid permanent retirement by allowing the agent to suspend effort temporarily. However, when the agent is equally patient as the principal, our result implies that the agent eventually reaches permanent retirement in the optimal contract.[4]

Of course, retirement and other dynamic properties of the optimal contract depend on the contractual environment. The agent cannot be forced to stop consuming at the low retirement point if he has acceptable outside opportunities. Then, the agent *quits* instead of retiring at the low endpoint. If the agent is *replaceable*, the principal hires a new agent when the old agent reaches retirement. The high retirement point may also be replaced with *promotion*, an event that allows the agent to gain greater human capital, and manage larger and more important projects with higher expected output.[5]

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to control volatility as well. Hellwig and Schmidt (2002) look at the conditions for a discrete-time principal-agent model to converge to the HM solution. See also Bolton and Harris (2001), Ou-yang (2003) and Detemple, Govindaraj and Loewenstein (2001) for further generalization and analysis of the HM setting.

[3]See also Wang (2006) for an extension of Spear and Wang (2005) to equilibrium in labor markets.

[4]See DeMarzo and Sannikov (2006), Farhi and Werning (2006a) and Sleet and Yeltekin (2006) for examples where the agent is less patient than the principal.

[5]I thank the editor Juuso Valimaki for encouraging me to investigate this possibility.

The contractual envirnoment matters for the dynamics of the agent's wages. We already mentioned that the drift of the agent's continuation value always points in the direction where it is cheaper to create incentives. Since better outside options make it more difficult to motivate and retain the agent, it is not surprising that wages become more backloaded with better outside options. Lower payments up front cause the agent's continuation value to drift up, away from the point where he is forced to quit. When the employer can offer better promotion opportunities, the agent's wages also become backloaded in the optimal contract. The agent is willing to work for lower wages up front when he is motivated by future promotions. These findings highlight the factors that may affect the agent's wage structure in internal labor markets.

Is the agent motivated more by wages in the short run, or those in the long run in an optimal contract? Contracts in practice use both short-term incentives, as piecework and bonuses, and long-term ones, as promotions and permanent wage increases. In our model, the ratio of the volatilities of the agent's consumption and his continuation value measures the mix of short-term and long-term incentives. We find that the optimal contract uses stronger short-term incentives when the agent has better outside options, which interfere with the agent's incentives in the long run. In contrast, when the principal has greater flexibility to replace or promote the agent, the optimal contract uses stronger long-term incentives and keeps the agent's wages more constant in the short run. We find that the agent puts higher effort and the principal gets greater profit when the optimal contract relies on stronger long-term incentives.

It is difficult to study these dynamic properties of the optimal contract using discrete-time models. Without the flexibility of the differential equation that describes the dynamics of the optimal contract in continous time, traditional discrete-time models produce a more limited set of results. Spear and Srivastava (1987) show how to analyze dynamic principal-agent problems in discrete time using recursive methods, with the agent's continuation value as a state variable.[6] Assuming that the agent's consumption is nonnegative and that his utility is separable in consumption and effort, they show that the inverse of the agent's marginal utility of consumption is a martingale. An earlier paper of Rogerson (1985) demonstrates this result on a two-period model.[7] However, this restriction is not

[6]Abreu, Pearce and Stacchetti (1986) and (1990) study the recursive structure of general repeated games.

[7]This condition, called the *inverse Euler equation*, has received a lot of attention in recent macroeconomics literature. For example, see Golosov, Kocherlakota and Tsyvinski (2003) and Farhi and Werning (2006b).

very informative about the optimal path of the agent's wages, since a great diversity of consumption profiles in different contractual environments we study satisfy this restriction.

In its early stage, this paper was inspired by Phelan and Townsend (1991) who develop a method for computing optimal long-term contracts in discrete time. There are strong similarities between our continuous-time solutions and their discrete-time example, implying that ultimately the two approaches are different ways of looking at the same thing. Their computational method relies on linear programming and multiple iterations to converge to the principal's value function. While their method is quite flexible and applicable to a wide range of settings, it is far more computationally intensive than our method of solving a differential equation. Also, as we discussed earlier, our characterization allows for analytic comparisons to study how the optimal contract depends on the contractual environment.

Because general discrete-time models are difficult to deal with, one may be tempted to restrict attention to the special tractable case when the agent is patient. As the agent's discount rate converges to 0, efficiency becomes attainable, as shown in Radner (1985) and Fudenberg, Holmstrom and Milgrom (1990).[8] However, we find that optimal contracts when the agent is patient do not deliver many important dynamic insights: the agent's continuation value becomes driftless, and the agent's effort, determined without taking the cost of incentives into account, is decreasing in the agent's value.

Concurrently with this paper, Williams (2003) also develops a continuous-time principal-agent model. The aim of that paper is to present a general characterization of the optimal contract using a partial differential equation and forward and backward stochastic differential equations. The resulting contract is written recursively using several state variables, but due to greater generality, the optimal contract is not explored in as much detail.

More recently, a number of other papers started using continuous-time methodology. DeMarzo and Sannikov (2006) study the optimal contract in a cash-flow diversion model using the methods from this paper. Biais et al. (2006) show that the contract of DeMarzo and Sannikov (2006) arises in the limit of discrete-time models as the agent's actions become more frequent. Cvitanic, Wan and Zhang (2006) study optimal contracts when the agent gets paid once, and Westerfield (2006) develops an approach that uses the agent's wealth, as opposed to his continuation value, as a state variable.

The paper is organized as follows. Section 2 presents the benchmark setting and formulates the principal's problem. Section 3 presents an optimal contract and discusses its properties: the agent's effort and consumption, the drift and volatility of his continuation

[8]Also, Fudenberg, Levine and Maskin (1994) prove a Folk Theorem for general repeated games.

value and retirement points. The formal derivation of the optimal contract is deferred to the Appendix. Section 4 explores how the agent's outside options and the possibilities for replacement and promotion affect the dynamics of the agent's wages, effort and incentives. Section 5 studies optimal contracts for small discount rates. Section 6 concludes the paper.

2 The Benchmark Setting.

Consider the following dynamic principal-agent model in continuous time. A standard Brownian motion $Z = \{Z_t, \mathcal{F}_t; 0 \leq t < \infty\}$ on $(\Omega, \mathcal{F}, \mathcal{Q})$ drives the output process. The total output X_t produced up to time t evolves according to

$$dX_t = A_t\, dt + \sigma dZ_t,$$

where A_t is the agent's choice of effort level and σ is a constant. The agent's effort is a stochastic process $A = \{A_t \in \mathcal{A}, 0 \leq t < \infty\}$ progressively measurable with respect to $\mathcal{F}_t$, where the set of feasible effort levels $\mathcal{A}$ is compact with the smallest element 0. The agent experiences cost of effort $h(A_t)$, measured in the same units as the utility of consumption, where $h : \mathcal{A} \to \Re$ is continuous, increasing and convex. We normalize $h(0) = 0$ and assume that there is $\gamma_0 > 0$ such that $h(a) \geq \gamma_0 a$ for all $a \in \mathcal{A}$.

The output process X is publicly observable by both the principal and the agent. The principal does not observe the agent's effort A, and uses the observations of X to give the agent incentives to make costly effort. Before the agent starts working for the principal, the principal offers him a contract that specifies a nonnegative flow of consumption $C_t(X_s; 0 \leq s \leq t) \in [0, \infty)$ based on the principal's observation of output. The principal can commit to any such contract. We assume that the agent's utility is bounded from below and normalize $u(0) = 0$. Also, we assume that $u : [0, \infty) \to [0, \infty)$ is an increasing, concave and C^2 function that satisfies $u'(c) \to 0$ as $c \to \infty$.

For simplicity, assume that both the principal and the agent discount the flow of profit and utility at a common rate r. If the agent chooses effort level $A_t, 0 \leq t < \infty$, his average expected utility is given by

$$E\left[r \int_0^\infty e^{-rt} (u(C_t) - h(A_t))\, dt \right],$$

and the principal gets average profit

$$E\left[r\int_0^\infty e^{-rt}\,dX_t - r\int_0^\infty e^{-rt}C_t\,dt\right] = E\left[r\int_0^\infty e^{-rt}(A_t - C_t)\,dt\right].$$

Factor r in front of the integrals normalizes total payoffs to the same scale as flow payoffs.

We say that an effort process $\{A_t, 0 \le t < \infty\}$ is *incentive compatible* with respect to $\{C_t, 0 \le t < \infty\}$ if it maximizes the agent's total expected utility.

2.1 The Formulation of The Principal's Problem.

The principal's problem is to offer a contract to the agent: a stream of consumption $\{C_t, 0 \le t < \infty\}$ contingent on the realized output and an incentive-compatible advice of effort $\{A_t, 0 \le t < \infty\}$ that maximizes the principal's profit

$$E\left[r\int_0^\infty e^{-rt}(A_t - C_t)\,dt\right] \tag{1}$$

subject to delivering to the agent a required initial value of at least $\hat{W}$

$$E\left[r\int_0^\infty e^{-rt}(u(C_t) - h(A_t))dt\right] \ge \hat{W}. \tag{2}$$

We assume that the principal can choose not to employ the agent, so we are only interested in contracts that generate nonnegative profit for the principal.

3 The Optimal Contract.

In this section, we heuristically derive an optimal contract and discuss its basic properties. In Appendix A we formally show that an optimal contract takes the form presented in this section. Only for this section, assume that an optimal contract can be written in terms of the agent's continuation value W_t as a single state variable. The continuation value W_t is the total utility that the principal expects the agent to derive from the future after a given moment of time t. In the optimal contract, W_t will play the role of the unique state descriptor that determines how much the agent gets paid, what effort level he is supposed to choose, and how W_t itself changes with the realization of output. The principal must design a contract that specifies functions $c(W)$, the flow of the agent's consumption, $a(W)$, the recommended effort level, and the law of motion of W_t driven by the output path X_t. Three objectives must be met. First, the agent must have sufficient incentives to choose the recommended effort levels. Second, payments, recommended effort and the law of motion must be consistent, so that the state descriptor W_t represents the agent's *true* continuation value. Lastly, the contract must maximize the principal's profit.

Before we describe the dynamic nature of the contract, note that the principal has the option to retire the agent with any value $W \in [0, u(\infty))$, where $u(\infty) = \lim_{c\to\infty} u(c)$. To retire the agent with value $u(c)$, the principal offers him constant consumption c and allows him to choose zero effort. Denote the principal's profit from retiring the agent by

$$F_0(u(c)) = -c.$$

Note that the principal cannot deliver any value less than 0, because the agent can guarantee himself nonnegative utility by always taking effort 0. In fact, the only way to deliver value 0 is through retirement. Indeed, if the future payments to the agent are not always 0, the agent can guarantee himself a strictly positive value by putting effort 0. We call F_0 the principal's *retirement profit.*

Given the agent's consumption $c(W)$ and recommended effort $a(W)$, the evolution of the agent's continuation value W_t can be written as

$$dW_t = r\left(W_t - u(c(W_t)) + h(a(W_t))\right)\, dt + rY(W_t)\, \underbrace{(dX_t - a(W_t)dt)}_{\sigma dZ_t}, \tag{3}$$

where $rY(W)$ is the sensitivity of the agent's continuation value to output. When the agent takes the recommended effort level, $dX_t - a(W_t)\, dt$ has mean 0, and so $r(W_t - u(c(W_t)) + h(a(W_t)))$ is the drift of the agent's continuation value. To account for the value that the principal owes to the agent, W_t grows at the interest rate r and falls due to the flow of repayments $r(u(c(W_t)) - h(a(W_t)))$. We will see later that in the optimal contract, W_t responds to output in the employment interval $(0, W_{gp})$ and stops when it reaches 0 or W_{gp}, leading to retirement. After the agent is retired, he stops putting effort and receives constant consumption utility $ru(c(W_t)) = rW_t$.

