Random Processes and Systems

GCOE conference

Organizer: T. Kumagai (Kyoto)

DATES: February 16 (Mon.) – 19 (Thur.), 2009

PLACE: Room 110, Bldg. No 3, Department of Math., Kyoto University

Program

February 16 (Mon.)

10:00–10:50	G. Grimmett (Cambridge)
	The quantum Ising model via stochastic geometry
10:50–11:00	Tea Break
11:00–11:50	T. Hara (Kyushu)
	Q-lattice animals: a model which interpolates lattice animals and percolation
13:20–13:50	B. Graham (UBC)
	Influence and statistical mechanics
14:00–14:30	R. Fukushima (Kyoto)
	Brownian survival and Lifshitz tail in perturbed lattice disorder
14:30–14:50	Tea Break
14:50–15:40	R.W. van der Hofstad (TUE)
	Incipient infinite clusters in high-dimensions
16:00–16:50	G. Slade (UBC)
	Random walks and critical percolation

February 17 (Tue.)

10:00–10:50	A.S. Sznitman (ETH)
	Disconnection of discrete cylinders and random interlacements
10:50–11:00	Tea Break
11:00–11:50	D. Croydon (Warwick)
	Random walks on random paths and trees
13:20–13:50	J. Goodman (UBC)
	Exponential growth of ponds in invasion percolation
14:00–14:50	M.T. Barlow (UBC)
	Convergence of random walks to fractional kinetic motion
14:50–15:10	Tea Break
15:10–16:00	A. Nachmias (Microsoft)
	The Alexander-Orbach conjecture holds in high dimensions
16:10–17:00	G. Kozma (Weizmann)
	Arm exponents for high-d percolation
18:30–20:30	Conference Dinner at Ganko Takasegawa Nijoen

February 18 (Wed.)

10:00–10:50	T. Funaki (Tokyo)
	Scaling limits for weakly pinned random walks with two large deviation minimizers
10:50–11:00	Tea Break
11:00–11:50	H. Tasaki (Gakushuin)
	Extended thermodynamic relation and entropy for nonequilibrium steady states
13:20–13:50	M. Sasada (Tokyo)
	Hydrodynamic limit for two-species exclusion processes with one conserved quantity
14:00–14:30	H. Sakagawa (Keio)
	Behavior of the massless Gaussian field interacting with the wall
14:40–15:10	S. Bhamidi (UBC)
	Gibbs measures on combinatorial structures
15:10–15:30	Tea Break
15:30–16:20	N. Yoshida (Kyoto)
	Phase transitions for linear stochastic evolutions
16:30–17:20	S. Olla (Paris)
	From Hamiltonian dynamics to heat equation: a weak coupling approach

February 19 (Thur.)

10:00–10:50	D. Brydges (UBC)
	Self-avoiding walk and the renormalisation group
10:50–11:00	Tea Break
11:00–11:50	A. Sakai (Hokkaido)
	Critical behavior and limit theorems for long-range oriented percolation in high dimensions
13:20–14:10	H. Tanemura (Chiba)
	Non-equilibrium dynamics of determinantal processes with infinite particles
14:20–15:10	H. Osada (Kyushu)
	Interacting Brownian motions with $2D$ Coulomb potentials
15:10–15:20	Tea Break
15:20–16:10	H. Spohn (TU München)
	Kinetic limit of the weakly nonlinear Schrödinger equation with random initial data

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HI*: Head Investigator

Abstracts

THE QUANTUM ISING MODEL VIA STOCHASTIC GEOMETRY

GEOFFREY GRIMMETT

We report on joint work with Jakob Björnberg, presented in [5], concerning the phase transition of the quantum Ising model with transverse field on the d-dimensional hypercubic lattice. The basic results are as follows. There is a sharp transition at which the two-point functions are singular. There is a bound on the associated critical exponent that is presumably saturated in high dimensions. These results hold also in the ground-state. The value of the ground-state critical point may be calculated rigorously in one dimension, and the transition is then continuous.

The arguments used are largely geometrical in nature. Geometric methods have been very useful in the study of d-dimensional lattice models in classical statistical mechanics. In contrast, graphical methods for *quantum* lattice models have received less attention. In the case of the quantum Ising model, one may formulate corresponding 'continuum' Ising and random-cluster models in $d + 1$ dimensions (see [3, 4, 7] and the references therein). The continuum Ising model thus constructed may be studied via a version of the random-current representation of [1, 2]. We call this the 'random-parity representation', and we use it to prove the sharpness of the phase transition for this model in a general number of dimensions.

The random-parity representation allows the proof of a family of differential inequalities for the magnetization, viewed as a function of the underlying parameters. A key step is the formulation and proof of the so-called switching lemma. Switching lemmas have been proved and used elsewhere: in [1, 2] for the classical Ising model, and recently in [6, 8] for the quantum Ising model.

The following theorems for the continuum Ising model are examples of the main results. The magnetization is written as M, with ρ the underlying parameter, and ρ_c the critical point.

Theorem 0.1. Let $u, v \in \mathbb{Z}^d$ where $d \geq 1$, and $s, t \in \mathbb{R}$.

(i) if $0 < \rho < \rho_c$, the two-point correlation function $\langle \sigma_{(u,s)} \sigma_{(v,t)} \rangle$ of the Ising model on $\mathbb{Z}^d \times \mathbb{R}$ decays exponentially to 0 as $|u-v| + |s-t| \to \infty$,

(ii) if $\rho \geq \rho_c$, $\langle \sigma_{(u,s)} \sigma_{(v,t)} \rangle \geq M(\rho)^2 > 0$.

Theorem 0.2. In the notation of Theorem 0.1, there exists $c = c(d) > 0$ such that
$$M(\rho) \geq c(\rho - \rho_c)^{1/2} \qquad \text{for } \rho > \rho_c.$$

It is a corollary that the quantum two-point functions decay exponentially to 0 when $\rho < \rho_c$, and are bounded below by $M(\rho)^2$ when $\rho > \rho_c$.

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Statistical Laboratory, Centre for Mathematical Sciences, University of Cambridge, Wilberforce Road, Cambridge CB3 0WB, U.K.

E-mail address: g.r.grimmett@statslab.cam.ac.uk

URL: http://www.statslab.cam.ac.uk/~grg/

q-lattice animals: a model which interpolates percolation and lattice animals[1]

Takashi Hara[2]

In this talk, we introduce a variant of lattice animals, tentatively called q-lattice animals, which interpolate ordinary lattice animals and percolation. Basic properties of the model, such as the existence of critical behavior, as well as correlation inequalities which resemble van den Berg-Kesten inequality in percolation, are derived. We also show, using lace expansion, that the model exhibits mean-field like critical behavior (whose critical exponents are identical with those of lattice animals) in dimensions greater than eight (if the model is sufficiently spread out). Our analysis suggests that the upper critical dimension of the model is eight.

0. Backgrounds.

Critical behavior of certain stochastic geometric systems, such as percolation and lattice animals are now rather well understood in high dimensions. In particular, there is a dimension, called *upper critical dimension*, d_c, above which these models exhibit mean-field like critical behavior.

For percolation, mean-field values for the critical exponents are $\gamma = 1, \nu = 1/2, \delta = 2, \beta = 1, \eta = 0$ [1, 2, 3]; and d_c is supposed to (almost has been proven to) be six [4, 5]. On the other hand for lattice animals, mean-field values of critical exponents are $\gamma = 1/2, \nu = 1/4, \eta = 0$ [6, 7]; and d_c is supposed to be eight. (Rigorously speaking, there is still no proof of the conjecture $d_c \geq 8$ for lattice animals.)

Despite these two distinct critical behavior, lattice animals and percolation are intimately connected. In particular, stochastic geometric objects which appear in these models (lattice animals = connected clusters) are the same; only the weights attached are different. In this talk, we introduce a model which interpolates lattice animals and percolation, and investigate its critical behavior.

1. Definition of the model.

We work on a d-dimensional hypercubic lattice, $\mathbb{Z}^d$. A *bond* is an unordered pair of distinct sites $\{x, y\} \subset \mathbb{Z}^d$, with some restrictions. We consider the following two situations.

- Nearest neighbor model: bond is an unordered pair $\{x, y\}$, with $|x - y| = 1$.
- Spread-out model: bond is an unordered pair $\{x, y\}$, with $0 < |x - y| \leq L$ for some $L \geq 1$.

Two bonds b_1, b_2 are said to be *connected*, when they share at least one endpoint.

