Hey,
What’s the Big Idea?

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Six Anecdotes in Search of an Author

- Trevor Hawkes, group theory, twenty-four situations
- David Tall, complex analysis, two boards
- Jim Eells, topics in mathematical physics, actually about the birth of methods from problems
- Anil Nerode, measure theory, juggling two non-equivalent definitions at the same time
- John Mason, calculus, ‘technique bashing’
- Me, linear algebra, ten years on, assent, assert

Christopher Zeeman on the value of (generic) examples

I look for an example that captures the quintessence of a whole branch of mathematics, that you can constantly refer back to fruitfully as you go deeper into that subject. Each example should naturally generate a few theorems around itself to prove its key properties. But even before you do this, it should be sufficiently intriguing to capture the attention.

Richard Courant on David Hilbert's consciously used problem-solving practice

If you want to solve a problem, first strip the problem of everything that is not essential. Simplify it, specialize it as much as you can without sacrificing its core. Thus it becomes simple, as simple as it can be made, without losing any of its punch, and then you solve it. The generalization is a triviality which you don't have to pay much attention to.

First question

If we believed this, and acted in its spirit, how could/would/might we teach differently?

All mathematical pedagogy even if scarcely coherent, rests on a philosophy of mathematics.

(*René Thom*)

One ‘pedagogic’ principle about methods and techniques

See one (*n*),

do one (*m*),

teach one (*p*).

Cultures of Generality and their Associated Pedagogies

- Arithmetic, geometric and algebraic
- How is generality of claim (of method) gestured at, otherwise indicated, worked on or alluded to in writing or in speech? What do **you** say?
- E.g. Egyptian papyrus ("Do the same thing in any example like this.")
- E.g. Babylonian tablet ("1 is the square of 1")
- E.g. Euclid's (generic) proof of the infinity of primes (exceptionally, nothing!)

Prime numbers are more than any assigned number of prime numbers

Let A, B, C be the assigned prime numbers. I say that there are more prime numbers than A, B, C. For let the least number measured by A, B, C be taken and let it be DE; let the unit DF be added to DE. Then EF is either prime or not. First let it be prime; then the prime numbers A, B, C, EF have been found which are more than A, B, C.

Next, let EF not be prime; therefore it is measured by some prime number. Let it be measured by the prime number G. I say that G is not the same with any of the numbers A, B, C. For, if possible, let it be so. Now A, B, C measure DE. Therefore G will also measure DE. But it also measures EF. Therefore G, being a number, will measure the remainder, the unit DF: which is absurd. Therefore G is not the same with any one of the numbers A, B, C. And by hypothesis it is prime. Therefore the prime numbers A, B, C, G have been found …

On examples and counterexamples

Questioning examples

- What are some of the different purposes that examples can serve in mathematics?
- What do students get from seeing an example worked in front of them? Where and to what do they attend and why?
- What do they think they are doing if they do not have a sense of what the ‘example’ is an example of? What can different examples be examples of?
- How can we know if a student is seeing a situation or example generically?

Edwina Michener on varied teacher purposes for examples

- *Start-up examples* – from which basic problems, definitions and fundamental results can be conjectured or motivated (e.g. circles curve – small ones more than big ones – and lines don’t, in differential geometry);
- *Reference* (or touchstone) *examples* – standard cases that are widely applicable, linked to several concepts and results (e.g. in linear algebra, the sixteen 2 x 2 matrices with entries 0 or 1);

Michener continued

- *Model examples* – paradigmatic or generic cases that summarise expectations and default assumptions (e.g. for cubics, $y = x(x2)$, $y = x(x2–1)$ and $y = x(x2+1)$) – w.l.o.g.;
- *Counterexamples* – sharpen distinctions between concepts and demonstrate non-universality of certain results (where to look?) e.g. Cantor's ternary set is a counterexample to the claim that only countable sets can have measure zero.
  e.g. $2 = (1–i)(1+i)$ is a counterexample to the claim that a prime in Z must be prime in Z[i].

(Big? Good?) question about example spaces and learner-generated examples

How might we find out about students' example spaces and how to engage students more actively in example generation?

Some of MacHale's 'theorems' about counterexamples

- The function $f(x) = \text{mod}(x)$ is the *only* real function that is continuous but not differentiable.
- The interval [0,1] is the *only* uncountable set.
- The function f defined on [0,1] which equals
  0 when x is rational and 1 when x is irrational
  is the *only* function which is not Riemann integrable.
- $\sqrt{2}$ is the *only* irrational number.

On definitions and defining

Imre Lakatos' 'Teacher' observes:

"Definitions are frequently proposed and argued about when counterexamples emerge."