The sensitivity $rY(W_t)$ of the agent's value to output affects the agent's incentives. If the agent deviates to a different effort level, his *actual* effort affects only the drift of X_t. The agent has incentives to choose effort that maximizes the expected change of W_t minus the cost of effort

$$rY(W_t)a - rh(a).$$

Since it is costly to expose the agent to risk, in the optimal contract $Y(W_t)$ is set at the *minimal* level that induces effort level $a(W_t)$. We denote this level by

$$\gamma(a) = \min\{y \in [0, \infty) : \ a \in \arg\max_{a' \in \mathcal{A}} ya' - h(a')\}. \tag{4}$$

Function $\gamma(a)$ is increasing in a. For the binary action case $\mathcal{A} = \{0, a\}$, $\gamma(a) = h(a)/a$. When $\mathcal{A}$ is an interval and h is a differentiable function, $\gamma(a) = h'(a)$ for $a > 0$.

We come to the crucial part where we describe the optimal choice of payments $c(W)$ and effort recommendations $a(W)$. Consider the highest profit $F(W)$ that the principal can derive when he delivers to the agent value W. Function $F(W)$ together with the optimal choices of $a(W)$ and $c(W)$ satisfy the HJB equation

$$rF(W) = \max_{a>0,c} \ r(a-c) + F'(W)r(W - u(c) + h(a)) + \frac{F''(W)}{2} r^2 \gamma(a)^2 \sigma^2. \tag{5}$$

The principal is maximizing the expected current flow of profit $r(a - c)$ plus the expected change of future profit due to the drift and volatility of the agent's continuation value.

Equation (5) can be rewritten in the following form suitable for computation

$$F''(W) = \min_{a>0,c} \frac{F(W) - a + c - F'(W)(W - u(c) + h(a))}{r\gamma(a)^2\sigma^2/2}. \tag{6}$$

To compute the optimal contract, the principal must solve this differential equation by setting

$$F(0) = 0 \tag{7}$$

and choosing the largest slope $F'(0) \geq F_0'(0)$ such that the solution F satisfies

$$F(W_{gp}) = F_0(W_{gp}) \quad \text{and} \quad F'(W_{gp}) = F_0'(W_{gp}) \tag{8}$$

at some point $W_{gp} \geq 0$, where $F'(W_{gp}) = F_0'(W_{gp})$ is called the *smooth-pasting condition.*[9] Let functions $c : (0, W_{gp}) \to [0, \infty)$ and $a : (0, W_{gp}) \to \mathcal{A}$ be the minimizers in equation (6). A typical form of the value function $F(0)$ together with $a(W)$, $c(W)$ and the drift of the agent's continuation value are shown in Figure 1.

Theorem 1, which is proved formally in Appendix A, characterizes optimal contracts.

Theorem 1. *The unique concave function $F \geq F_0$ that satisfies (6), (7) and (8)*

[9]If r is sufficiently large, then the solution of (6) with boundary conditions $F(0) = 0$ and $F'(0) = F_0'(0)$ satisfies $F(W) > F_0(W)$ for all $W > 0$. In this case $W_{gp} = 0$ and contracts with positive profit do not exist.

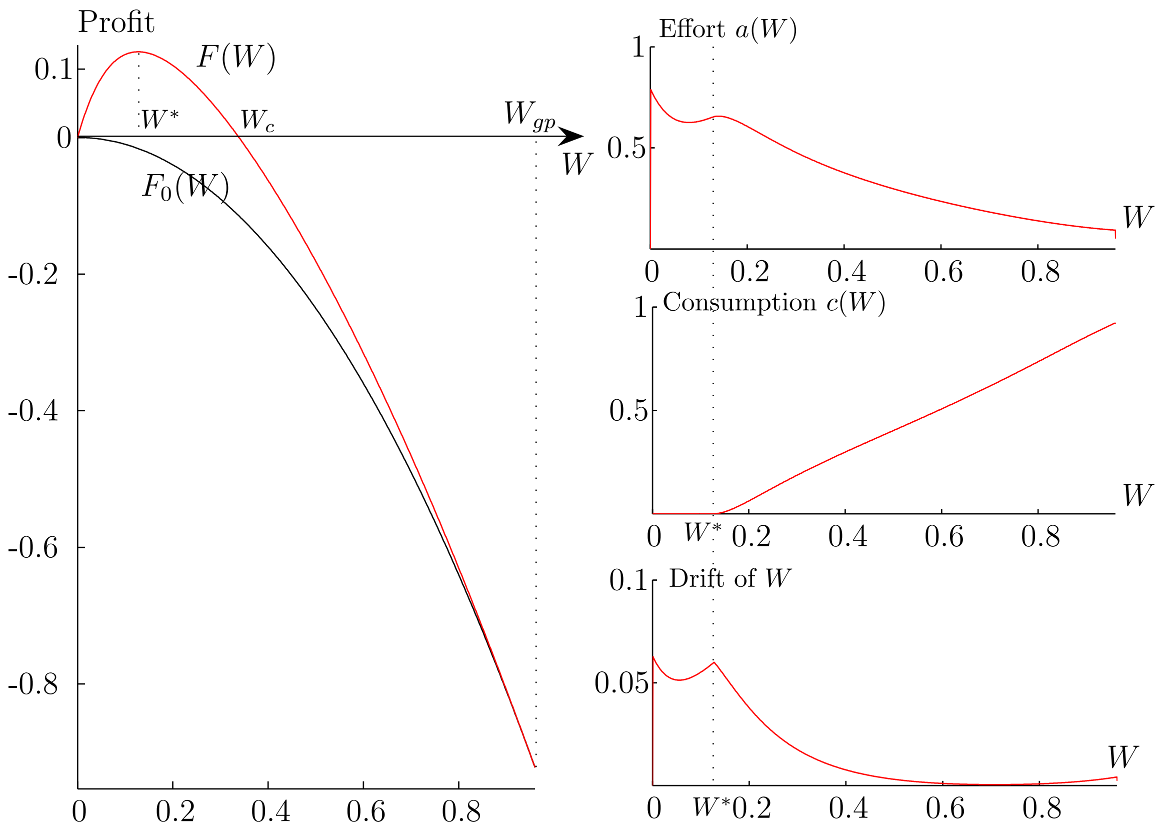

Figure 1: Function F for $u(c) = \sqrt{c}$, $h(a) = 0.5a^2 + 0.4a$, $r = 0.1$ and $\sigma = 1$. Point W^* is the maximum of F.

characterizes any optimal contract with positive profit to the principal. For the agent's starting value of $W_0 > W_{gp}$, $F(W_0) < 0$ *is an upper bound on the principal's profit. If* $W_0 \in [0, W_{gp}]$, *then the optimal contract attains profit* $F(W_0)$. *Such a contract is based on the agent's continuation value as a state variable, which starts at* W_0 *and evolves according to*

$$dW_t = r\left(W_t - u(C_t) + h(A_t)\right)\, dt + r\gamma(A_t)\left(dX_t - A_t\, dt\right) \tag{9}$$

under payments $C_t = c(W_t)$ *and effort* $A_t = a(W_t)$, *until the retirement time* τ. *Retirement occurs when* W_t *hits* 0 *or* W_{gp} *for the first time. After retirement the agent gets constant consumption of* $-F_0(W_\tau)$ *and puts effort 0.*

The intuition why the agent's continuation value W_t completely summarizes the past history in the optimal contract is the same as in discrete time. The agent's incentives are unchanged if we replace the continuation contract that follows a given history with a different contract that has the same continuation value W_t.[10] Therefore, to maximize the principal's profit after any history, the continuation contract must be optimal given W_t. It follows that the agent's continuation value W_t completely determines the continuation contract.

Let us discuss optimal effort and consumption using equation (5). The optimal effort maximizes

$$ra + rh(a)F'(W) + r^2\sigma^2\gamma(a)^2\frac{F''(W)}{2}, \tag{10}$$

where ra is the expected flow of output, $-rh(a)F'(W)$ is the cost of compensating the agent for his effort, and $-r^2\sigma^2\frac{\gamma(a)^2}{2}F''(W)$ is the cost of exposing the agent to income uncertainty to provide incentives. These two costs typically work in opposite directions, creating a complex effort profile (see Figure 1). While $F'(W)$ *decreases* in W because F is concave, $F''(W)$ *increase* over some ranges of W.[11] However, as $r \to 0$, the cost of exposing the agent to risk goes away and the effort profile becomes decreasing in W, except possibly near endpoints 0 and W_{gp} (see Section 5).

The optimal choice of consumption maximizes

$$-c - F'(W)u(c). \tag{11}$$

Thus, the agent's consumption is 0 when $F'(W) \geq -1/u'(0)$, in the *probationary* interval $[0, W^{**}]$, and it is increasing in W according to $F'(W) = -1/u'(c)$ above W^{**}. Intuitively, $1/u'(c)$ and $-F'(W)$ are the marginal costs of giving the agent value through current consumption and through his continuation payoff, respectively. Those marginal costs must be equal under the optimal contract, except in the probationary interval. There, consumption zero is optimal because it maximizes the drift of W_t away from the inefficient low retirement point.

Why is it optimal for the principal to retire the agent if his continuation payoff becomes sufficiently large? This happens due to the income effect: when the flow of payments to the agent is large enough, it costs the principal too much to compensate the agent for his effort, so it is optimal to allow effort 0. When the agent gets richer, the monetary cost of

[10] This logic would fail if the agent had hidden savings. With hidden savings, the agent's past incentives to save depend not only on his current promised value, but also on how his value would change with savings level. Therefore, the problem with hidden savings has a different recursive structure, as discussed in the conclusions.

[11] $F''(W)$ increases at least on the interval $[0, W^*]$, where $c = 0$ and sign $F'''(W) = \text{sign}\,(rW - u(c) + h(a)) > 0$ (see Theorem 2). When W is smaller, the principal faces a greater risk of triggering retirement by providing stronger incentives.

delivering utility to the agent rises indefinitely (since $u'(c) \to 0$ as $c \to \infty$) while the utility cost of output stays bounded above 0 since $h(a) \geq \gamma_0 a$ for all a. High retirement occurs even before the cost of compensating the agent for effort exceeds the expected output from effort, since the principal must compensate the agent not only for effort, but also for risk (see (10)). While it is *necessary* to retire the agent when W_t hits 0 and it is *optimal* to do so if W_t reaches W_{gp}, there are contracts that prevent W_t from reaching 0 or W_{gp} by allowing the agent to suspend effort temporarily. Those contracts are suboptimal: in the optimal contract the agent puts positive effort until he is retired forever.[12]

The drift of W_t is connected with the allocation of the agent's wages over time. Section 5 shows that the drift of W_t becomes zero when the agent becomes patient, to minimize intertemporal distortions of the agent's consumption. In general, the drift of W_t is nonzero in the optimal contract. Theorem 2 shows that the drift of W_t always points in the direction where it is cheaper to provide incentives.

Theorem 2. *Until retirement, the drift of the agent's continuation value points in the direction in which $F''(W)$ is increasing.*

Proof. From (5) and the Envelope Theorem, we have

$$r(W - u(c) + h(a))F''(W) + r^2\sigma^2\gamma(a)^2\frac{F'''(W)}{2} = 0 \tag{12}$$

Since $F''(W)$ is always negative, $W - u(c) + h(a)$ has the same sign as $F'''(W)$. QED

By Ito's lemma, (12) is the drift of $-1/u'(C_t) = F'(W_t)$ on $[W^{**}, W_{gp}]$. Thus, in our model the inverse of the agent's marginal utility is a martingale when the agent's consumption is positive. The analogous result in discrete time was first discovered by Rogerson (1985).

In the next subsection we discuss the paths of the agent's continuation value and income, and the connections between the agent's incentives, productivity and income distribution in the example in Figure 1. Before that, we make three remarks about possible extensions of our model.

Remark 1: Retirement. If the agent's utility was unbounded from below (e.g. exponential utility), our differential equation would still characterize the optimal contract, but the agent may never reach retirement at the low endpoint. To take care of this possibility,

[12]This conclusion depends on the assumption that the agent's discount rate is the same as that of the principal. If the agent's discount rate was higher, the optimal contract may allow the agent to suspend effort temporarily.

the boundary condition $F(0) = 0$ would need to be replaced with a regularity condition on the asymptotic behavior of F. Of course, the low retirement point does not disappear if the agent has an outside option at all times (see Section 4). Similarly, if the agent's utility had no income effect, the high retirement point may disappear as well. This would be the case if we assumed exponential utility with a monetary cost of effort, as in Holmstrom and Milgrom (1987).[13]

Remark 2: Savings. We assume in this model that the agent cannot save or borrow, and is restricted to consume what the principal pays him at every moment of time. What would happen if the agent actually *could* save and borrow at rate r, but the principal did not know it? What would he do? Since $1/u'(c(W_s))$ is a martingale, the agent's marginal utility of consumption $u'(c(W_s))$ must be a submartingale. Since the agent's marginal utility increases in expectation, he is tempted to save for the future. The conclusion discusses the difficulties connected with finding the optimal contract when the agent can save secretly.