A *lattice animal* (LA) is a finite connected set of bonds. For $n \geq 1$ and $x_1, x_2, \ldots, x_n \in \mathbb{Z}^d$, we denote by $\mathcal{A}(x_1, x_2, \ldots, x_n)$ the set of all lattice animals containing $x_1, x_2, \ldots, x_n$.

The *q-lattice animal model* is a model of lattice animals, whose n-point function is defined as (p and q are parameters which satisfy $0 \leq p \leq 1$ and $1 - p \leq q \leq 1$)

$$G_{p,q}(x_1, x_2, \ldots, x_n) := \sum_{A \in \mathcal{A}(x_1, x_2, \ldots, x_n)} w(A) \qquad \text{where} \qquad w(A) := p^{|A|}\, q^{|\partial A|}. \tag{1}$$

Here ∂A denotes the set of *boundary bonds* of A (i.e. bonds which touch A, but are not in A), and $|A|$ and $|\partial A|$ denote the number of bonds in A and ∂A, respectively.

This model interpolates ordinary lattice animals and ordinary percolation. When $q = 1$, equation (1) is nothing but the n-point function of ordinary lattice animals. When $q = 1 - p$, equation (1) is exactly the n-point function of ordinary percolation, because for percolation with parameter p, if a lattice animal A contains the origin 0,

$$\mathbb{P}[A \text{ is the connected cluster of } 0] = p^{|A|}(1 - p)^{|\partial A|}. \tag{2}$$

The above weight is equal to our $w(A)$ with $q = 1 - p$. Our model has been introduced to investigate what happens when $1 - p < q < 1$.

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[1]Based on a joint work with Keita Tamenaga

[2]Faculty of Mathematics, Kyushu University, Fukuoka 810-8560, Japan. E-mail:hara@math.kyushu-u.ac.jp

2. Critical Behavior.

Let p_c^{LA} and p_c^{perc} be the critical points of ordinary lattice animals and ordinary percolation, respectively. Using subadditivity arguments, it is easily seen:

Proposition 1. Fix q so that $1 - p_c^{\mathrm{perc}} < q < 1$.
(i) The n-point function of the q-lattice model is finite for $p < p_c^{\mathrm{LA}}$, and is infinite for $p > p_c^{\mathrm{perc}}$.
(ii) There is a critical point $p_c(q) \in (0,1)$ such that $G_{p,q}(x_1, x_2, \ldots, x_n)$ is finite for $p < p_c(q)$ but is infinite for $p > p_c(q)$.
(iii) The above critical point $p_c(q)$ is nonincreasing in q.

Around the critical point, we expect some critical behavior. Define the *susceptibility* $\chi_{p,q}$ as

$$\chi_{p,q} := \sum_{x \in \mathbb{Z}^d} G_{p,q}(0,x) = \sum_{A \in \mathcal{A}(0)} w(A)\,|A|. \tag{3}$$

Proposition 2. Fix q so that $1 - p_c^{\mathrm{perc}} < q < 1$. As $p \uparrow p_c(q)$, the susceptibility $\chi_{p,q}$ diverges, i.e.

$$\lim_{p \uparrow p_c(q)} \chi_{p,q} = \infty. \tag{4}$$

The proof is based on a correlation inequality stated in Proposition 4.

In high dimensions, we expect that the model exhibits mean-field type critical behavior. This is the claim of the following theorem:

Theorem 3. Fix q so that $1 - p_c^{\mathrm{perc}} < q < 1$. The following two models
(i) nearest-neighbor q-lattice animals in sufficiently high dimensions,
(ii) spread-out q-lattice animals in $d > 8$, with sufficiently large L,
exhibit mean-field like critical behavior. More precisely, the susceptibility $\chi_{p,q}$ and the correlation length $\xi_{p,q}$ diverge as $p \uparrow p_c(q)$

$$\chi_{p,q} \approx (p_c(q) - p)^{-1/2}, \qquad \xi_{p,q} \approx (p_c(q) - p)^{-1/4}. \tag{5}$$

Also the critical two-point function $G_{p_c(q),q}(0,x)$ decays like $|x|^{2-d}$.

The proof relies on the lace expansion, and the following correlation inequality.

3. Correlation inequality.

Now it is time to state the key correlation inequality. We consider one of the following events:

- a simple "connection" event, such as x and y are in a single lattice animal. This event is written as $x \longleftrightarrow y$.
- intersections of simple connection events, such as $\{x \longleftrightarrow y\} \cap \{z \longleftrightarrow w\}$.
- "disjoint occurrence" of the above.

If E is one of the above events, we define the "correlation function" $G(E)$ as

$$G(E) = \sum_A w(A) I[E \text{ occurs on } A] = \sum_A p^{|A|}\, q^{|\partial A|}\, I[E \text{ occurs on } A]. \tag{6}$$

Now the following inequality, which resembles the van den Berg - Kesten (BK) inequality in percolation, holds.

Proposition 4. Let E, F be two of the above events. Then we have

$$G(E \circ F) \le G(E)\, G(F). \tag{7}$$

Details will be presented in [8].

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Influence and statistical mechanics

B. T. Graham

1984 Mathematics Road, V6T 1Z2, UBC, Vancouver, Canada.

E-mail:ben@math.ubc.ca

Statistical mechanics is the branch of physics that seeks to explain the properties of matter that emerge from microscopic scale interactions. Probabilistic models such as percolation help describe various physical phenomena. The models are generally not exactly solvable; simple local interactions produce complex long range behaviour.

The technology of influence provides a way to study these processes. We will look at applications of influence to percolation, directed and undirected first passage percolation, the Ising model, and the random-cluster model.

This is joint work with Geoffrey Grimmett.

Percolation and the random-cluster model

Bond percolation was introduced by Simon Broadbent and John Hammersley in 1957. It is one of the simplest spatial models with a phase transition. Despite its simplicity, it seems to encompass many of the most important features of the phase transition phenomenon.

Bond percolation, defined say on the hypercubic lattice $\mathbb{Z}^d$ with 'nearest-neighbour' edges, is controlled by a parameter $p \in [0,1]$. Each edge is *open* with probability p, and *closed* otherwise. The edge states are independent. Two points $x, y \in \mathbb{Z}^d$ are said to *connected* if they are joined by a chain of open edges. The *open clusters* are the maximal sets of connected vertices. Let $\{0 \leftrightarrow \infty\}$ denote the event that the origin belongs to an unbounded open cluster. The model is said to percolate, or to be supercritical, if $\mathbb{P}_p(0 \leftrightarrow \infty) > 0$. There is a threshold value, the critical probability $p_c(d)$, such that there is a unique infinite cluster if $p > p_c(d)$, and no infinite cluster if $p < p_c(d)$.

Statistical physicists have devised stochastic models for ferromagnetic spin systems in thermal equilibrium; two of the most famous are the Ising and Potts models. They exhibit phase transitions in two and higher dimensions, providing a simple setting for the study of the phase transition phenomenon.

A key step in the understanding of the Ising and Potts models was the development of the random-cluster model by Fortuin and Kasteleyn [F]. It provides a connection between bond percolation and the magnetic spin models. It is an edge model like bond percolation, but the edge states are not independent. The connected clusters of the random-cluster model correspond to cluster of like spins in the Ising and Potts models.

Influence and sharp thresholds

Influence is a concept that has proved to be closely related to the phenomenon of sharp thresholds. Let $X = (X_1, \dots, X_N)$ be a collection of independent Bernoulli(p) random variables. Given $x \in \{0,1\}^N$, let $U_k x$ represent the configuration with x_k replaced by $1 - x_k$. For an event A, define the *influence* of the k-th coordinate,

$$I_k(A) = \mathbb{P}_p(1_A(X) \neq 1_A(U_k X)).$$

Theorem.[BKKKL, Tal] *There is constant C such that for increasing events A,*

$$\frac{d}{dp}\mathbb{P}_p(A) = \sum_{k=1}^{N} I_k(A) \geq C\,\mathbb{P}_p(A)[1 - \mathbb{P}_p(A)] \log\left[\frac{1}{2\max_k I_k(A)}\right].$$

We extend this result from product measure to monotonic measures. This allows us to demonstrate sharp thresholds in the random-cluster and Ising models.

Influence and first passage percolation

The technology of influence has also been applied to first passage percolation. Instead of the edges simply being open or closed, each edge is assigned a cost $\omega(e) \in \mathbb{R}$. This induces the first-passage percolation metric,

$$d_\omega(x,y) = \inf_\gamma \sum_{v \in \gamma} \omega(v), \qquad x, y \in V;$$

this is also called the traversal time from x to y. The infimum is taken over all paths γ from x to y.