David Fowler

In contemporary mathematics, more and more power is being packed into the definitions and the proofs are becoming relative trivialities. How are students to contend with this?

Does it not serve to render students more passive in the face of mathematics and for mathematics to seem ever more authoritarian?

If I know the number of sides of a polygon, can I work out the number of diagonals it must have?

Diagonals of some regular polygons

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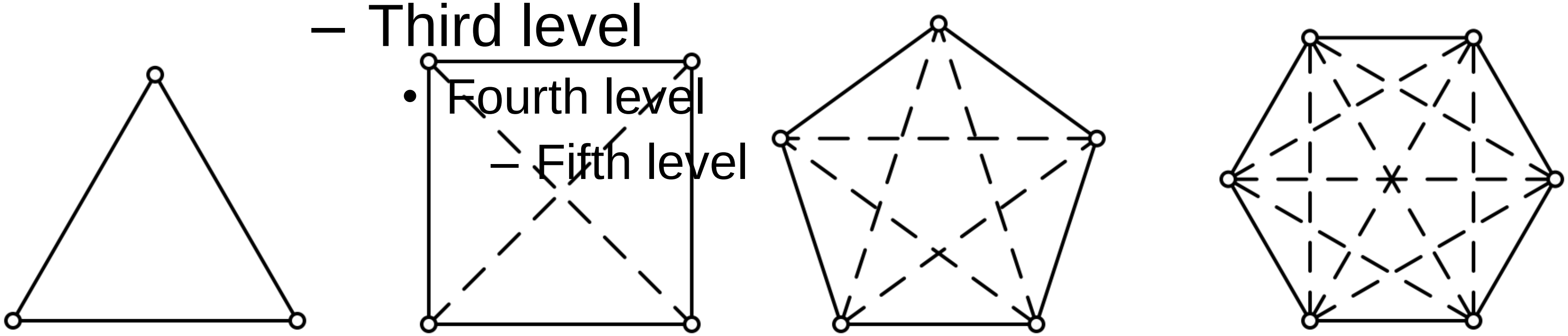

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Some diagonals of some irregular polygons

- Click to edit Master text styles
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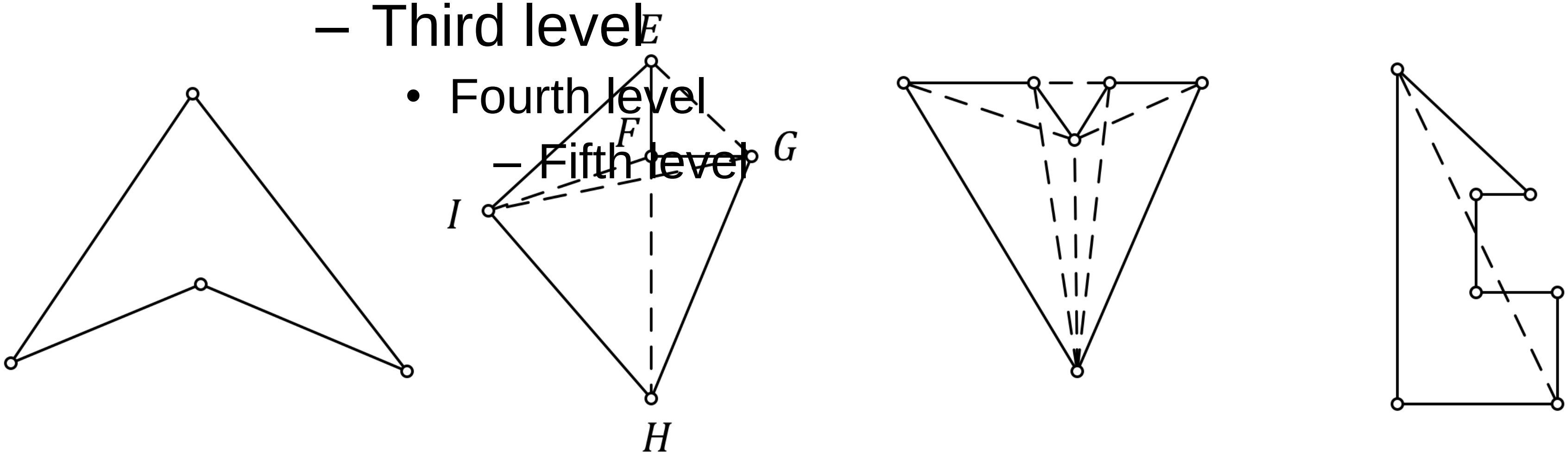


Tall and Vinner

Concept *image* versus concept *definition*

And later Leo Corry on the body and the image of mathematics.