Remark 3: A risk-averse principal. Although our model assumes a risk-neutral principal, there are a number of ways to examine the principal's risk aversion. First, if the risk-averse principal has access to perfect capital markets he should be able to diversify the idiosyncratic risk connected with the agent's output.[14] This makes the principal effectively risk-neutral to output and our solution applies. Second, if we allow the principal to be explicitly risk-averse we can expect the qualitative features of the optimal contract (including retirement) to be the same as with risk-neutrality. The agent would probably have to bear higher risk and take higher effort. Formally, this problem can be dealt with by introducing an additional state variable, the principal's wealth. Although an additional state variable would involve a partial differential equation and a more complicated solution, some simplifications are possible for specific functional forms of utilities and production technologies. Williams (2005) has an example with log utilities that reduces to a single state variable.[15]

[13] DeMarzo and Sannikov (2006) study a dynamic agency problem without the income effect. In their setting the moral hazard problem is that the agent may secretly divert cash from the firm, so his benefit from the hidden action is measured in monetary terms. The optimal contract has a low absorbing state, since the agent's utility is bounded from below, but no upper absorbing state.

[14] After the principal subtracts from output the noise correlated with market risk factors, the remaining statistic for the agent's effort contains only idiosyncratic noise.

[15] Because log utilities are unbounded from below, Williams (2005) does not have a low retirement point.

3.1 Income Dispersion, Effort and Productivity.

The left panel of Figure 2 shows how the distribution of the agent's continuation values W_t changes with time when W_0 is the largest value for which the principal's profit is nonnegative, for the example in Figure 1.[16] The distribution becomes absorbed at the retirement points over time. In this example, the agent is more likely to retire at W_{gp} than at 0, despite the fact that W_0 is at the lower end of the interval $[0, W_{gp}]$. This outcome is not surprising, given that the drift of W_t is positive on the entire interval $[0, W_{gp}]$ in this example.

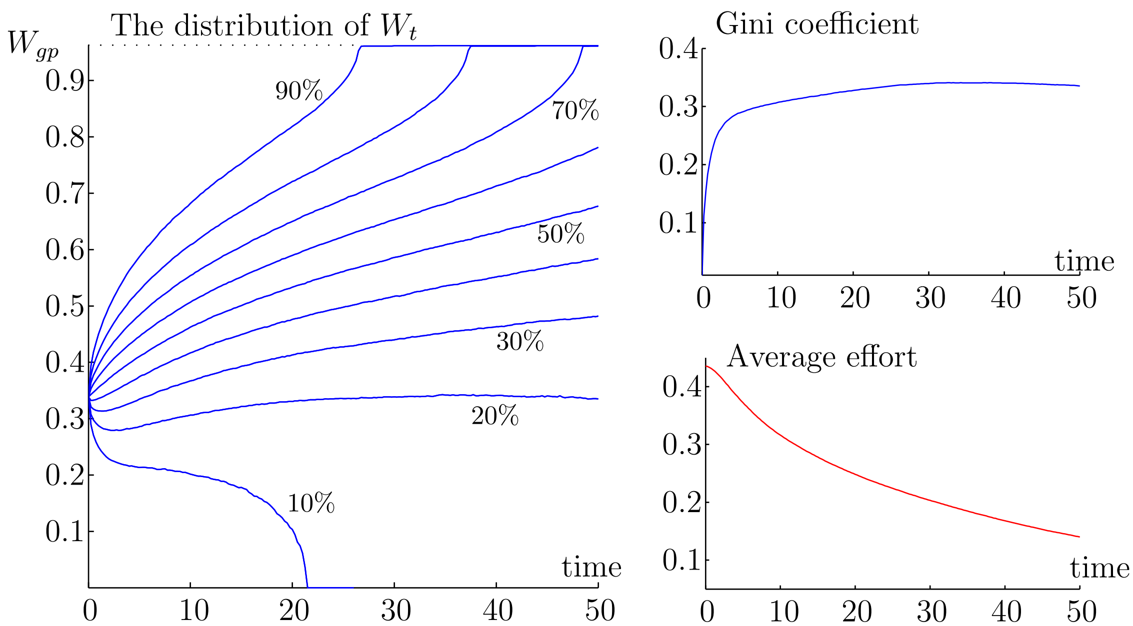

Figure 2: Consumption, inequality and expected output for the example in Figure 1.

Besides the likely paths of the agent's continuation value, Figure 2 also explores the distribution of income and effort for a population of identical agents who start with value W_0 under the optimal contract. Income inequality widens over time due to the provision of incentives and the Gini coefficient in the population generally increases (top right panel). However, the average effort *decreases* (bottom right panel). This fact is consistent with empirical findings that economic growth is *negatively* related to income inequality.[17] Those

[16] This distribution is computed using Monte Carlo simulations.

[17] For example, see Alesina and Rodrik (1994), Persson and Tabellini (1994), Perrotti (1996) and an excellent survey Aghion, Caroli and Garcia-Penalosa (1999).

findings appear to be at odds with the traditional idea that there is a trade-off between efficiency and equality (see Mirrlees (1971)). Our model suggests that incentives are connected with the *rate* at which the income distribution widens, rather than the income inequality itself.[18]

The example in Figure 2 is also related to empirical findings in personnel economics (see Spitz (1991) and Lazear (1999)) that workers' wages typically rise faster than their productivity. This work is motivated by Lazear (1979), who argues that deferred compensation can serve as a motivating factor for workers early in their career. In our example the agent's puts high effort early in his career and his continuation value tends to increase over time. However, the increasing pattern of the agent's wages and the extent to which the agent is motivated by distant future are really *different* issues. Indeed, if the agent expects his wages to increase no matter what, he would not be motivated to put effort. In the next section we explore the factors that determine the time profiles of the agent's wages and incentives.

4 The Agent's Career Path.

In this section we extend our model to study how the contractual environment affects the dynamics of the agent's wages, effort and incentives in the optimal contract. Specifically, we investigate the agent's outside opportunities, the principal's options to replace the agent, and the possibilities for the agent's career development.[19]

Lazear (1979) argues that employers can strengthen the employment relationship by offering a rising pattern of wages. Our optimal contracts weigh this benefit of backloaded wages against the costs of the income effect, earlier retirement and distortions of the agent's consumption. The optimal resolution of this trade-off depends on the contractual environment. In our optimal contract wages become more backloaded when the agent has better outside options, or when promotion, instead of retirement, can be used to reward the agent. Technically, we estimate the agent's wage profile by studying how his current wage compares with his continuation value.[20]

[18]In the long run, average effort converges to zero as inequality becomes extreme, a fact related to the immizeration result of Atkeson and Lucas (1992). However, on Figure 2 we see that in the short run also, average effort decreases while inequality widens.

[19]Our methods can be also applied to study the optimal contract when the principal cannot commit. Such a contract must be a public equilibrium of the game between the principal and the agent (see Levin (2003)).

[20]This measure has its drawbacks, since the agent's continuation value depends not only on his future

The dynamic provision of incentives is also an important characteristic of employment relationships. Although both short-term and long-term incentives have been studied individually, the dynamic properties of incentives have not received sufficient attention in the literature. Short-term incentives include piece rates and year-end bonuses, while long-term incentives include terminations, permanent wage increases and promotions. Lazear and Rosen (1981) study the long-term incentives created by promotions, and Holmstrom and Milgrom (1987) and (1991) study the optimality of piece rates as short-term incentives. We study the optimal mix of short-term and long-term incentives. This issue is of practical importance. For example, Lazear (2000) documents the example, in which the productivity has increased by 44 percent when the company started using piece rates to motivate its workers. Prior to the switch to piecerates, autoglass installers were motivated only by terminations and promotions, which provide long-term incentives.

We measure the optimal mix of short-term and long-term incentives by the ratio of the volatilities of the agent's consumption and his continuation value. Long-term incentives are more difficult to create when the agent has better outside options, and easier, when the principal can replace or promote the agent. In the case of autoglass installers at Safelite, short-term incentives turned out to be vital for the optimal contract, perhaps because terminations and promotions play a minor role in their contractual environment in comparison, for example, with junior faculty, whose contracts do not include a significant piecerate component. Generally, our results imply that the agent works harder when the optimal contract relies more on long-term incentives.

The rest of this section is organized as follows. Subsection 4.1 presents Theorem 3, which characterizes the optimal contract in a general contractual environment. Subsections 4.2 and 4.3 apply Theorem 3 to settings with outside options, replacements and promotions. Subsection 4.4 provides comparisons across the different environments.

4.1 Additional Contractual Possibilities.

We can summarize the contractual environment, with the agent's outside option $\tilde{W}$, by a function $\tilde{F}_0 : [\tilde{W}, \infty) \to \Re$. This function captures the principal's options to deliver to the agent value $W \in [\tilde{W}, \infty)$ at profit $\tilde{F}_0(W)$. In our benchmark setting retirement is the only contractual option, so that $\tilde{W} = 0$ and $\tilde{F}_0 = F_0$. If the agent has an outside option of $\tilde{W} > 0$, then $\tilde{F}_0(\tilde{W}) = 0$ as the principal can deliver to the agent value $\tilde{W}$ by firing

wages, but also the cost of effort and income uncertainty. However, in our setting this measure produces by far the cleanest analytical results in comparisons with other measures, e.g. the drift of consumption.

him. Function $\tilde{F}_0$ can capture many other contractual possibilities, including replacement and promotion. In general, we assume that $\tilde{F}_0$ is an upper semi-continuous function that satisfies $\tilde{F}_0(W) \geq F_0(W)$ for all $W \in [\tilde{W}, \infty)$, with equality when W is sufficiently large.

A contract in this environment specifies the agent's consumption $\{C_t, t \leq \tau\}$, an incentive-compatible effort recommendation $\{A_t, t \leq \tau\}$ and a stopping time τ when the agent receives value W_τ at profit $\tilde{F}_0(W_\tau)$. We consider the problem of maximizing the principal's profit

$$E\left[r\int_0^\tau e^{-rt}(A_t - C_t)\,dt + e^{-r\tau}\tilde{F}_0(W_\tau)\right]$$

subject to giving the agent a specific value of $W_0 \geq \tilde{W}$

$$E\left[r\int_0^\tau e^{-rt}(u(C_t) - h(A_t))\,dt + e^{-r\tau}W_\tau\right] = W_0$$

and the agent's participation constraint for all $t \leq \tau$

$$E\left[r\int_t^\tau e^{-r(s-t)}(u(C_s) - h(A_s))\,ds + e^{-r(\tau-t)}W_\tau \mid \mathcal{F}_t\right] \geq \tilde{W}.$$

The value of W_0 is determined by the relative bargaining powers of the principal and the agent.

Theorem 3 shows that the optimal contract in the presence of new contracting possibilities is described by the same HJB equation as before, but with new boundary conditions. This characterization allows us to perform clean comparisons across the different environments.

Theorem 3. *Suppose that a concave function $\tilde{F} \geq \tilde{F}_0$ satisfies the HJB equation on $[\tilde{W}, \infty)$ and coincides with $\tilde{F}_0$ at W_L and $W_H > W_L$. Then the optimal contract with any value $W_0 \in [W_L, W_H]$ to the agent has profit $\tilde{F}(W_0)$ to the principal. Let $\tilde{a} : [W_L, W_H] \to \mathcal{A}$ and $\tilde{c} : [W_L, W_H] \to [0, \infty)$ be the optimizers in the HJB equation associated with the function $\tilde{F}$. In the optimal contract the agent's value starts at W_0 and evolves according to*

$$dW_t = r\left(W_t - u(C_t) + h(A_t)\right)\,dt + r\gamma(A_t)\left(dX_t - A_t\,dt\right) \tag{13}$$

under payments $C_t = \tilde{c}(W_t)$ and effort $A_t = \tilde{a}(W_t)$, until W_t hits W_L or W_H for the first time at time τ. At time τ the principal follows the contractual possibility that delivers to the agent value W_τ at profit $\tilde{F}_0(W_\tau)$.