In one dimension, the variance of $d_\omega(0,n)$ grows linearly with n. In two and higher dimensions, for a wide range of edge weight distributions, the variance of $d_\omega(0,x)$ is sublinear as a function of $|x|$ [BKS, RB]. We extend this result to the case of directed first passage percolation with Bernoulli edge weights.

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Brownian survival and Lifshitz tail in perturbed lattice disorder

Ryoki Fukushima (Kyoto University)

1 Introduction

We consider a Brownian motion moving in a randomly distributed killing traps, which is usually referred to as the "trapping problem". A quantity of primary interest in this problem is the survival probability of the Brownian motion up to time t. For the traps attached around the Poisson point process, there are extensive studies and, among others, Donsker and Varadhan [2] proved that

$$\text{the annealed survival probability} = \exp\Big\{-c\,t^{\frac{d}{d+2}}(1+o(1))\Big\} \quad \text{as} \quad t\to\infty,$$

and Sznitman [3] proved that almost surely

$$\text{the quenched survival probability} = \exp\big\{-c' t\,(\log t)^{-2/d}(1+o(1))\big\} \quad \text{as} \quad t\to\infty,$$

where $c, c' > 0$ are constants having explicit expressions.

In this talk, we discuss another model where the configuration of the traps is given by

$$V_\xi(x) = \sum_{q\in\mathbb{Z}^d} W(x-q-\xi_q).$$

Here $W \geq 0$ is supported on a compact set with nonempty interior and $((\xi_q)_{q\in\mathbb{Z}^d}, \mathbb{P}_\theta)$ is a collection of i.i.d. random variables with common distribution

$$\mathbb{P}_\theta(\xi_q \in dx) = N(d,\theta)\exp\{-|x|^\theta\}\,dx$$

for some $\theta > 0$. This is a model of the so-called "Frenkel defects" in a crystal and well-known as the "random displacement model" in the theory of random Schrödinger operator. However, the mathematical studies are restricted to the bounded displacements case so far.

2 Results

Let $((B_t)_{t\geq 0}, P_0)$ be the standard Brownian motion on $\mathbb{R}^d$. Then, the survival probability $S_{t,\xi}$ of the Brownian motion killed by V_ξ is expressed as follows:

$$S_{t,\xi} = E_0\Big[\exp\Big\{-\int_0^t V_\xi(B_s)\,ds\Big\}\Big].$$

We have both annealed and quenched asymptotics of this quantity in a weak sense. In the following statements, the symbol $f(t) \asymp g(t)$ means that there exists a constant $C > 0$ such that $C^{-1}g(t) \leq f(t) \leq Cg(t)$ for all sufficiently large t.

Theorem 1. (Annealed asymptotics) *For any $\theta > 0$, we have*

$$\log \mathbb{E}_\theta[S_{t,\xi}] \asymp \begin{cases} -t^{\frac{2+\theta}{4+\theta}}(\log t)^{-\frac{\theta}{4+\theta}} & (d=2), \\[2ex] -t^{\frac{d^2+2\theta}{d^2+2d+2\theta}} & (d\geq 3). \end{cases}$$

Theorem 2. (Quenched asymptotics) *For any $\theta > 0$, we have*

$$\log S_{t,\xi} \asymp \begin{cases} -t\,(\log t)^{-\frac{2}{2+\theta}}(\log\log t)^{-\frac{\theta}{2+\theta}} & (d=2), \\[2ex] -t\,(\log t)^{-\frac{2d}{d^2+2\theta}} & (d\geq 3), \end{cases}$$

with $\mathbb{P}_\theta$-probability one.

Remark. These results coincide with the case of the Poissonian traps in the limit $\theta \to 0$ for all $d \geq 2$, and with the case of the periodic traps as $\theta \to \infty$ for $d \geq 3$. When $d = 2$ and $\theta \to \infty$, there remain lower order corrections.

Let us define the density of states (of $-1/2\Delta + V_\xi$) by

$$\ell(d\lambda) = \lim_{N \to \infty} \frac{1}{(2N)^d} \sum_{i \geq 1} \delta_{\lambda_i^{\xi,N}}(d\lambda),$$

where $\lambda_i^{\xi,N}$ is the i-th smallest Dirichlet eigenvalue of $-1/2\Delta + V_\xi$ in $(-N, N)^d$ (counted with multiplicity). As is well known, we can relate this measure to $\mathbb{E}_\theta[S_{t,\,\xi}]$ through the Laplace transform and therefore, using a Tauberian argument, can derive the asymptotics of $\ell([0, \lambda])$ as $\lambda \to 0$. In the following statements, the symbol $f(\lambda) \asymp g(\lambda)$ is used for the similar meaning as before, but "for all *small* λ" this time.

Corollary 1. *For any $\theta > 0$, we have*

$$\log \ell[0, \lambda] \asymp \begin{cases} -\lambda^{-1-\frac{\theta}{2}} \left(\log \lambda^{-1}\right)^{-\frac{\theta}{2}} & (d = 2), \\ -\lambda^{-\frac{d}{2}-\frac{\theta}{d}} & (d \geq 3). \end{cases} \tag{1}$$

This result says that the density of states is exponentially thin around the origin, which is referred to as the "Lifshitz tail effect".

3 Some remarks on the methods

3.1 Annealed asymptotics

The proof of the annealed asymptotics of the survival probability proceeds in two steps:

(1) Showing that only the best survival strategy counts (the Laplace principle),

(2) Estimating the probability of $((B_t)_{t \geq 0}, (\xi_q)_{q \in \mathbb{Z}^d})$ taking the best strategy.

In both of the steps, we need an expression for the "emptiness probability", that is, $\mathbb{P}_\theta(\xi(U) = 0)$ $(U \in \mathbb{R}^d)$. For the Poisson point process with intensity $\nu > 0$, it equals $e^{-\nu|U|}$. This simple expression allows one to use large deviation techniques to show the Laplace principle. On the other hand, it has the expression $\exp\{-\int_U \mathrm{dist}(x, \partial U)^\theta dx\}$ (for a wide class of sets) in our model. Due to this complicated form, I could not find a way to use the large deviation techniques. Instead, I have used a coarse graining method which is a slightly altered version of Sznitman's "method of enlargement of obstacles" (cf. [4]) to prove the Laplace principle. The second step will be explained in the talk.

3.2 Quenched asymptotics

In existing works [1, 4], the asymptotics of quenched survival probability is derived by a rather complicated localizing argument. In this talk, I will introduce a novel way which deduces the upper estimate directly from the Lifshitz tail effect. It is not only simple but also works in general settings. Therefore, the upper bound for the quenched asymptotics turns out to be a corollary of that for the annealed asymptotics. This kind of implication is quite unusual in the theory of random media.

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Incipient infinite clusters in high-dimensions

Remco van der Hofstad *

In this talk, we discuss the recent results on the existence of the *incipient infinite cluster* in *high-dimensional percolation*.

Percolation is a paradigm model in statistical physics obtained by independently keeping and removing edges of a (finite or infinite) graph. The classical model is nearest-neighbor bond percolation on $\mathbb{Z}^d$, in which bonds $\{x, y\}$ with $|x - y| = 1$ are independently occupied or vacant with probability p. When the dimension $d \geq 2$, there is a phase transition in the parameter p such that below the critical value there is no infinite component and above it, there is a unique infinite component (see e.g. [2] and the references therein).

Despite the simplicity of the model, percolation shows a fascinatingly rich critical behavior. It is expected (and in some cases proved) that the critical behavior is governed by so-called *critical exponents* indicating that many power laws are present close to criticality. For example, it is widely believed that, at criticality, the probability that the origin is connected to the boundary of a cube of width n decays like $n^{-1/\rho}$, where $\rho \geq 0$ is a *critical exponent*. It is expected, though in many cases unproved, that this critical behavior is *universal*, i.e., the behavior close to criticality is rather insensitive to the precise details of the model, such as the chosen bond set. Despite the tremendous amount of research on percolation, many aspects are still ill understood. For example, the key question whether there exists an infinite component for nearest-neighbor critical percolation has only been answered negatively in two dimensions and for $d \geq 19$ for the nearest-neighbor model, even though physics arguments predict that, for finite range models, it should be the case as soon as $d \geq 2$. This is the celebrated *continuity of the percolation function* problem.