Which brings us back to example spaces once again.

A Teacher and a student

What is the definition of a straight line?

Monge’s criteria for a good definition

- Notions used in the definitions should not be more complicated than that being defined.
- A defining property needs to be found (an ‘if an only if’), but it should be the simplest and easiest one to conceive.
- Over and above this, it should provide an image.

Molland on two types of definitions

Definition by **genesis**

versus

definition by **property**

Example of conics and their symptoms and naming by symptom

Frege

Never let us take a description of the origin of an idea for a definition.

Frege (to Hilbert) on Definitions

There is already widespread confusion in mathematics with regard to definitions in mathematics and some seem to act according to the rule

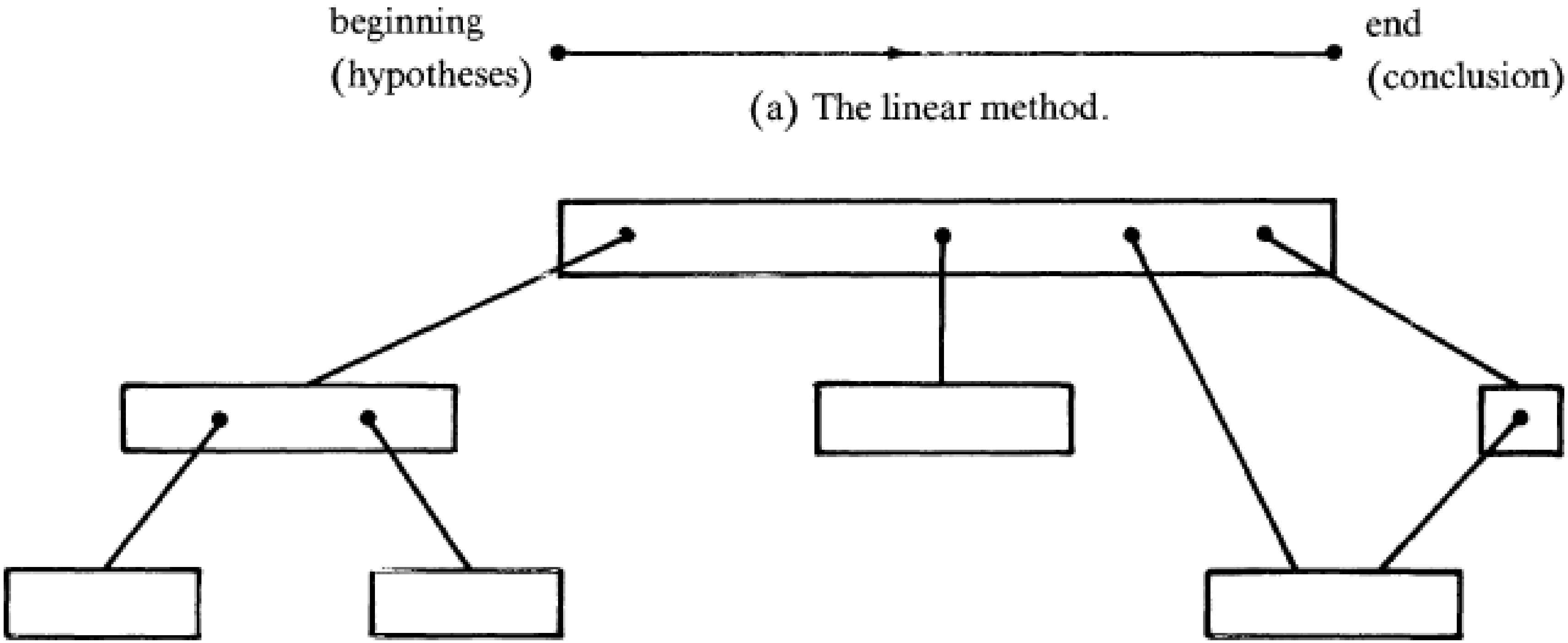
If you can’t prove a proposition,
Then treat it as a definition.

I think it is about time that we came to an understanding about what a definition is supposed to be and do, and accordingly about the principles to be followed in defining a term.

Question on proof and proving

- How are students to come to value theorems if that is all they see all day? Rarity versus commonplace?
- Yuri Manin asserts, “A good proof is one that makes us wiser”. How can we communicate that wisdom to students?

Leron’s two images



(a) The linear method.

(b) The structural method.

FIG. 1. The two methods of presentation.

Final question, final slide

So what's the big idea?

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A Few References

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