Proof. See Appendix B.

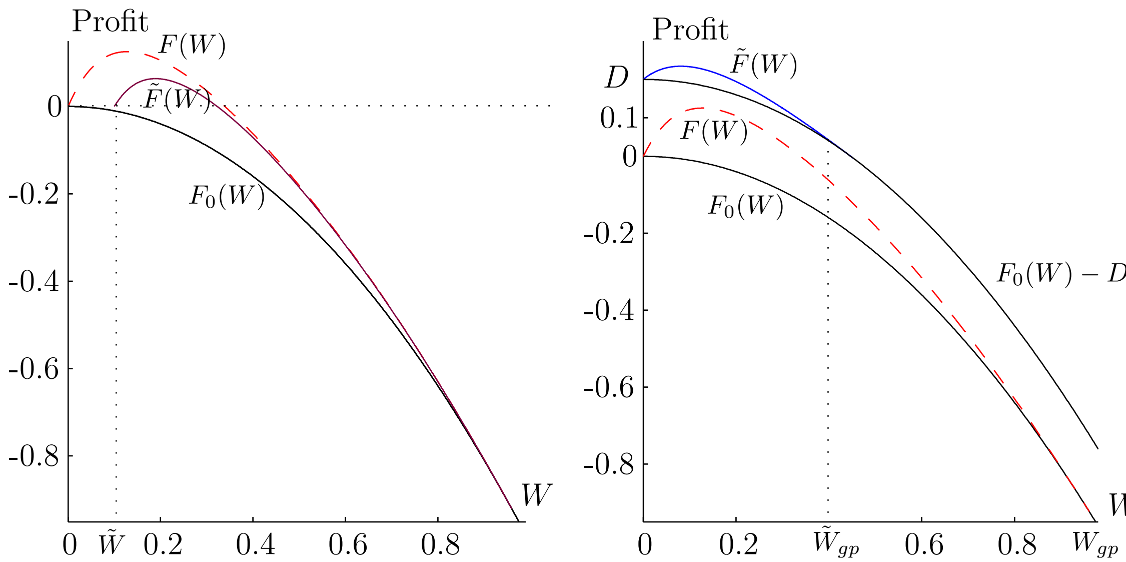

Figure 3: The principal's profit when the agent has an outside option of $\tilde{W} = 0.1$ (left panel) or the principal can replace him for profit $D = 0.2$ (right panel).

Lemma 5 in Appendix B shows that for any $W_0 \in [\tilde{W}, \infty)$, if a function $\tilde{F}$ with the desired properties does not exist, then the principal cannot do better than to randomize among the points of $\tilde{F}(W)$ at time 0. Also, note that if $\tilde{F}$ does exist, then it characterizes the optimal contract for any value in the interval $[W_L, W_H]$.

4.2 Quitting and Replacement.

In this subsection we use Theorem 3 to describe the optimal contract when the agent may quit or the principal may replace the agent.

What if the agent can quit working for the principal at any time and receive a value of $\tilde{W} \geq 0$? We can interpret $\tilde{W}$ as the value from new employment minus the search cost. Then $\tilde{F}_0(\tilde{W}) = 0$ and $\tilde{F}_0(W) = F_0(W)$ for $W \in (\tilde{W}, \infty)$. The function $\tilde{F} \geq \tilde{F}_0$ in the left panel of Figure 3, which satisfies the boundary conditions $\tilde{F}(\tilde{W}) = 0$, $\tilde{F}(\tilde{W}_{gp}) = \tilde{F}_0(\tilde{W}_{gp})$ and $\tilde{F}'(\tilde{W}_{gp}) = \tilde{F}_0'(\tilde{W}_{gp})$ for some $\tilde{W}_{gp} \geq \tilde{W}$, represents the maximal profit that the principal can achieve from any contract with value $W_0 \in [\tilde{W}, \tilde{W}_{gp}]$. The principal's profit would be negative for any contract with value $W_0 > \tilde{W}_{gp}$, since function $\tilde{F}$ is an upper bound on the principal's profit there. Figure 3 (both panels) are computed for the same functional forms of the agent's utility and cost of effort as in Figure 1 in Section 3.

What if the agent cannot freely quit, but the principal can replace him with a new agent? If the principal derives value D from the new agent, then $\tilde{F}_0(W) = F_0(W) + D$ for all $W \in [0, \infty)$. The optimal contract is characterized by the function $\tilde{F} \geq \tilde{F}_0$ that satisfies the HJB equation with the boundary conditions $\tilde{F}(0) = 0$, $\tilde{F}(\tilde{W}_{gp}) = \tilde{F}_0(\tilde{W}_{gp})$ and $\tilde{F}'(\tilde{W}_{gp}) = \tilde{F}_0'(\tilde{W}_{gp})$, for some $\tilde{W}_{gp} \geq 0$. The right panel of Figure 3 shows the typical form of $\tilde{F}$.

Note that if the market is full of identical agents with reservation value 0, and if it costs the principal C to replace the agent, then D is determined endogenously by $D = \max \tilde{F}(W) - C$.

4.3 Promotion.

Suppose that the principal has an opportunity to promote the agent to a new position by training him at a cost $K \geq 0$. Training permanently increases the agent's productivity of effort from a to θa, for some $\theta > 1$, and his outside option from 0 to $W_p \geq 0$. The agent's new skills may be valuable to other firms. What does the optimal contract look like in this setting? Does the principal train the agent immediately upon hiring, or with a delay?

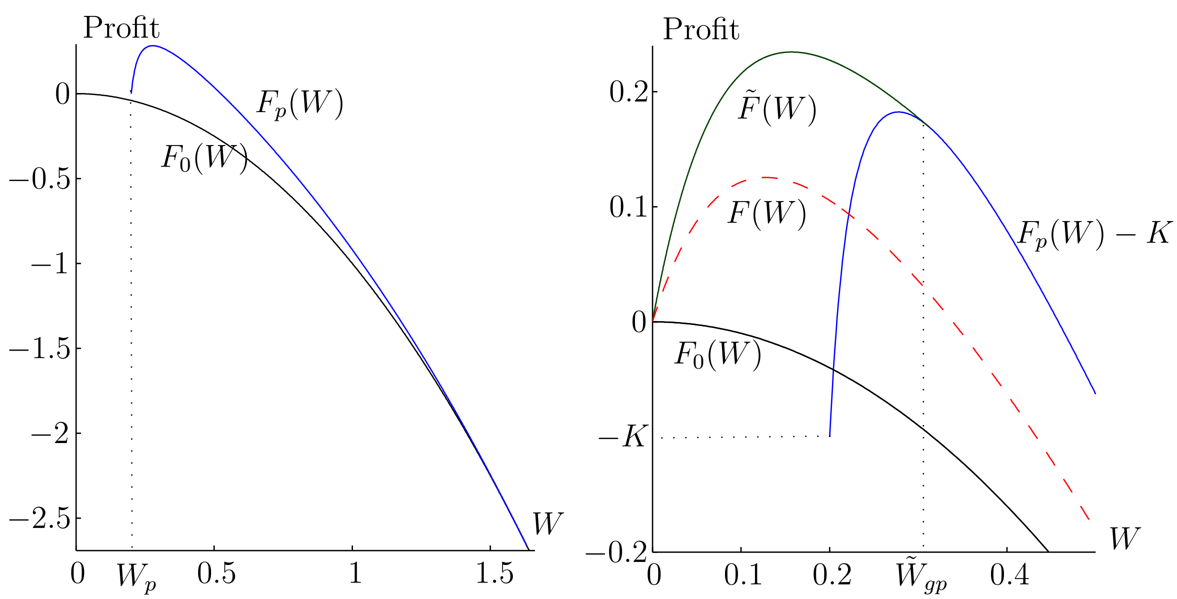

Figure 4: The principal's profit with promotion when $\theta = 1.25$ and $K = 0.1$.

To solve the problem, denote by $F_p(W)$ the principal's profit function from a trained agent, the largest solution of equation

$$F_p''(W) = \min_{a>0,c} \frac{F_p(W) - \theta a + c - F_p'(W)(W - u(c) + h(a))}{r\sigma^2\gamma(a)^2/(2\theta^2)}$$

with boundary conditions $F_p(\tilde{W}_p) = 0$, $F_p(W_{gp}) = F_0(W_{gp})$ and $F_p'(W_{gp}) = F_0'(W_{gp})$ for some $W_{gp} \geq 0$, where $r\sigma\gamma(a)/\theta$ is the volatility of the agent's continuation value required to provide incentives for productivity θa.

For an untrained agent, the principal must decide on the timing of promotion or retirement. Function $\tilde{F}_0(W) = \max(F_0(W), F_p(W) - K)$ summarizes these options. Figure 4 shows a typical form of the principal's profit from an agent before promotion, which satisifes

$$\tilde{F}''(W) = \min_{a>0,c} \frac{\tilde{F}(W) - a + c - \tilde{F}'(W)(W - u(c) + h(a))}{r\sigma^2\gamma(a)^2/2}$$

with boundary conditions $\tilde{F}(0) = 0$, $\tilde{F}(\tilde{W}_{gp}) = F_p(\tilde{W}_{gp}) - K$ and $\tilde{F}'(\tilde{W}_{gp}) = F_p'(\tilde{W}_{gp})$.

The right panel of Figure 4 compares the principal's profit with and without the possibility of promotion. Promotion, which happens when the agent's value hits W_{gp}, provides an efficient way to reward the agent.

This example sheds light on how firms use complex employment hierarchies to create incentives, and why employees climb up the hierarchial ladder gradually. Two reasons why promotion is delayed in our example are the cost of training and the improvement of the agent's outside option.

4.4 Comparisons.

We can study how the contractual environment affects the agent's career by exploring how the solutions of the HJB equation change with the boundary conditions. Theorem 4 is the main tool that allows us to perform comparisons.

Theorem 4. *Consider two concave solutions of the HJB equation $\tilde{F}_1(W)$ and $\tilde{F}_2(W)$. Denote by $\tilde{c}_i(W)$ and $\tilde{a}_i(W)$ the agent's consumption and effort that correspond to value W in the contract associated with function $\tilde{F}_i(W)$.*

(a) If $\tilde{F}_1(W) = \tilde{F}_2(W)$ and $\tilde{F}_1'(W) < \tilde{F}_2'(W)$ for some W, then the contract associated with $\tilde{F}_2$ involves more backloaded payments. That is, for any w, $\tilde{c}_1(w) \geq \tilde{c}_2(w)$.

(b) If $\tilde{F}_1(W) \leq \tilde{F}_2(W)$ for all W, then for each pair of values w and w' such that $\tilde{F}_1'(w) = \tilde{F}_2'(w')$ (and so $\tilde{c}_1(w) = \tilde{c}_2(w')$) we have

(i) $\tilde{a}_1(w) \leq \tilde{a}_2(w')$. Therefore, for each positive wage level the contract associated with $\tilde{F}_2$ involves weakly higher effort.

(ii) the ratio of volatilities of the agent's consumption and continuation values is greater in the contract associated with $\tilde{F}_2$ when $\tilde{c}_1(w) = \tilde{c}_2(w') > 0$. Therefore, for each positive wage level, the contract associated with $\tilde{F}_2$ relies less on short-term incentives.

Proof. See Appendix B.

Theorem 4 compares the agent's continuation values, efforts and the mix of short and long-term incentives in two different contracts at points that correspond to the *same* wage level. This method of comparisons is not only convenient analytically, but also relevant empirically, since wages are much easier to observe than continuation payoffs.