When, at criticality, there does not exist an infinite cluster, then large critical percolation clusters should have a fractal, self-similar nature. The *incipient infinite cluster* (IIC) can be thought of as the infinite cluster which is on the verge of appearing at criticality. The existence of the IIC is not obvious, which can be understood since we in fact only know in certain cases that there is no infinite cluster at criticality. A way to describe large critical percolation clusters is through Kesten's IIC [20], which is the law of an infinite *critical* cluster. When the percolation function is continuous, there are no infinite clusters, and therefore, this object needs to be constructed by an appropriate limiting argument. As a result, the IIC is not full-dimensional and has a fractal structure.

Kesten [20] constructed the IIC in $d = 2$ in two ways: (i) by conditioning the cluster of the origin to be connected to the boundary of a cube of width n centered around the origin and letting $n \to \infty$, and (ii) by taking $p > p_c$, conditioning the origin to be in the infinite connected component and letting $p \downarrow p_c$. Kesten proved that these two constructions give rise to a measure on infinite connected components, and the two constructions give rise to the *same* measure. Járai [19, 18] proved that several more natural constructions, for example by taking a uniform point in the largest critical cluster inside a large cube and shifting this to the origin followed by letting the cube tend to infinity, give rise to the same IIC measure. These results show that in two-dimensions, the IIC is a natural and robust object.

In this talk, we discuss the construction of the IIC in high-dimensions, both for oriented as well as unoriented percolation. In high-dimensional percolation, a lot of progress has been made in the past two decades. The application of the *lace expansion* in the seminal paper [6] has incited a wealth of results being proved [1, 3, 4, 5, 7, 8, 9, 9, 16]. As a result, we now have good insight in the critical nature

*Department of Mathematics and Computer Science, Eindhoven University of Technology, P.O. Box 513, 5600 MB Eindhoven, The Netherlands. `rhofstad@win.tue.nl`

of percolation in high dimensions, even though particularly the supercritical percolation regime in high dimensions is still not so well understood. For example, it is not know how the expected cluster size, when conditioning the cluster to be finite, diverges as the percolation parameter approaches the critical value from above.

The high-dimensional IIC is proved to exist in [16] through two different constructions, both for the spread-out setting for $d > 6$, as well as for the nearest-neighbor setting for d sufficiently large. Several properties have been investigated, such as the fact that the IIC has an essentially unique infinite path, in the sense that every pair of infinite paths share infinitely many bonds. The collection of vertices contained in infinite paths is two dimensional, while the IIC itself is four dimensional. The results in [9, 10] suggest that large critical clusters are like large components of critical branching random walk, suggesting that large critical clusters should scale to a measure-valued diffusion. As a result, it is conjectured in [12] that the scaling limit of the high-dimensional IIC is super-Brownian motion conditioned to survive forever.

For the oriented setting, the picture of the IIC is even more complete. The results in [14, 15, 17] show that the scaling limit of critical spread-out oriented percolation is super-Brownian motion (SBM). See also [11] for an expository discussion of the relation between SBM and oriented percolation. These results imply that the scaling limit of the IIC is SBM conditioned to survive forever [12, 13].

In this talk, we discuss the various different constructions of the IIC in high-dimensional percolation, discuss the fact that the limits agree, and study properties of the IIC.

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Random walks and critical percolation

Gordon Slade, University of British Columbia

We survey recent developments that prove precise estimates showing subdiffusive behaviour (Alexander–Orbach conjecture) for random walks on four different random environments related to critical percolation. These environments are:

1. the incipient infinite cluster (IIC) on a tree (a critical branching process conditioned to survive forever),

2. the invasion percolation cluster on a tree,

3. the IIC for oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d > 6$ (spread-out model),

4. the IIC for percolation on $\mathbb{Z}^d$ for $d > 6$ (spread-out model).

In each case, the IIC or invasion percolation cluster provides a random infinite graph on which simple random walk is performed. The behaviour of the random walk is subdiffusive in all cases. For example, and roughly speaking, the time to exit a ball of radius R centred at the starting point is almost surely of order R^3 in all four examples.

The behaviour of the random walk is obtained as a consequence of geometric properties of the environment. A general theorem of [2] (see [12] for an extension) shows that in a large class of random environments, if the growth of the volume of a ball of radius R is typically R^2, and if the effective resistance between the exterior of a ball of radius R and its centre typically grows like R, then the above-mentioned R^3 exit time and other related results hold for the random walk in the random environment. An analysis of the geometry of each environment shows that this general theorem applies to each of the above four examples:

1. the IIC on the binomial tree [3],

2. the invasion percolation cluster on a regular tree [1],

3. the IIC for oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ in dimensions $d > 6$ (spread-out model) [2] (the IIC itself is constructed in [7, 8]),

4. the IIC for percolation on $\mathbb{Z}^d$ for $d > 6$ (spread-out model) [11] (the IIC itself is constructed in [9]).

The results for example 3 depend on the lace expansion [10], while the results of example 4 depend on both the lace expansion [5, 6] and the critical behaviour of the magnetization [4].

The construction of the IIC will be discussed in the lecture of Remco van der Hofstad. The results of [11] will be discussed in the lecture of Asaf Nachmias.

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Disconnection of discrete cylinders and random interlacements

Alain-Sol Sznitman

The disconnection by random walk of a discrete cylinder with a large finite connected base has been a recent object of interest. It has to do with the way paths of random walks can create interfaces. In this talk we give an overview of some current results and explain how this problem is related to questions of percolation and to the model of random interlacements.

Random walks on random paths and trees

David Croydon

There are many examples of sequences of random graphs that can be rescaled to yield a non-trivial continuum limit. The existence of a structural scaling result, however, only leads to further questions regarding the asymptotic behaviour of other properties of the sequence of graphs considered, such as: how can we describe the scaling limit of the associated simple random walks? In this talk, I will discuss recent results that provide some answers to this question for particular sequences of random graphs, including those generated by random walk paths, branching processes and branching random walk, and briefly outline how attempting to understand the random walk on a critical percolation cluster provides some motivation for this work.

The range of the simple random walk on the integer lattice is perhaps the simplest example of a random graph that admits a non-deterministic scaling limit. To define this graph, first let $S=(S_n)_{n\geq 0}$ be the simple random walk on $\mathbb{Z}^d$ starting from 0, and define the range of the random walk S to be the graph $\mathcal{G}=(V(\mathcal{G}),E(\mathcal{G}))$ with vertex set $V(\mathcal{G}):=\{S_n : n\geq 0\}$, and edge set $E(\mathcal{G}):=\{\{S_n,S_{n+1}\} : n\geq 0\}$. In low dimensions, when $d=1,2$, the recurrence of S implies that the graph $\mathcal{G}$ is equal to the whole lattice, and so the associated simple random walk, X say, will converge to a standard Brownian motion. Conversely, for $d\geq 3$ the random walk S is transient and does not explore all of $\mathbb{Z}^d$, and so it becomes an interesting problem to determine the behaviour of X. In the first section of the talk, I will explain how to establish quenched and annealed scaling limits for the process X when $d\geq 5$, which show that the intersections of the original simple random walk path are essentially unimportant. For $d = 4$, the results I will present are less precise, but they do show that any scaling limit for X will require logarithmic corrections to the polynomial scaling factors seen in higher dimensions. Furthermore, I will demonstrate that when $d = 4$ similar logarithmic corrections are necessary in describing the asymptotic behaviour of the return probability of X to the origin.

In the context of random walks on graphs, it is often the case that graph trees are particularly tractable for the reason that the absence of circuits make it easy to understand the connections in the graphs. Exploiting this loop-free property, it is possible to deduce scaling limits for random walks on sequences of graph trees generated by conditioned Galton-Watson processes. In particular, if $(T_n)_{n=1}^{\infty}$ is a family of graph trees such that, for each n, T_n is a critical Galton-Watson tree conditioned on its total progeny being equal to n, then it is known that under mild technical conditions the sequence $(T_n)_{n=1}^{\infty}$ can be rescaled to converge to a so-called α-stable tree, where the $\alpha \in (1, 2]$ is an index that depends on the tail of the offspring distribution. In the second part of the talk, I will summarise how the corresponding simple random walks have as a scaling limit a natural Brownian motion on the limiting trees, and how embedding this result into $\mathbb{Z}^d$ results in a scaling limit for the simple random walk on the graph generated by branching random walk.