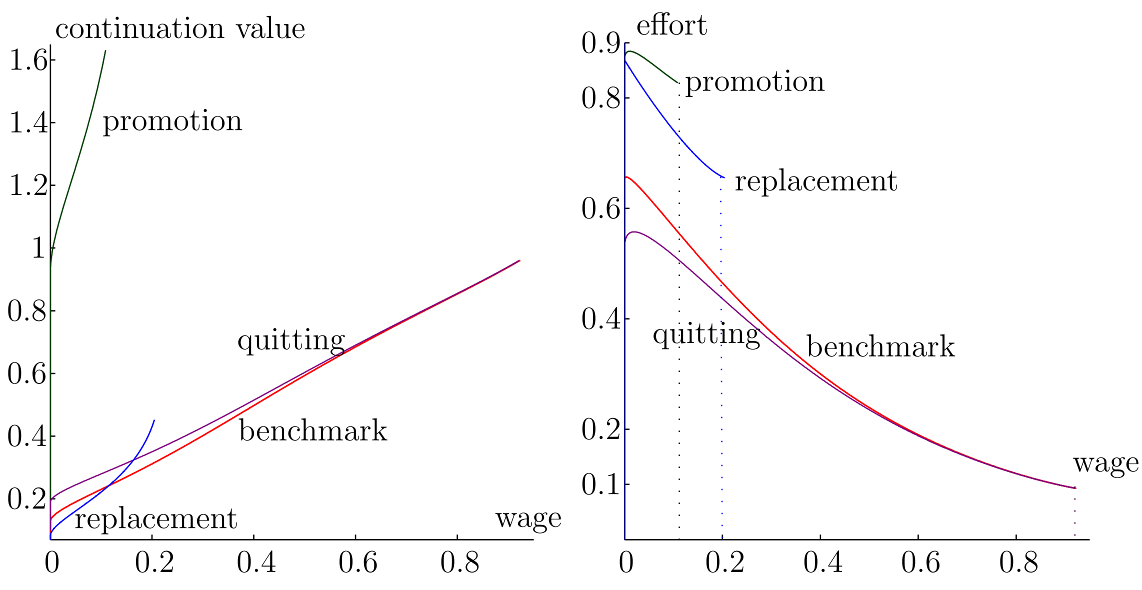

Figure 5: The agent's value and effort as functions of wage under different contractual possibilities.

To apply Theorem 4, we need to compare the principal's profit in different settings and find points where profit functions intersect. In terms of the principal's profit (part (b) of Theorem 4), the settings we have considered compare as follows:

$$\text{quitting} \; < \; \text{benchmark} \; < \; \text{replacement, promotion}$$

where the last two settings cannot be ranked.[21] For each wage level, both the agent's effort and the use of long-term incentives are greater when the principal's profit is higher.

Regarding the relationship between the agent's consumption and his continuation value, the profit function $\tilde{F}$ with promotion intersects F at $W = 0$, where $\tilde{F}'(0) \geq F'(0)$. Also, the profit function $\tilde{F}$ when the agent has an outside option must intersect F at a point $W \in (\tilde{W}_{gp}, W_{gp})$, where $\tilde{F}'(W) \geq F'(W)$. Therefore, both when the agent can get promoted or when he has an outside option, the agent's wages are more backloaded relative to the benchmark contract.

The following table summarizes how contracts in different settings compare to the benchmark, and Figure 5 illustrates them for our examples.

possibility	payments	effort	incentives
outside option	more backloaded	smaller	more immediate
replacement	either	greater	more distant
promotion	more backloaded	greater	more distant

Note that we perform comparisons only for positive wage levels, which correspond to a single point on the principal's value function.

Our final comparative static has to do with the nature of human capital the agent gains *after* promotion. The human capital is less firm-specific when W_p is higher. When the agent gains better outside options due to promotion, then the principal's profit after promotion F_p is lower and the slope $\tilde{F}'(0)$ before promotion becomes lower. As a result, less firm-specific human capital after promotion results in more frontloaded wages and incentives in the optimal contract, and it also motivates the agent to put lower effort.

5 Optimal Contracts when the Agent is Patient.

Discrete-time models have focused considerable attention on the case when the agent is patient. The standard result is that efficiency is attainable as the agent's discount rate

[21] Also, for the case of quitting it is assumed that $\tilde{W} < W_c$, where $W_c > 0$ is defined by $F(W_c) = 0$. Then $\tilde{F} \leq F$ on $[\tilde{W}, \tilde{W}_{gp}]$ because $\tilde{F}(\tilde{W}) < F(\tilde{W})$ and $\tilde{F}(\tilde{W}_{gp}) = F_0(\tilde{W}_{gp})$.

goes to 0 (or the discount factor between periods goes to 1). In this section, we confirm this result in continuous time. More importantly, we argue that the case when the agent is patient may be of limited interest if one wants to understand the dynamic properties of optimal contracts. When r is close to 0, optimal contracts have straightforward dynamic properies. In the limit, the agent's wages are neither backloaded nor frontloaded and his continuation value is driftless. The agent's effort decreases in his continuation value, and stays roughly constant over time. The dynamic properties of the contract related to the optimal incentive provision disappear.

We focus on the case where $\mathcal{A} = [0, \bar{A}]$ and the cost of effort $h(a)$ is twice continuously differentiable. Denote by $\bar{F} : [-h(\bar{A}), u(\infty)) \to (-\infty, \bar{A}]$ the first-best average profit, which the principal would be able to achieve if he could control both the agent's consumption and effort. When $W \geq W_{gp}^*$, we have $\bar{F}(W) = F_0(W)$, where W_{gp}^* is defined by $F_0'(W_{gp}^*) = \gamma_0 = 1/v'(0)$. When $W < W_{gp}^*$, profit $\bar{F}(W) = \bar{a}(W) - \bar{c}(W)$ is achieved by constant effort and consumption that satisfy $W = u(\bar{c}(W)) - h(\bar{a}(W))$ and (assuming $v'(\bar{A}) = u'(0)$) the first-order condition

$$\bar{F}'(W) = u'(\bar{c}(W)) = v'(\bar{a}(W)). \tag{14}$$

Theorem 5 characterizes the principal's profit and the optimal contract when r is small.

Theorem 5. *As $r \to 0$, $W_{gp} \to W_{gp}^*$ and the principal's average profit F converges to first best pointwise on $(0, W_{gp}^*)$. The agent's consumption and effort converge to $\bar{c}(W)$ and $\bar{a}(W)$ pointwise on $(0, W_{gp}^*)$, and the principal's profit is*

$$F(W) = \bar{F}(W) + r\gamma(\bar{a}(W))^2\sigma^2 \frac{\bar{F}''(W)}{2} + o(r).$$

Proof. See Appendix C.

What is the intuition behind Theorem 5? When the agent is patient, incentives require very little variation in the agent's income. As a result, the optimal contract implements first-best effort $\bar{a}(W)$ and uses only a local portion of the agent's utility function. The average inefficiency is

$$-\frac{r\gamma(\bar{a}(W))^2\sigma^2\bar{F}''(W)}{2} = -\frac{rv'(\bar{a}(W))^2\sigma^2 u''(\bar{c}(W))}{2u'(\bar{c}(W))^3} = \frac{r\alpha^2\sigma^2\delta}{2},$$

where $\delta = -u''(\bar{c}(W))/u'(\bar{c}(W))$ is the agent's coefficient of absolute risk aversion at $\bar{c}$ and

$\alpha = v'(\bar{a})/u'(\bar{c})$ is the piece-rate. This form of losses is similar to that in the multi-task model of Holmstrom and Milgrom (1991), confirming once again that the case of a patient agent captures only short-run properties of the optimal contract.

6 Conclusion.

This paper develops a new flexible method of analyzing optimal dynamic contracts between a principal and an agent when the agent's effort is not directly observable. Contracts in continuous-time are conveniently characterized by the drift and volatility of the agent's continuation value. The drift is related to the intertemporal pattern of the agent's wages and the volatility to incentives and effort. The provision of incentives, the agent's effort and the allocation of payments over time depend on many factors, including the agent's outside options, the cost of replacing the agent and promotion opportunities.

Despite the abstract form of the model, the properties of the optimal contract we investigate have a real economic meaning. It is possible to estimate empirically both the rate at which employee wages tend to increase and the mix of short-term and long-term incentives from employment data.[22] While some of the issues we study have already received attention in the personnel economics literature (e.g. Spilz (1991) confirms empirically that employee wages tend to rise faster than their productivity), others have not received due attention because of the lack of theory. For example, it would be very difficult to study without continuous-time methods the optimal mix of short-term and long-term incentives. Our analysis of optimal dynamic incentives relies on the differentiability properties that continuous time delivers.

While interpreting our results, one has to be conscious of our assumption that the agent cannot save. Generally, our intuitions about the dynamics of the agent's wages and incentives do not appear to depend on this assumption. However, the assumption that the agent cannot save matters if one addresses the practically important issue that the principal's commitment may be limited. A contract cannot have a high retirement point if the principal cannot commit to make high payments to the agent in exchange for zero effort and the agent cannot save. Yet, with savings a high retirement point can be implemented by letting the agent accumulate wealth during employment, and then consume the annuity value of his wealth during retirement. This paper does not explore the degree of the

[22]For example, one could use the dataset like that of Meyersson-Milgrom, Petersen and Asplund (2002) to estimate these properties of actual contracts.

principal's commitment.

We finish by discussing the technical issues connected with the no-savings assumption. If the agent's savings were observable and contractible, then the principal would be able to achieve the same profit as if the agent could not save or borrow. However, if the principal cannot observe the agent's savings, then the contracting problem becomes extremely difficult.[23] The contract proposed in this paper would be vulnerable to many deviations if the agent could save and borrow. The agent would save his income to insure himself against future manipulations by the principal. One way to approach this problem is to add a restriction on the contract that the payments to the agent must induce a martingale marginal utility of consumption. Then the agent would not be able to improve his welfare by deviating only with his effort, or only with his savings. This idea, called the first-order aproach, was taken up in discrete time in Werning (2002), who finds a computational way to check the validity of this approach for specific examples. In general, this approach always provides a useful upper bound on the principal's profit, since the first-order conditions are necessary for incentive compatibility. However, nobody has found a satisfactory set of conditions that guarantee full incentive-compatibility under the first-order approach to dynamic contracts with hidden state variables like savings, although limited progress has been made in this direction. For example, Williams (2003) presents one set of useful sufficient conditions, although they are not comprehensive enough to deal with hidden savings. It is also known that the first-order approach fails in a few reasonable cases, e.g. when the agent has binary effort choice or a linear cost of effort.[24] Kocherlakota (2004) points out that the first order approach is invalid when the agent's cost of effort is linear in the unemployment insurance problem, and develops a number of new elegant ideas to solve the problem.

[23]The problem with hidden savings can be solved for the case when the agent is risk-neutral but has limited liability. For example, see the continuous-time model of DeMarzo and Sannikov (2006).

[24]There is a simple verbal argument to show that the first-order approach fails with binary effort. Under any contract derived using the first order approach, the agent's marginal utility of consumption would be a martingale. When the agent is supposed to put positive effort, the first order condition implies that he is indifferent between positive effort and effort zero, given that he does not alter his consumption pattern. The agent's deviation to effort zero would modify the underlying probability measure, so that with the original consumption pattern his marginal utility of consumption will be a submartingale. Therefore, by saving appropriately the agent can strictly improve his utility. We conclude that under an optimal contract subject to just first order incentive compatibility conditions, the agent always has a profitable deviation, which involves choosing effort zero and increased savings.

7 Appendix A: Proof of Theorem 1.

Here we provide a formal derivation of the optimal contract. We organize the derivation into five steps:

1. Define the agent's continuation value $W_t(C, A)$ for an arbitrary contract (C, A).

2. Using the Martingale Representation Theorem, find a stochastic representation for $W_t(C, A)$.

3. Derive necessary and sufficient conditions for the optimality of the agent's effort in terms the sensitivity of $W_t(C, A)$ to output.

4. Investigate regularity properties of the HJB equation, and prove existence and uniqueness of an appropriate solution to conjecture an optimal contract.

5. Verify that the conjectured contract is indeed optimal using a martingale argument that relies on the properties of the HJB equation.

7.1 The agent's continuation value.

Fix an arbitrary consumption process $C = \{C_t\}$ and an effort strategy $A = \{A_t\}$, which may or may not be optimal for the agent given C. The agent's continuation value, his expected future payoff from (C, A) after time t, is

$$W_t(C, A) = E^A \left[r \int_t^\infty e^{-r(s-t)} (u(C_s) - h(A_s))\, ds \mid \mathcal{F}_t \right], \tag{15}$$

where E^A denotes the expectation under the probability measure $\mathcal{Q}^A$ induced by the agent's strategy A.

7.2 The evolution of $W_t(C, A)$.

If A_t and C_t are completely determined by the path of output $\{X_s, s \in [0, t]\}$ for all $t \geq 0$, i.e. the contract does not use randomization, then the agent's continuation value $W_t(C, A)$ is completely determined by the path of output as well. The motion of $W_t(C, A)$ is described by a stochastic differential equation. Proposition 1 derives the drift and introduces notation for the volatility of $W_t(C, A)$, which describe the law of motion of $W_t(C, A)$. The volatility of $W_t(C, A)$, which comes from its sensitivity towards the output X_t, plays an important role for the agent's incentives.

Proposition 1. Representation of the agent's value as a diffusion process. *There exists a progressively measurable process $Y = \{Y_t, \mathcal{F}_t; 0 \le t < \infty\}$ in $\mathcal{L}^*$ such that*[25]

$$W_t(C,A) = W_0(C,A) + \int_0^t r(W_s(C,A) - u(C_s) + h(A_s))\,ds + \int_0^t rY_s(dX_s - A_s\,ds) \quad (16)$$

for every $t \in [0,\infty)$.

Proof. Note that the agent's *total* expected payoff from (C,A) given the information at time t,

$$V_t = r\int_0^t e^{-rs}(u(C_s) - h(A_s))\,ds + e^{-rt}W_t(C,A), \quad (17)$$

is a $\mathcal{Q}^A$-martingale. Assuming that the filtration $\{\mathcal{F}_t\}$ satisfies the usual conditions, the $\mathcal{Q}^A$-martingale V must have a RCLL modification by Theorem 1.3.13 of Karatzas and Shreve (1991) (from now on KS). Then by the Martingale Representation Theorem (KS, p.182), we get the representation

$$V_t = V_0 + r\int_0^t e^{-rs}Y_s\sigma\,dZ_s^A, \quad 0 \le t < \infty, \quad (18)$$

where

$$Z_t^A = \frac{1}{\sigma}\left(X_t - \int_0^t A_s\,ds\right) \quad (19)$$

is a Brownian motion under $\mathcal{Q}^A$ and the factor $re^{-rt}\sigma$ that multiplies Y_t is just a convenient rescaling. Differentiating (17) and (18) with respect to t we find that

$$dV_t = re^{-rt}Y_t\sigma\,dZ_t^A = re^{-rt}(u(C_t) - h(A_t))\,dt\,\underbrace{-e^{-rt}W_t(C,A)\,dt + e^{-rt}\,dW_t(C,A)}_{d(e^{-rt}W_t(C,A))} \quad (20)$$

$$\Rightarrow \quad dW_t(C,A) = r(W_t(C,A) - u(C_t) + h(A_t))\,dt + rY_t\sigma\,dZ_t^A,$$

which implies (16). QED

The value of $W_t(C,A)$ depends only on the path $\{X_s; 0 \le s \le t\}$ and the processes (C,A) *after* time t, but not the agent's effort *before* time t. Therefore, equation (16) describes how $W_t(C,A)$ is determined by the output path $\{X_s; 0 \le s \le t\}$ regardless of the agent's effort *before* time t.[26] This distinction is crucial for Proposition 2, which derives a necessary and sufficient condition for the optimality of the agent's effort.

[25] A process Y is in $\mathcal{L}^*$ if $E^A\left[\int_0^t Y_s^2\,ds\right] < \infty$ for all $t \in [0,\infty)$.

[26] For simplicity, we suppress the dependence of Y_t on the processes (C,A) from time t onward in equation (16).

7.3 Incentive-compatibility condition.

Proposition 2. The agent's incentives. *For a given strategy A, let Y_t be the process from Proposition 1 that represents $W_t(C, A)$. Then A is optimal if and only if*

$$\forall a \in \mathcal{A}, \quad Y_t A_t - h(A_t) \geq Y_t a - h(a), \quad 0 \leq t < \infty \tag{21}$$

almost everywhere.

PROOF. Consider an arbitrary alternative strategy A^*. Define by

$$\hat{V}_t = r \int_0^t e^{-rs}(u(C_s) - h(A_s^*))\, ds + e^{-rt} W_t(C, A), \tag{22}$$

the time-t expectation of the agent's total payoff if he experienced the cost of effort from the strategy A^* before time t, and plans to follow the strategy A after time t. Let us identify the drift of the process $\hat{V}_t$ under the probability measure $\mathcal{Q}^{A^*}$. We have

$$d\hat{V}_t = re^{-rt}(u(C_t) - h(A_t^*))\, dt \underbrace{-re^{-rt}(u(C_t) - h(A_t^*))\, dt + re^{-rt} Y_t dZ_t^A}_{d(e^{-rt}W_t(C,A)) \text{ by } (20)}$$

$$= \underbrace{re^{-rt}(h(A_t) - h(A_t^*) + Y_t(A_t^* - A_t))}_{\text{drift of } \hat{V}}\, dt + re^{-rt} Y_t \sigma\, dZ_t^{A^*},$$

where the Brownian motions under $\mathcal{Q}^A$ and $\mathcal{Q}^{A^*}$ are related by $\sigma Z_t^A = \sigma Z_t^{A^*} + \int_0^t (A_s^* - A_s) ds$.

If (21) does not hold on a set of positive measure, choose A_t^* that maximizes $Y_t A_t^* - h(A_t^*)$ for all $t \geq 0$. Then the drift of $\hat{V}$ (under $\mathcal{Q}^{A^*}$) is nonnegative and positive on a set of positive measure. Thus, there exists a time $t > 0$ such that

$$E^{A^*}[\hat{V}_t] > \hat{V}_0 = W_0(C, A).$$

Because the agent gets utility $E^{A^*}[\hat{V}_t]$ if he follows A^* until time t and then switches to A, the strategy A is suboptimal.

Suppose (21) holds for the strategy A. Then $\hat{V}_t$ is a $\mathcal{Q}^{A^*}$-supermartingale for any alternative strategy A^*. Moreover, since the process $W(C, A)$ is bounded from below, we can add

$$\hat{V}_\infty = r \int_0^\infty e^{-rs}(u(C_s) - h(A_s^*))\, ds$$

as the last element of the supermartingale $\hat{V}$.[27] Therefore,

$$W_0(C, A) = \hat{V}_0 \geq E^{A^*}\left[\hat{V}_\infty\right] = W_0(C, A^*),$$

so the strategy A is at least as good as any alternative strategy A^*. QED

[27]See Problem 3.16 in Karatzas and Schreve (1991). Note that $\hat{V}_\infty \leq \lim_{t\to\infty} \hat{V}_t$.

From Proposition 2 it follows that the minimal volatility of the agent's continuation value required to induce action $a \in \mathcal{A}$ is given by $r\gamma(a)\sigma$, where $\gamma : \mathcal{A} \to [0, \infty)$ is defined by (4).

7.4 The HJB Equation.

Next, we conjecture an optimal contract using the HJB equation. To show that the HJB equation has an appropriate solution, we start by investigating its regularity properties (Lemmas 1 and 2) in order to prove existence and uniqueness of an appropriate solution (Lemma 3). From that solution, Proposition 3 conjectures an optimal contract.

To ensure regularity, we start with a version of the HJB equation in which the sensitivity parameter Y is bounded from below by γ_0 and consumption is bounded from above by the level $\bar{C}$ such that $u'(\bar{C}) = \gamma_0$.[28] Letting $\gamma_1 = \max_a \gamma(a)$, consider equation

$$F''(W) = \min_{(a,Y)\in\Gamma, c\in[0,\bar{C}]} \underbrace{\frac{F(W) - a + c - F'(W)(W - u(c) + h(a))}{r\sigma^2 Y^2/2}}_{H_{a,Y,c}(W,F(W),F'(W))}, \tag{23}$$

where Γ is the set of pairs $(a, Y) \in \mathcal{A} \times [\gamma_0, \gamma_1]$ that satisfy the incentive constraints (21) of Proposition 2. Note that Γ is a compact set that contains the set of pairs $\{(a, \gamma(a)), a > 0\}$. Lemma 1 proves continuity and concavity properties of the solutions to (23), and Lemma 2 proves monotinicity properties of the phase diagram of solutions.

Lemma 1. *The solutions to the HJB equation (23) exist and are unique and continuous in initial conditions $F(W)$ and $F'(W)$. Moreover, initial conditions with $F''(W) < 0$ result in a concave solution.*

Proof. Functions $H_{a,Y,c}(W, F, F')$ are differentiable in all of its arguments, with uniformly bounded derivatives over all $(a, Y) \in \Gamma$ and $c \in [0, \bar{C}]$. Therefore, the right hand side of (23) is Lipschitz continuous. It follows that the solutions to (23) exist globally and are unique and continuous in initial conditions.[29]

We still need to prove that a solution F with a negative second derivative at one point must be concave everywhere. Let us show that the second derivative of the solution F can never reach 0, and therefore must remain negative. If $F''(W) = 0$, it follows that the entire solution must be a straight line $F(W') = F(W) + F'(W)(W' - W)$ since

$$\frac{F(W) + F'(W)(W' - W) - a + c - F'(W)(W' - u(c) + h(a))}{r\sigma^2 Y^2/2} \tag{24}$$

takes the same value for all W', for all a, Y and c. QED

[28] It can be shown that whenever the tangent to F at W passes above F_0 and $F''(W) < 0$, then the optimal choice of consumption in the HJB equation is less than $\bar{C}$. Thus, the restriction to $c < \bar{C}$ is without loss of generality in the relevant range.

[29] Note that the linear growth conditions hold as well.

Lemma 2. *Consider two solutions F and $\tilde{F}$ of equation (23) that satisfy $F(W) = \tilde{F}(W)$ and $F'(W) < \tilde{F}'(W)$. Then $F'(W') < \tilde{F}'(W')$ for all W'. Therefore, $F(W') < \tilde{F}(W')$ for $W' > W$ and $F(W') > \tilde{F}(W')$ for $W' < W$.*

Proof. Without loss of generality, let us prove the claim for $W' > W$. If the claim is false, let $W' \in (W, \infty)$ be the smallest point for which $F'(W') = \tilde{F}'(W')$. Then $F'(V) < \tilde{F}'(V)$ for $V \in (W, W')$, so $F(W') < \tilde{F}(W')$ and

$$F''(W') \leq H_{a,Y,c}(W', F(W'), F'(W')) < H_{a,Y,c}(W', \tilde{F}(W'), \tilde{F}'(W')) = \tilde{F}''(W')$$

for the triple (a, Y, c) that serves as a minimizer in the HJB equation for function $\tilde{F}$ at point W. Note that $H_{a,Y,c}(W, F, F')$ is increasing in its second argument. It follows that $F'(W' + \varepsilon) < \tilde{F}'(W' + \varepsilon)$ for all sufficiently small ε, a contradiction. QED

The optimal contract is constructed from a specific solution of the HJB equation, which satisfies $F(W) \geq F_0(W)$ for all $W \geq 0$ and boundary conditions

$$F(0) = 0, \;\; F(W_{gp}) = F_0(W_{gp}) \;\; \text{and} \;\; F'(W_{gp}) = F_0'(W_{gp}) \tag{25}$$

for some $W_{gp} \in [0, W_{gp}^*]$. Lemma 3 shows that there is a unique function with these properties.