One incentive for considering the above problems is understanding the random walk on a critical percolation cluster conditioned to be large. It is becoming clearer how in high dimensions such a random graph has the same asymptotic structure as the range of the integrated super-Brownian motion $\mathcal{S}$ (or a suitably conditioned version of this set), and so one might also expect that the associated random walks rescale to a diffusion on $\mathcal{S}$. The canonical nature of the diffusion scaling limit of the simple random walks on the graphs generated by branching random walk, which also rescale to the random set $\mathcal{S}$, mean that it would not be surprising for the random walks on the critical percolation clusters to share the same diffusion as a scaling limit, at least in high dimensions $d \geq 8$ where $\mathcal{S}$ is a dendrite (tree-like topological space). However, the critical percolation cluster is more difficult than the tree case, because it is not a loop-free graph, and to understand it we need to extend the theory to be able to deal with random walks on graphs that are only asymptotically tree-like. Working on the case of the random walk on the range of the random walk is a (very small) first step in this direction.

Exponential growth of ponds in invasion percolation

Jesse Goodman

Consider an infinite, connected, locally finite graph G. To each edge attach a weight chosen uniformly and independently from $[0, 1]$. Starting from a distinguished root vertex, grow a subgraph according to the following rule. Among all the edges on the boundary of the current cluster, select the one of lowest weight and add it to the cluster. The finished object - i.e., the increasing union of the subgraphs constructed above - is called the invasion percolation cluster. By construction, it is an infinite connected subgraph a.s.

As its name suggests, invasion percolation is closely linked with ordinary (Bernoulli) percolation. For instance, consider the weight X_n of the n^{th} invaded edge. Then for any (quasi-)transitive graph, $\limsup X_n = p_c$, where p_c is the percolation threshold (see [1]; the result was proved earlier for $\mathbb{Z}^d$ in [2]). This suggests that, apart from some initial edges, the invasion percolation cluster behaves like a critical percolation cluster, but will be infinite a.s. The fact that invasion percolation is linked to critical percolation, even though it contains no external parameter, makes it an example of self-organized criticality.

The first *outlet* is defined to be the invaded edge of greatest weight. For $i > 1$, the i^{th} outlet is the invaded edge invaded after the $(i-1)^{\text{st}}$ outlet having greatest weight. The outlets divide the invasion cluster into *ponds*: the i^{th} pond consists of the edges invaded after the $(i-1)^{\text{st}}$ outlet and before the i^{th} outlet. From the construction, it follows that the ponds form a chain of disjoint sub-clusters linked by the outlets, and that the weights of the outlets decrease toward p_c.

We consider invasion percolation on regular trees, where independence and symmetry allow the invasion cluster to be studied with great precision. In [3], extending work of [4], a structural representation of the cluster is given in terms of the outlet weights. It is shown that the weights of outlets are all drawn from the same distribution, with later outlets conditioned to have lower weights than earlier ones.

This characterization is extended to show that numerous statistics of the ponds - including radius, number of edges, and distance of outlet weights from p_c - grow exponentially. Indeed, logarithms of these quantities obey strong laws of large numbers, invariance principles and large deviation principles, with parameters that can be calculated explicitly and that are independent of the degree of the tree.

The asymptotics of a fixed pond are also studied and compared with results on $\mathbb{Z}^2$ from [5] and [6]. We find that the radius of a pond has similar tail behaviour to in $\mathbb{Z}^2$. Unlike Z^2, however, we find that on a tree the invasion cluster exhibits a surprising and marked difference from percolation with defects.

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Convergence of random walks to fractional kinetic motion

Martin T. Barlow* **and Jiří Černý**

Let E^d be the set of all nearest-neighbour edges in the Euclidean lattice $\mathbb{Z}^d$, (where $d \geq 2$), and let μ_e, $e \in E^d$, be i.i.d. random variables (defined on a space $(\Omega, \mathcal{F}, \mathbb{P})$) with $\mathbb{P}(\mu_e > t) \sim t^{-\alpha}$ and $\mu_e \geq 1$ a.s. We call μ_e the *conductance* of the edge e, and for $x \in \mathbb{Z}^d$ write μ_x for the sum of conductances of all edges adjacent to x

Let X_t be the continuous time random walk which jumps from x to $y \sim x$ with rate μ_{xy}/μ_x. Write P_ω^x for the law of X started at x in the random environment given by $(\mu_e(\omega))$. Let

$$X_t^{(n)} = n^{-1} X_{n^2 t}, \quad t \geq 0.$$

In [BD08] a quenched invariance principle was proved for X: for $\mathbb{P}$-a.a. ω, $X^{(n)}$ converges in law to $C_0 W$, where W is a standard Brownian motion in $\mathbb{R}^d$. Further $C_0 > 0$ if and only if $\mathbb{E}\mu_e < \infty$.

In this paper we consider the case when $\alpha < 1$, so that $\mathbb{E}\mu_e = \infty$. We consider the processes

$$\widetilde{X}_t^{(n)} = n^{-1} X_{n^{2/\alpha} t}, \quad t \geq 0.$$

Let W be a standard Brownian motion in $\mathbb{R}^d$, and V_t be an independent stable subordinator with index α. The time change of X by V is a common procedure, and yields a symmetric stable process of index 2α. Let V_t^1 be the inverse of V, so that V^1 is a continuous non-decreasing process. The *fractional kinetics process of index* α is defined by

$$Z_t = W_{V_t^{-1}}, \quad t \in [0, \infty).$$

This is a continuous non-Markovian process: its trajectory follows that of a Brownian motion, but it has long intervals of constancy.

Theorem 1. *The processes* $\widetilde{X}^{(n)}$ *converge* $\mathbb{P}$ *- a.s. to a non-zero multiple of the fractional kinetics process of index* α.

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THE ALEXANDER-ORBACH CONJECTURE HOLDS IN HIGH DIMENSIONS

ASAF NACHMIAS

In this talk we discuss the behavior of the simple random walk on the incipient infinite cluster (IIC) of critical percolation on $\mathbb{Z}^d$. The IIC is a random infinite connected graph containing the origin which can be thought of as a critical cluster conditioned to be infinite. The *spectral dimension* d_s of an infinite connected graph G is defined by

$$d_s = d_s(G) = -2 \lim_{n\to\infty} \frac{\log \mathbf{p}_{2n}(x,x)}{\log n} \qquad \text{(if this limit exists)}\,,$$

where $x \in G$ and $\mathbf{p}_n(x,x)$ is the return probability of the simple random walk on G after n steps (note that if the limit exists, then it is independent of the choice of x). Alexander and Orbach [1] conjectured that $d_s = 4/3$ for the IIC in all dimensions $d > 1$, but their basis for conjecturing this in low dimensions was mostly rough correspondence with numerical results and it is now believed that the conjecture is false when $d < 6$ [11, 7.4]. In a joint work with Gady Kozma [14] we verify this conjecture in high dimensions.

Theorem 0.1. *Let $\mathbf{P}_{\rm IIC}$ be the IIC measure of critical percolation on $\mathbb{Z}^d$ with large d ($d \geq 19$ suffices) or with $d > 6$ and sufficiently spread-out lattice and consider the simple random walk on the IIC. Then $\mathbf{P}_{\rm IIC}$-a.s.*

$$\lim_{n\to\infty} \frac{\log \mathbf{p}_{2n}(0,0)}{\log n} = -\frac{2}{3}\,, \qquad \lim_{r\to\infty} \frac{\log \mathbb{E}\tau_r}{\log r} = 3\,, \qquad \lim_{n\to\infty} \frac{\log |W_n|}{\log n} = \frac{2}{3}\,,$$

where τ_r is the hitting time of distance r from the origin (the expectation $\mathbb{E}$ is only over the randomness of the walk) and W_n is the range of the random walk after n steps.

Our main contribution is the analysis of the geometry of the IIC. The IIC admits *fractal* geometry which is dramatically different from the one of the infinite component of *supercritical* percolation. The latter behaves in many ways as $\mathbb{Z}^d$ after a "renormalization" i.e. ignoring the local structure [9] (see also [8] for a comprehensive exposition). In particular, the random walk on the supercritical infinite cluster has an invariance

Microsoft Research, One Microsoft way, Redmond, WA 98052-6399, USA.
asafnach@math.berkeley.edu.

principle, the spectral dimension is $d_s = d$ and other $\mathbb{Z}^d$-like properties hold, see [7, 5, 2, 18, 6, 15].

Our analysis establishes that balls of radius r in the IIC typically have volume of order r^2 and that the *effective resistance* between the center of the ball and its boundary is of order r. These facts alone suffice to control the behavior of the random walk and yield Theorem 0.1, as shown by Barlow, Járai, Kumagai and Slade [3]. The key ingredient of our proofs is establishing that the critical exponents dealing with the *intrinsic* metric (i.e., the metric of the percolated graph) attain their mean-field values. In a joint work with Yuval Peres [16], we first demonstrated that these exponents yield analogous statements to the Alexander-Orbach conjecture in the *finite* graph setting. In particular, in [16], the diameter and mixing time of critical clusters in mean-field percolation on finite graphs were analyzed.

In different settings the Alexander-Orbach conjecture was proved by various authors. When the underlying graph is an infinite regular tree, this was proved by Kesten [13] and Barlow and Kumagai [4] and in the setting of *oriented* spread-out percolation with $d > 6$, this was proved recently in the aforementioned paper [3].

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ARM EXPONENTS FOR HIGH-d PERCOLATION

GADY KOZMA

Examine a d-dimensional lattice with $d > 6$. Let p_c be the critical probability for the existence of an infinite percolation cluster. Assume the lattice satisfies

$$\mathbb{P}_{p_c}(0 \leftrightarrow x) \approx |x|^{2-d} \tag{1}$$

where $\approx$ means that the ration between the right- and left-hand sides is bounded between two constants (which might depend on the lattice and in particular on d, but not on x). It is known that (1) holds when $d > 6$ and the lattice is sufficiently spread out, or for $\mathbb{Z}^d$ when d is sufficiently large [HvdHS03, H08]. In this lecture we will discuss the following theorem (with Asaf Nachmias)

Theorem. *Every lattice in dimension $d > 6$ satisfying (1) satisfies at p_c*

$$\mathbb{P}(0 \leftrightarrow \partial B(0, r)) \approx r^{-2}$$

The notation here is standard: $B(v, r)$ stands for the ball $\{x \in \mathbb{Z}^d : |x - v| < r\}$ and ∂A for any set of vertices A stands for its external boundary, i.e. all vertices in $\mathbb{Z}^d \setminus A$ with a neighbor in A.

The 2 in the exponent is often called the 1-arm exponent or the cluster radius exponent. A relate quantity is often denoted by $1/\rho$ so we get that if ρ exists it is equal to $\frac{1}{2}$. In dimension 2 it is of great interest to examine the probability of k disjoint connections between 0 and $\partial B(0, r)$, and call the related exponent the "k-arm exponent", denoted by δ_k. However, in high dimension it is more-or-less standard to deduce from (1) that existence of several disjoint connections are "approximately independent" events and hence $\delta_k = k\delta_1$ so determining the 1-arm exponent determines all the k-arm exponents.

To put the theorem in context, one must note that on a tree, $\mathbb{P}(0 \leftrightarrow \partial B(0, r)) \approx r^{-1}$. The discrepancy is explained by the "integrated super-Brownian excursion" heuristic, which claim that a critical percolation cluster looks like a critical branching process embedded in $\mathbb{Z}^d$ using random walks. Hence to reach distance r in $\mathbb{Z}^d$, one must reach distance r^2 in the tree.

This means that one cannot prove that $\rho = \frac{1}{2}$ assuming only the triangle condition

$$\sum_{x,y \in \mathbb{Z}^d} \mathbb{P}(0 \leftrightarrow x)\mathbb{P}(0 \leftrightarrow y)\mathbb{P}(x \leftrightarrow y) < \infty. \tag{2}$$

The triangle condition follows from (1) by a simple sum (when $d > 6$), and a significant body of literature exists that shows that merely assuming the triangle condition gives many critical exponents. This is true for the sub-critical expected cluster size exponent γ [AN84], the super-critical percolation probability exponent β [BA91], the critical volume exponent δ [BA91], and the intrinsic 1-arm exponent [KN]. The reason for the success of the triangle condition in all these cases is that it bounds the effect of the "past" on the "future" when exploring the cluster, and hence allows to use branching process arguments. It is of great interest to prove results directly from the triangle condition, since it might hold in many cases where neither $|\cdot|$ nor d are obvious to define, so (1) might be meaningless or false. Typical scenarios include long-range percolation and percolation on general graphs and groups. Unfortunately, this strength is in our case a weakness — the triangle condition does not "see" the Euclidean structure, and hence cannot distinguish between the tree case (when $\rho = 1$) and the $\mathbb{Z}^d$ case (when $\rho = \frac{1}{2}$).

Lace expansion is not used directly in the proof, only through [HvdHS03, H08]. The proof revolves around the fact that the size of the largest percolation cluster in a box has exponential tails [A97]. This fact is used to prove "structure theorems" for the cluster. The various structure theorems, hold with such high probability, that they also hold when conditioning on the events of interest. This allows to pull through many heuristic arguments which were previously not easy to do.

Attending the lectures of van der Hofstad, Slade or Nachmias is not required to understand this lecture, which is planned to be self-contained.

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Scaling limits for weakly pinned random walks with two large deviation minimizers

Tadahisa Funaki (Univ. Tokyo)

The scaling limits for d-dimensional random walks perturbed by an attractive force toward the origin are studied under the critical situation that the rate functional of the corresponding large deviation principle (LDP) admits two minimizers.

Let $\phi = \{\phi_i\}_{i=0}^N$ be the Markov chain on $\mathbb{R}^d$ with the transition probability density $p(x)$ starting at aN. Then, it is easy to see that the law of large numbers $h^N(t) := \frac{1}{N}\phi_{[Nt]} \to a + mt, \ t \in [0,1]$ holds, where $m = \int_{\mathbb{R}^d} xp(x)\,dx$ is the mean drift. We consider Markov chains modified by adding a pinning effect at 0 to ϕ (i.e., allowing occasional jumps to 0). Our goal is to study the scaling limit mentioned above for such modified chains, in particular when the corresponding LD rate functional admits two minimizers. The paper [1] discusses the case where $p(x)$ is mean-zero Gaussian, and we extend the results to the general case under the absence of the wall.

We will first define the weakly pinned Markov chains on $\mathbb{R}^d$ imposing the Dirichlet or free conditions for the arriving points at the final time $i = N$. Then, we will state our results on (1) LDP, (2) the free energies $\xi^{D,\varepsilon}$ and $\xi^{F,\varepsilon}$, (3) the minimizers of the LD rate functional, (4) the scaling limits under critical situation that two minimizers exist, and finally on (5) the critical exponents for the free energies.

The original motivation comes from the study of the (1+1)-dimensional interface model, or $(1+d)$-dimensional directed polymer model. The so-called Wulff shape (for crystal) is characterized by a variational formula and our interest is to see what happens if the variational problem has non-unique solutions. This is a joint work with T. Otobe.

1 Markov chains with weak pinning

Let $\varepsilon \geq 0$ be a parameter representing the strength of pinning at 0. We define two probability measures $\mu_N^{D,\varepsilon}$ (called the Dirichlet case: $\phi_0 = aN$, $\phi_N = bN$, $a, b \in \mathbb{R}^d$) and $\mu_N^{F,\varepsilon}$ (called the free case: $\phi_0 = aN$) on $(\mathbb{R}^d)^{N+1}$, respectively, by

$$\mu_N^{D,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,b,\varepsilon}}\delta_{aN}(d\phi_0)\prod_{i=1}^{N-1}\big(\varepsilon\,\delta_0(d\phi_i) + d\phi_i\big)\delta_{bN}(d\phi_N),$$

$$\mu_N^{F,\varepsilon}(d\phi) = \frac{\mathfrak{p}_N(\phi)}{Z_N^{a,F,\varepsilon}}\delta_{aN}(d\phi_0)\prod_{i=1}^{N}\big(\varepsilon\,\delta_0(d\phi_i) + d\phi_i\big),$$

where $Z_N^{a,b,\varepsilon}$ and $Z_N^{a,F,\varepsilon}$ are the normalizing constants and $\mathfrak{p}_N(\phi) = \prod_{i=1}^N p(\phi_i - \phi_{i-1})$.

Let $h^N = \{h^N(t), t \in [0,1]\} \in \mathcal{C} = C([0,1], \mathbb{R}^d)$ be the macroscopic path of the Markov chain defined as the polygonal approximation of $\{\frac{1}{N}\phi_{Nt}, t = 0, 1/N, \dots, N/N\}$. We assume that $\sup_{x\in\mathbb{R}^d} e^{\lambda\cdot x}p(x) < \infty$ for all $\lambda \in \mathbb{R}^d$ and the condition on the Legendre transform of $\Lambda(\lambda) = \log\int_{\mathbb{R}^d} e^{\lambda\cdot x}p(x)dx$: $\Lambda^*(v) = \sup_{\lambda\in\mathbb{R}^d}\{\lambda\cdot v - \Lambda(\lambda)\} < \infty$ for all $v \in \mathbb{R}^d$.