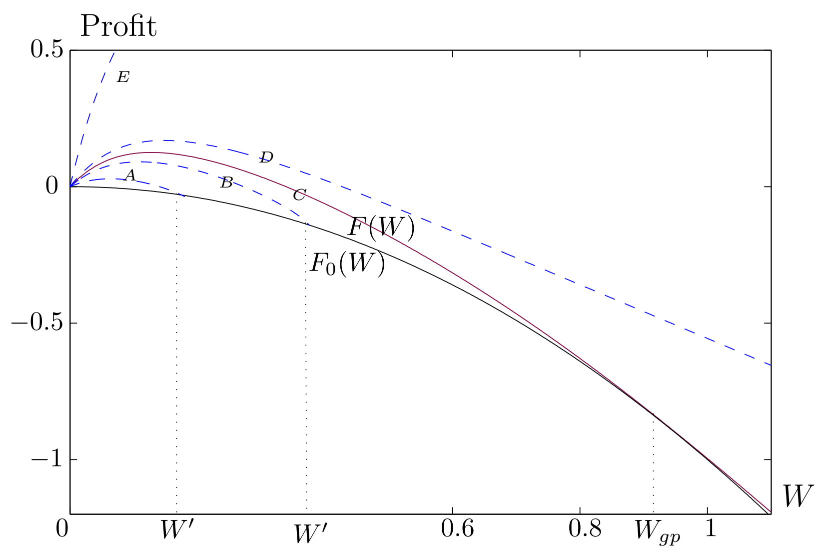

Figure 6: Typical solutions of the HJB equation.

Lemma 3. *There exists a unique function $F \geq F_0$ that solves the HJB equation and satisfies the boundary conditions (25) for some $W_{gp} \in [0, W_{gp}^*]$.*

Proof. In this proof we consider the solutions of equation (23) with $F(0) = F_0(0)$ and various slopes $F'(0) \geq F_0'(0)$. By Lemma 1, all of these solutions are concave and continuous in $F'(0)$. If the solution F with $F'(0) = F_0'(0)$ satisfies $F(W) \geq F_0(W)$ for all $W \geq 0$, then it is the desired solution, and $W_{gp} = 0$.

Otherwise, as we increase $F'(0)$ above $F_0'(0)$, initially the resulting solutions F must reach F_0 at some point $W' \in (0, \infty)$, as shown by curves A and B in Figure 6. By Lemma 2, point W' is moving to the right as $F'(0)$ is increasing. When $F'(0)$ becomes sufficiently large the resulting solution never reaches F_0 on $(0, \infty)$. Indeed, by Lipschitz continuity of (23), if $F'(0)$ is large enough then F grows above the level $\max \mathcal{A}$. Such a solution must grow forever, since if it ever reaches slope 0 at a point W, then $F(W) > \max \mathcal{A}$ together with $F'(W) = 0$ would imply that $F''(W) > 0$, contradicting Lemma 1 (see solution E in Figure 6).

Now, as we increase $F'(0)$ above $F_0'(0)$, point W' stays bounded by W_{gp}^*. Otherwise, if $W' > W_{gp}^*$ then for some $W'' < W'$, $F'(W'') = F_0'(W_{gp}^*) = -1/u'(\bar{C}) = -1/\gamma_0$ and

$$F(W'') + F'(W'')W'' > F_0(W_{gp}^*) + F_0'(W_{gp}^*)W_{gp}^* = -\bar{C} + F'(W'')u(\bar{C}).$$

But then

$$F''(W'') = \frac{F(W'') - a + \bar{C} - F'(W'')(W'' - u(\bar{C}) + h(a))}{r\sigma^2 Y^2/2} > \frac{-a + h(a)/\gamma_0}{r\sigma^2 Y^2/2} \geq 0,$$

a contradiction to Lemma 1.

Since W' cannot escape to infinity, by continuity it follows that there is a largest slope $F'(0) > F_0'(0)$ for which the resulting solution reaches $F_0(0)$ at a point $W_{gp} = W' \leq W_{gp}^*$ (solution C in Figure 6). Because solutions with a larger slope at 0 would never reach F_0 on $(0, \infty)$, it follows that $F(W) \geq F_0(W)$ for all $W \in [0, \infty)$, and thus F is tangent to F_0 at W_{gp}.

Thus, we have constructed a function $F \geq F_0$ that satisfies conditions (25) for some $W_{gp} \geq 0$. To see that this function is unique, note that any solution with a larger slope at 0 would be strictly greater than F_0 on $(0, \infty)$. QED

Proposition 3 conjectures an optimal contract from the solution of equation (23) we have just constructed (see solution C in Figure 6).

Proposition 3. *Consider the unique solution $F(W) \geq F_0(W)$ of equation (23) that satisfies conditions (25) for some $W_{gp} \in [0, W_{gp}^*]$. Let $a : [0, W_{gp}] \to \mathcal{A}$, $Y : [0, W_{gp}] \to [\gamma_0, \gamma_1]$ and $c : [0, W_{gp}] \to [0, \bar{C}]$ be the minimizers in (23).*

For any starting condition $W_0 \in [0, W_{gp}]$ there is a unique in the sense of probability law weak solution to equation

$$dW_t = r\left(W_t - u(c(W_t)) + h(a(W_t))\right)\, dt + rY(W_t)\, \underbrace{(dX_t - a(W_t)Y(W_t)\, dt)}_{\sigma dZ_t^{a(W)}} \tag{26}$$

until the time $\tau = \inf\{t : W_t = W_L \text{ or } W_H\}$. *The contract* (C, A) *defined by*

$$\begin{array}{ll} C_t = c(W_t), \text{ and } A_t = a(W_t), & \text{for } t \in [0, \tau) \\ C_t = -F_0(W_\tau), \text{ and } A_t = 0, & \text{for } t \geq \tau \end{array} \tag{27}$$

is incentive-compatible, and it has value W_0 *to the agent and profit* $F(W_0)$ *to the principal.*

Proof. There is a weak solution of (26) unique in the sense of probability law by Theorem 5.5.15 from KS because the drift and the volatility of W are bounded on $[0, W_{gp}]$, and the volatility is bounded above 0 by $r\gamma_0\sigma$. Define $W_t = W_\tau$ for $t > \tau$, and let us show that $W_t = W_t(C, A)$, where $W_t(C, A)$ is the agent's true continuation value in the contract (C, A).

From the representation of $W_t(C, A)$ in Proposition 1, we have[30]

$$d(W_t(C, A) - W_t) = r(W_t(C, A) - W_t)\, dt + r(Y_t - Y(W_t))\sigma\, dZ_t^A \quad \Rightarrow$$

$$E_t[W_{t+s}(C, A) - W_{t+s}] = e^{rs}(W_t(C, A) - W_t).$$

Note that $E_t[W_{t+s}(C, A) - W_{t+s}]$ must remain bounded, because both W (by 0 and W_{gp}) and $W(C, A)$ (since C_t is bounded) are bounded. We conclude that $W_t = W_t(C, A)$ for all $t \geq 0$, and in particular, the agent gets value $W_0 = W_0(C, A)$ from the entire contract. Also, the contract (C, A) is incentive compatible, since $(A_t, Y_t) \in \Gamma$ for all t.

To see that the principal gets profit $F(W_0)$, consider

$$G_t = r \int_0^t e^{-rs}(A_s - C_s)\, ds + e^{-rt} F(W_t)$$

By Ito's lemma, the drift of G_t is

$$re^{-rt} \left((A_t - C_t - F(W_t)) + F'(W_t)(W_t - u(C_t) + h(A_t)) + r\sigma^2 Y_t^2 \frac{F''(W_t)}{2} \right).$$

The value of this expression is 0 before time τ by the HJB equation. Therefore, G_t is a bounded martingale until τ and the principal's profit from the entire contract is

$$E\left[r \int_0^\tau e^{-rs}(A_s - C_s)\, ds + e^{-r\tau} F_0(W_\tau) \right] = E[G_\tau] = G_0 = F(W_0),$$

since $F(W_\tau) = F_0(W_\tau)$. QED

[30]When one constructs a weak solution W_t, one may need a filtration bigger than that generated by underlying the Brownian motion. Then, while generally the Martingale Representation Theorem (used in Proposition 1 to represent $W_t(C, A)$) may fail, it holds on the minimal filtration that contains the Brownian motion and the solution if the solution is unique in the sense of probability law (see Jacod and Yor (1977)).

7.5 Verification.

Our last step is to verify that the contract presented in Proposition 3 is optimal. We start with a lemma that bounds from above the principal's profit from contracts that give the agent a value higher than W_{gp}^*.

Lemma 4. *The profit from any contract (C, A) with the agent's value $W_0 \geq W_{gp}^*$ is at most $F_0(W_0)$.*

Proof. Define c by $u(c) = W_0$. Then $W_0 \geq W_{gp}^* \;\Rightarrow\; u'(c) \leq \gamma_0$. We have

$$W_0 = E\left[r\int_0^\infty e^{-rt}(u(C_t) - h(A_t))\,dt\right] \leq E\left[r\int_0^\infty e^{-rt}(u(c) + (C_t - c)u'(c) - \gamma_0 A_t)\,dt\right] \leq$$

$$u(c) - u'(c)\left(E\left[r\int_0^\infty e^{-rt}(A_t - C_t)\,dt\right] + c\right),$$

where $u(c) = W_0$ and $c = -F_0(W)$. It follows that the profit from this contract is at most $F_0(W)$. QED

Now, note that function F from which the contract is constructed satisfies

$$\min_{W' \in [0, u(\infty))} F(W) - F_0(W') - F'(W)(W - W') = \min_{c \in [0,\infty)} F(W) + c - F'(W)(W + u(c)) \geq 0 \tag{28}$$

for all $W \geq 0$. For any such solution, the optimizers in the HJB equation satisfy $a(W) > 0$ and $c(W) < \bar{C}$. If either of these conditions failed, (28) would imply that $F''(W) \geq 0$. Also, we have $Y(W) = \gamma(a(W))$.

Proposition 4. *Consider a concave solution F of the HJB equation that satisfies (28). Any incentive-compatible contract (C, A) achieves profit at most $F(W_0(C, A))$.*

Proof. Denote the agent's continuation value by $W_t = W_t(C, A)$, which is represented by (16) using the process Y_t. By Lemma 4, the profit is at most $F_0(W_0) \leq F(W_0)$ if $W_0 \geq W_{gp}^*$. If $W_0 \in [0, W_{gp}^*]$, define

$$G_t = r\int_0^t e^{-rs}(A_s - C_s)\,ds + e^{-rt}F(W_t)$$

as in Proposition 3. By Ito's lemma, the drift of G_t is

$$re^{-rt}\left((A_t - C_t - F(W_t)) + F'(W_t)(W_t - u(C_t) + h(A_t)) + r\sigma^2 Y_t^2 \frac{F''(W_t)}{2}\right).$$

Let us show the drift of G_t is always non-positive. If $A_t > 0$ then Proposition 2 and the definition of γ imply that $Y_t \geq \gamma(A_t)$. Then equation (23) together with $F''(W_t) \leq 0$ imply that the drift of G is non-positive. If $A_t = 0$, then $F''(W_t) < 0$ (28) imply that the drift of G_t is non-positive.

It follows that G_t is a bounded supermartingale until the stopping time τ' (possibly ∞) when W_t reaches W^*_{gp}. At time τ' the principal's future profit is less than or equal to $F_0(W^*_{gp}) \leq F(W^*_{gp})$ by Lemma 4. Therefore, the principal's expected profit at time 0 is less than or equal to

$$E^A\left[\int_0^{\tau'} e^{-rt}(dX_t - C_t\,dt) + e^{-r\tau'}F(W_{\tau'})\right] = E^A[G_{\tau'}] \leq G_0 = F(W_0).$$

QED

Remark. A concave function F that satisfies (28) is still an upper bound on the principal's profit even if we allowed randomization. With this possibility, the representation for W would involve an extra term, a martingale orthogonal to the Brownian motion. One can easily adjust the arguments of Proposition 4 to accommodate this extra term.

8 Appendix B: Additional Contractual Possibilities.

Proof of Theorem 3. First, since $F \geq \tilde{F}_0 \geq F_0$ is a concave function, it follows that (28) holds for all $W \in [\tilde{W}, \infty)$. Second, since $\tilde{F}_0(W) = F_0(W)$ for sufficiently large W, there exists a point $\tilde{W}^*_{gp} \geq W^*_{gp}$ such that $\tilde{F}_0(W) = F_0(W) \geq F_0(W) + (W - W')F_0'(W) - \tilde{F}_0(W')$ for all $W \geq \tilde{W}^*_{gp}$ and $W' \geq \tilde{W}$. Then by an argument similar to Lemma 4 it follows that the profit from any contract with value $W \geq \tilde{W}^*_{gp}$ is at most $F_0(W)$.