2 Sample path LDP

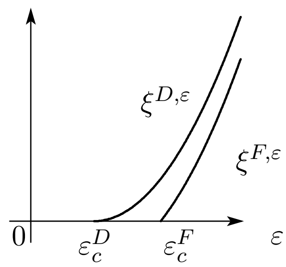
The LDP for $\mu_N = \mu_N^{D,\varepsilon}$ or $\mu_N^{F,\varepsilon}$, which is roughly stated as $\mu_N(h^N \sim h) \sim e^{-NI(h)}$, $h \in \mathcal{C}$, holds with the rate functional $I(h) = \Sigma(h) - \inf\Sigma$, where

$$\Sigma(h) = \int_0^1 \Lambda^*\big(\dot{h}(t)\big)\,dt - \xi^\varepsilon\Big|\{t\in[0,1]; h(t)=0\}\Big|.$$

The case without pinning (i.e. $\xi^\varepsilon = 0$) was studied by Mogul'skii for the free case. Here, the pinning free energies $\xi^\varepsilon = \xi^{D,\varepsilon}$ or $\xi^{F,\varepsilon} \geq 0$ are defined respectively by

$$\xi^{D,\varepsilon} = \lim_{N\to\infty}\frac{1}{N}\log\frac{Z_N^{0,0,\varepsilon}}{Z_N^{0,0,0}},$$

$$\xi^{F,\varepsilon} = \lim_{N\to\infty}\frac{1}{N}\log\frac{Z_N^{0,F,\varepsilon}}{Z_N^{0,F,0}}.$$



$d \geq 3,\ m \neq 0$

We point out the following properties of the free energies:

(1) $^\exists$two critical values $0 \leq \varepsilon_c^D \leq \varepsilon_c^F$ s.t. $\xi^{D,\varepsilon} > 0 \Leftrightarrow \varepsilon > \varepsilon_c^D$, $\quad \xi^{F,\varepsilon} > 0 \Leftrightarrow \varepsilon > \varepsilon_c^F$,

(2) $\varepsilon_c^D > 0$ $(d \geq 3)$, $\varepsilon_c^D = 0$ $(d = 1, 2)$,

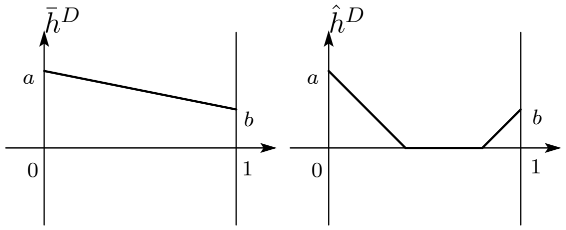
(3) $m = 0 \Rightarrow \varepsilon_c^D = \varepsilon_c^F, \xi^{D,\varepsilon} = \xi^{F,\varepsilon}$, $\quad m \neq 0 \Rightarrow \varepsilon_c^D < \varepsilon_c^F$, $\xi^{F,\varepsilon} < \xi^{D,\varepsilon}$ for $^\forall\varepsilon > \varepsilon_c^D$.

As an immediate consequence of the LDP, we see that the limits under μ^N are concentrated on the set of minimizers of the functional Σ.

3 Minimizers of Σ

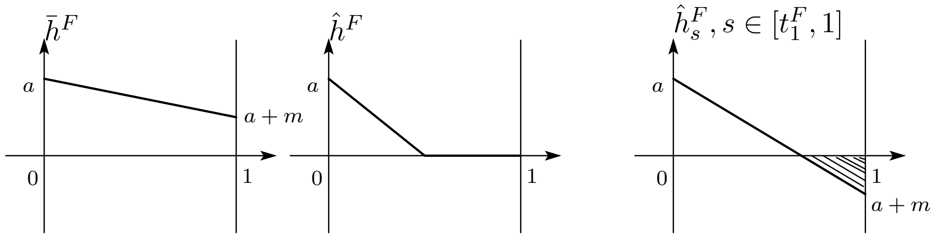
We assume $\xi^\varepsilon > 0$. Then, two (or more) possible minimizers : $\bar{h}, \hat{h}$ are given as in the following pictures.

Dirichlet case (i.e. $h(0) = a, h(1) = b$):



Free case (i.e. $h(0) = a$, $h(1)$ is free):

$\xi^{F,\varepsilon} > \Lambda^*(0)$ $\qquad\qquad$ $\xi^{F,\varepsilon} = \Lambda^*(0)$



The Young's relation (the free boundary condition) determining the times t_1^D or t_1^F when $\hat{h}^D$ or $\hat{h}^F$ first touch 0 is given by the formula: $-\frac{a}{t} \cdot \nabla\Lambda^*\big(-\frac{a}{t}\big) - \Lambda^*\big(-\frac{a}{t}\big) = \xi^\varepsilon - \Lambda^*(0)$, with $\xi^\varepsilon = \xi^{D,\varepsilon}$ or $\xi^{F,\varepsilon}$.

4 Main results

Scaling limits under the critical situation: $\Sigma(\bar{h}) = \Sigma(\hat{h})$ (we assume $\xi^{F,\varepsilon} > \Lambda^*(0)$ in the free case) are summarized in:

Theorem 1. (1) (Dirichlet case) The limits of h^N under $\mu_N^{D,\varepsilon}$ are $\hat{h}^D$ if $d = 1$, both $\bar{h}^D$ and $\hat{h}^D$ (coexistence) if $d = 2$, and $\bar{h}^D$ if $d \geq 3$.
(2) (Free case) The limits of h^N under $\mu_N^{F,\varepsilon}$ are both $\bar{h}^F$ and $\hat{h}^F$ (coexistence) if $d = 1$, and $\bar{h}^F$ if $d \geq 2$.

The central limit theorem for the first and the last hitting times of the weakly pinned Markov chains ϕ at 0 holds under a suitable scaling and conditioning (if necessary).

5 Critical exponents

Proposition 2. (1) (Dirichlet case) As $\varepsilon \downarrow \varepsilon_c^D$,

$$\xi^{D,\varepsilon} \sim \begin{cases} C(\varepsilon - \varepsilon_c^D)^2, & d = 1, 3, \\ e^{-2\pi\sqrt{\det Q}/\varepsilon}, & d = 2, \\ C(\varepsilon - \varepsilon_c^D)/\log(\varepsilon - \varepsilon_c^D), & d = 4, \\ C(\varepsilon - \varepsilon_c^D), & d \geq 5, \end{cases}$$

where C are different positive constants depending on d and Q is the covariance matrix of the Cramér transform of p in such a way that its mean becomes 0.
(2) (Free case) (i) If $m = 0$, $\xi^{F,\varepsilon} = \xi^{D,\varepsilon}$.
(ii) If $m \neq 0$, we have $\xi^{F,\varepsilon} \sim C(\varepsilon - \varepsilon_c^F)$ as $\varepsilon \downarrow \varepsilon_c^F$ for all $d \geq 1$.

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Extended thermodynamic relation and entropy for nonequilibrium steady states

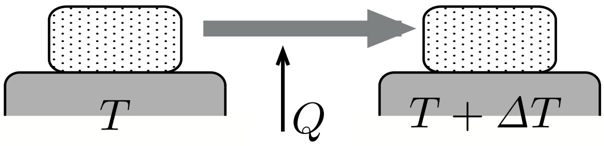
Hal Tasaki, *Department of Physics, Gakushuin University*

Thermodynamics is a theoretical framework that describes universal quantitative laws obeyed by macroscopic systems in equilibrium. Thermodynamics was also an essential theoretical guide when equilibrium statistical mechanics was constructed. Here we wish to address the question whether thermodynamics can be extended to nonequilibrium steady states which, like equilibrium states, lack macroscopic time-dependence.

To be concrete, we here consider a system of N classical particles (which generally interact with each other) confined in a fixed volume. (Our theory covers a much more general class of systems.) By $\boldsymbol{q} = (\boldsymbol{q}_1, \ldots, \boldsymbol{q}_N)$ and $\boldsymbol{p} = (\boldsymbol{p}_1, \ldots, \boldsymbol{p}_N)$, we collectively denote the coordinates and momenta, respectively, of the particles. We assume that the time evolution is governed by the deterministic Newton equation and stochastic dynamics (realized, for example, by thermal walls) which describes the interaction between the heat baths.