Now, consider an arbitrary incentive-compatible contract (C, A) with value to the agent $W_0 < \tilde{W}^*_{gp}$, in which the principal gives the agent value W_τ at profit $\tilde{F}_0(W_\tau)$ at time τ. Then the agent's continuation value follows

$$dW_t = r(W_t - u(C_t) + h(A_t))\,dt + rY_t\sigma\,dZ^A_t,$$

for some $Y_t \geq \gamma(A_t)$ until time τ. Denote by τ' the time when the process W_t hits value $\tilde{W}^*_{gp}$ for the first time. With this process W_t, as shown in the proof of Proposition 4, the process

$$G_t = r\int_0^t e^{-rs}\,(dX_s - C_s\,ds) + e^{-rt}F(W_t)$$

is a supermartingale until time $\min(\tau, \tau')$, and a martingale if until that time

$$W_t \in [W_L, W_H],\ A_t = a(W_t),\ C_t = c(W_t) \text{ and } Y_t = \gamma(A_t).$$

Thus, the principal's profit from this contract is less than or equal to

$$E[G_\tau + 1_{\tau\leq\tau'}e^{-r\tau}(\tilde{F}_0(W_\tau) - F(W_\tau)) + 1_{\tau'<\tau}e^{-r\tau'}(F_0(\tilde{W}^*_{gp}) - F(W_\tau))] \leq G_0 = F(W_0),$$

with equalities everywhere if G is a martingale, $W_\tau = W_L$ or W_H and $\tau \leq \tau'$ almost surely.

We conclude that $F(W_0)$ is an upper bound on the profit from any contract with any value $W_0 \in [\tilde{W}, \infty)$. For $W_0 \in [W_L, W_H]$ this upper bound is achieved by a contract of the form outlined in the statement of Theorem 3. QED

Lemma 5. *For any $W_0 \in [\tilde{W}, \infty)$, if there is no concave solution $F \geq \tilde{F}_0$ of the HJB equation that reaches $\tilde{F}_0$ at $W_L < W_0$ and $W_H > W_0$, then the best way to deliver to the agent value W_0 is to randomize among the contracts of from $\tilde{F}_0$.*

Sketch of Proof. Consider solutions to the HJB equation F that satisfy $F(W_0) = f \geq \tilde{F}_0(W_0)$ and $F'(W_0) = f'$. Of these solutions, let us replace those that are convex by a straight line through (W_0, f) with slope f'. Then functions F (concave solutions or straight lines) change continuously with f and f'. For any value f, the set of slopes f' for which F reaches $\tilde{F}_0$ at a point $W > W_0$ is a half-line $(-\infty, f'_H(f)]$, where $f'_H(f)$ is a continuous function decreasing function. Function $f'_L(f)$ defined similarly for $W < W_0$ is continuous and increasing.

If $f'_H(\tilde{F}_0(W_0)) \leq f'_L(\tilde{F}_0(W_0))$, then function F that corresponds to level $\tilde{F}_0(W_0)$ and slope $(f'_H(\tilde{F}_0(W_0)) + f'_L(\tilde{F}_0(W_0)))/2$ stays weakly above $\tilde{F}_0$, and it can be shown to be an upper bound on the principal's profit for all W by an argument similar to the proof of Theorem 3. Then the best contract with value W_0 gives profit $\tilde{F}_0(W_0)$ to the principal.

Otherwise, there is a value $f > \tilde{F}_0(W_0)$ such that $f'_H(f) = f'_L(f)$. Then the function F that corresponds to level f and slope $f'_H(f) = f'_L(f)$ stays weakly above $\tilde{F}_0$, so it is an upper bound on the principal's profit for all W. Moreover, by continuity F reaches $\tilde{F}_0$ at some points $W_H > W_0$ and $W_L < W_0$. If F is a straight line, then the best contract with value W_0 involves randomization between two contracts from $\tilde{F}_0$. If F is concave, then it satisfies the conditions of Lemma 5 (and Theorem 3). QED

Proof of Theorem 4. (a) According to Lemma 2, we have $\tilde{F}_1'(w) < \tilde{F}_2'(w)$ for all w. Then the HJB equation implies that $\tilde{c}_1(w) \leq \tilde{c}_2(w)$, with equality only when $\tilde{c}_1(w) = \tilde{c}_2(w) = 0$.

(b) Since profit from $\tilde{F}_1 < \tilde{F}_2$ are concave functions, it follows that at any two points w and w' with $\tilde{F}_1'(w) = \tilde{F}_2'(w')$, the line tangent to $\tilde{F}_2$ at w' should be above (and parallel to) the line tangent to $\tilde{F}_1$ at w. This amounts to

$$\tilde{F}_1(w) + \tilde{F}_2'(w)w < \tilde{F}_2(w') + \tilde{F}_2'(w')w'.$$

Because $\tilde{F}_1'(w) = \tilde{F}_2'(w')$ the HJB equation implies that $\tilde{F}_1''(w) < \tilde{F}_2''(w)$, so the cost of exposing the agent to risk is greater for $\tilde{F}_1$ than $\tilde{F}_2$. Then, (10) implies that $\tilde{a}_1(w) \leq \tilde{a}_2(w')$. Otherwise, we would have $\gamma(\tilde{a}_1(w))^2 > \gamma(\tilde{a}_2(w'))^2$ and

$$\tilde{a}_1(w)+h(\tilde{a}_1(w))\tilde{F}_1'(w)+r\sigma^2\frac{\gamma(\tilde{a}_1(w))^2}{2}\tilde{F}_1''(w) \geq \tilde{a}_2(w')+h(\tilde{a}_2(w'))\tilde{F}_1'(w)+r\sigma^2\frac{\gamma(\tilde{a}_2(w'))^2}{2}\tilde{F}_1''(w)$$

would imply that

$$\tilde{a}_1(w)+h(\tilde{a}_1(w))\tilde{F}_2'(w')+r\sigma^2\frac{\gamma(\tilde{a}_1(w))^2}{2}\tilde{F}_2''(w') > \tilde{a}_2(w')+h(\tilde{a}_2(w'))\tilde{F}_2'(w')+r\sigma^2\frac{\gamma(\tilde{a}_2(w'))^2}{2}\tilde{F}_2''(w'),$$

a contradiction.

Regarding volatilities, the volatility of W_t in contract i is $r\gamma(\tilde{a}_i(W_t))\sigma$. By Ito's Lemma, the volatility of $\tilde{F}_i'(W_t) = 1/u'(\tilde{c}_i(W_t))$ is $r\gamma(\tilde{a}_i(W_t))\sigma\tilde{F}_i''(W_t)$, and so the volatility of $\tilde{c}_i(W_t)$ is

$$r\gamma(\tilde{a}_i(W_t))\sigma\tilde{F}_i''(W_t)\,\frac{dc}{d1/u'(c)}\bigg|_{c=\tilde{c}_i(W_t)} = r\gamma(\tilde{a}_i(W_t))\sigma\tilde{F}_i''(W_t)u''(\tilde{c}_i(W_t)).$$

The ratio of the volatility of $\tilde{c}_i(W_t)$ to the volatility of W_t is

$$\tilde{F}_i''(W_t)u''(\tilde{c}_i(W_t)).$$

Now, since $\tilde{F}_1''(w) < \tilde{F}_2''(w')$ and $\tilde{c}_1(w) = \tilde{c}_2(w')$, we have

$$\tilde{F}_1''(w)u''(\tilde{c}_1(w)) > \tilde{F}_2''(w')u''(\tilde{c}_2(w')),$$

i.e. contract 1 relies more on short-term incentives. QED

9 Appendix C: The proof of Theorem 5.

Sketch of Proof. We will go loosely through the argument behind the proof, to spare the reader of long precise calculations. For a constant $k \approx 1$ that does not depend on r, consider the function

$$\tilde{F}(W) = \bar{F}(W) + kr\gamma(\bar{a}(W))^2\sigma^2\frac{\bar{F}''(W)}{2}.$$

Let us argue that for all sufficiently small r, $\tilde{F}(W)$ is an upper bound on the principal's profit from the optimal contract when $k < 1$ and a lower bound, when $k > 1$.

First, we claim that

$$\max_{a,c}\; r(a-c)+\tilde{F}'(W)r(W-u(c)+h(a))+\frac{\tilde{F}''(W)}{2}r^2\sigma^2\gamma(a)^2 = r\bar{F}(W)+r^2\sigma^2\gamma(a)^2\frac{\bar{F}''(W)}{2}+O(r^3) \tag{29}$$

Ignoring the terms of the order of r^2 the problem (29) becomes

$$\max_{a,c}\; r(a-c) + \bar{F}'(W)r(W - u(c) + h(a)).$$

This problem has solution $c = \bar{c}(W)$ and $a = \bar{a}(W)$ and value $r\bar{F}(W)$. Therefore, problem (29) has the solution $c = \bar{c}(W) + O(r)$ and $a = \bar{a}(W) + O(r)$, and value

$$r(\bar{a}(W) - \bar{c}(W)) + \tilde{F}(W)'r\underbrace{(W - u(\bar{c}(W)) + h(\bar{a}(W)))}_{0} + \frac{\tilde{F}''(W)}{2}r^2\sigma^2\gamma(\bar{a}(W))^2 + O(r^3) =$$

$$r\bar{F}(W) + r^2\sigma^2\gamma(\bar{a}(W))^2\frac{\bar{F}''(W)}{2} + O(r^3).$$

Now, for any $k > 1$,

$$r\tilde{F}(W) < r(\bar{a}(W)-\bar{c}(W))+\tilde{F}(W)'r(W-u(\bar{c}(W))+h(\bar{a}(W))+\frac{\tilde{F}''(W)}{2}r^2\sigma^2\gamma(\bar{a}(W))^2 \quad (30)$$

for all sufficiently small $r > 0$. A contract that has value $W_0 \in (0, W^*_{gp})$ for the agent and profit of at least $\tilde{F}(W_0)$ for the principal can be constructed by solving

$$dW_t = r\underbrace{(W_t - u(C_t) + h(A_t))}_{0}\,dt + r\gamma(A_t)\,(dX_t - A_t\,dt)$$

and letting $C_t = \bar{c}(W_t)$ and $A_t = \bar{a}(W_t)$, until the stopping time τ when W_t hits 0 or W^*_{gp}. After time τ, let $A_t = 0$ and $C_t = u^{-1}(W_\tau)$. Then the process

$$G_t = r\int_0^t e^{-rs}(A_s - C_s)\,ds + e^{-rt}\tilde{F}(W_t)$$

is a submartingale by (30), so the principal's profit under this contract is

$$E[G_\tau + e^{-r\tau}(F_0(W_\tau) - \tilde{F}(W_\tau))] > G_0 + O(E[e^{-r\tau}]) > \tilde{F}(W_0) + O(E[e^{-r\tau}]),$$

where $O(E[e^{-r\tau}])$ decays exponentially fast as $r \to 0$, since the volatility of W_t is of the order of r and the drift of W_t is 0.

For any $k < 1$,

$$r\tilde{F}(W) < \max_{a>0,c} r(a - c) + \tilde{F}(W)'r(W - u(c) + h(a)) + \frac{\tilde{F}''(W)}{2}r^2\sigma^2\gamma(a)^2. \quad (31)$$

By modifying $\tilde{F}$ slightly near W^*_{gp}, we can create a function that satisfies (31) and also

$$\min_{c\in[0,\infty)} \tilde{F}(W) + c - \tilde{F}'(W)(W + u(c)) \geq 0.$$

This function is an upper bound on the principal's profit, since for any incentive-compatible contract

$$G_t = r\int_0^t e^{-rs}(A_s - C_s)\,ds + e^{-rt}\tilde{F}(W_t)$$

is a supermartingale.

Our tight bounds on the principal's profit imply that as $r \to 0$, $W_{gp} \to W^*_{gp}$ and $F'(W) \to \bar{F}(W)$ pointwise on $(0, W_{gp})$, so the agent's consumption converges to $\bar{c}(W)$. From equation (6), the agent's effort converges to $\bar{a}(W)$. QED

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