Equilibrium states: First suppose that the system is attached to a heat bath with temperature T. When the system is in touch with the heat bath for a sufficiently long time, it reaches the equilibrium state with temperature T, which can be described by the canonical distribution $\rho^{(T)}_{\mathrm{eq}}(\boldsymbol{q},\boldsymbol{p}) \propto \exp[-H(\boldsymbol{q},\boldsymbol{p})/(k_{\mathrm{B}}T)]$. Here $H(\boldsymbol{q},\boldsymbol{p})$ is the energy (Hamiltonian) of the system and k_{B} is the Boltzmann constant.

Consider a *thermodynamic operation* in which one changes the temperature of the bath by a very small amount, from T to $T + \Delta T$. After a sufficiently long time, the system settles to a new equilibrium with temperature $T + \Delta T$. During this process the system absorbs the energy Q (as heat) from the bath.



The Clausius relation, which is at the core of the classical thermodynamics, states that the heat Q is related to the change of entropy as

$$S(T + \Delta T) - S(T) = \frac{Q}{T}, \tag{1}$$

where (and in what follows) we assumed that ΔT is small and omitted the terms of $O((\Delta T)^2)$.

It is also known from equilibrium statistical mechanics that the entropy of an equilibrium state is written as the (Gibbs-)Shannon entropy of the canonical distribution

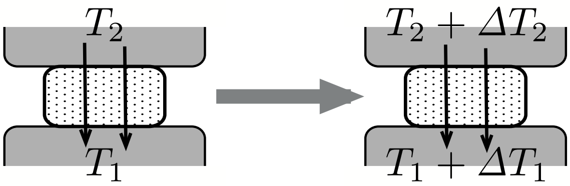
$$S(T) = -k_{\mathrm{B}} \int d\boldsymbol{q}d\boldsymbol{p}\, \rho_{\mathrm{eq}}^{(T)}(\boldsymbol{q},\boldsymbol{p}) \log \rho_{\mathrm{eq}}^{(T)}(\boldsymbol{q},\boldsymbol{p}), \tag{2}$$

where $d\boldsymbol{q}d\boldsymbol{p}$ is the Lebesgue measure of the phase space.

Nonequilibrium steady states: Our goal is to extend the relations (1) and (2) to nonequilibrium steady states.

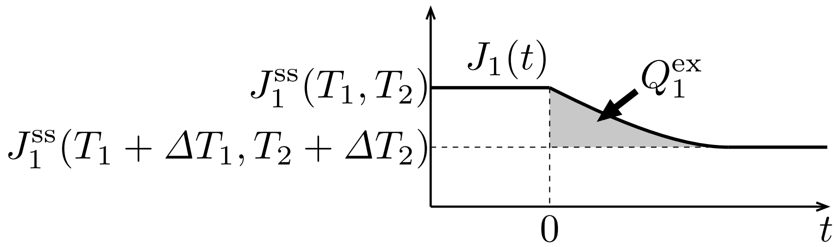
Now suppose that the system is attached to two heat baths with different temperatures T_1 and T_2. When the system is in touch with the baths for a sufficiently long time, it settles to a *nonequilibrium steady state* (abbreviated as NESS) with no macroscopically observable changes but with a steady flow of energy called heat current. We denote by $\rho_{\mathrm{ss}}^{(T_1,T_2)}(\boldsymbol{q},\boldsymbol{p})$ the probability density of the state $(\boldsymbol{q},\boldsymbol{p})$ in the NESS. Unlike the equilibrium, we do not have any general expressions for $\rho_{\mathrm{ss}}^{(T_1,T_2)}(\boldsymbol{q},\boldsymbol{p})$.

Let us consider a *thermodynamic operation* in the above NESS. We change the temperatures of the baths from T_1 and T_2 to $T_1 + \mathit{\Delta} T_1$ and $T_2 + \mathit{\Delta} T_2$, respectively.



We then ask if there is any relation like (1). Since there is a steady heat current, the naive heat Q_1, Q_2 (heat transferred from each bath to the system) diverge linearly in time. It is thus impossible to look for a relation which involves the heat itself.

Instead of the "bare" heat, we concentrate on the *excess heat* (or the "renormalized" heat), which characterizes the intrinsic heat transfer caused by the change of the temperatures. It is defined by subtracting steady "house-keeping heat" from the (diverging) total heat.



Suppose, for example, that the temperatures are changed suddenly at $t = 0$. Then the heat current $J_1(t)$ from the system to the first bath behaves as in the above figure, where

$J_1^{\rm ss}(T_1,T_2)$ denotes the steady heat current in the NESS. The are of the shaded region is the excess heat $Q_1^{\rm ex}$.

More generally, when the temperatures $T_1(t)$, $T_2(t)$ are given as functions of time t, the excess heat is defined by

$$Q_1^{\rm ex}=\int_{-\infty}^{\infty}dt\,\{J_1(t)-J_1^{\rm ss}(T_1(t),T_2(t))\}. \tag{3}$$

Recently, in T. S. Komatsu, N. Nakagawa, S. Sasa, and H. Tasaki, Phys. Rev. Lett. **100**, 230602 (2008) (archived as 0711.0246), we have shown (but not proved) the extended Clausius relation for NESS

$$S(T_1+\varDelta T_1,T_2+\varDelta T_2)-S(T_1,T_2)=\frac{Q_1^{\rm ex}}{T_1}+\frac{Q_2^{\rm ex}}{T_2}+O((T_2-T_1)^2\varDelta T), \tag{4}$$

which is a natural extension of the equilibrium relation (1). Moreover we found that the entropy here is given by the following "symmetrized Shannon" entropy

$$S(T)=-k_{\rm B}\int d\boldsymbol{q}d\boldsymbol{p}\,\rho_{\rm ss}^{(T_1,T_2)}(\boldsymbol{q},\boldsymbol{p})\,\log\sqrt{\rho_{\rm ss}^{(T_1,T_2)}(\boldsymbol{q},\boldsymbol{p})\,\rho_{\rm ss}^{(T_1,T_2)}(\boldsymbol{q},-\boldsymbol{p})}. \tag{5}$$

We hope that these relations become starting points of further rich exploration of thermodynamics and statistical mechanics of NESS.

Hydrodynamic limit for two-species exclusion processes with one conserved quantity

Makiko Sasada (The University of Tokyo)

1 Introduction

We discuss the hydrodynamical behavior of two-species exclusion processes. A special case was studied by Quastel [3]. Our results can be applied to establish the hydrodynamic limit for the evolution of height differences in interfaces governed by the 1-dimensional SOS dynamics.

We consider two-species exclusion processes on the d-dimensional discrete torus $\mathbb{T}_N^d := (\mathbb{Z}/N\mathbb{Z})^d = \{0, 1..., N-1\}^d$ taking the effects of exchange, creation and annihilation into account. The model is, in general, of nongradient type. We prove that the particle density converges to the solution of a certain nonlinear diffusion equation under a diffusive rescaling in space and time.

2 Model

The two-species exclusion process describes the evolution of a system of mechanically distinguishable particles, say $+$particles and $-$particles under the constraint that at most one particle can occupy each site. The state space of the process is given by $\chi_N^d := \{-1, 0, 1\}^{\mathbb{T}_N^d}$ and its elements (called configurations) are denoted by $\eta = (\eta(x), x \in \mathbb{T}_N^d)$, with $\eta(x) = 0$ or 1 or -1 depending on whether $x \in \mathbb{T}_N^d$ is empty or occupied by a $+$particle or a $-$particle. Each particle moves to a neighboring empty site with the constant jump rate $C_\pm > 0$, respectively. Two different types of neighboring particles exchange their location with the constant rate $C_E \geq 0$. Also they annihilate simultaneously when they are neighboring with the constant rate $C_A \geq 0$, and two different types of particles are created with the constant rate $C_C \geq 0$ if two empty sites are neighboring. The generator of this Markov process denoted by L_N is defined as

$$(L_N f)(\eta) = \sum_{x,y \in \mathbb{T}_N^d, |x-y|=1} L_{xy} f(\eta)$$

for $f : \chi_N^d \to \mathbb{R}$, where

$$\begin{aligned} L_{xy} f(\eta) = {} & C_+ 1_{\{\eta(x)=1, \eta(y)=0\}} (f(\eta^{x,y}) - f(\eta)) \; + \; C_- 1_{\{\eta(x)=0, \eta(y)=-1\}} (f(\eta^{x,y}) - f(\eta)) \\ & + C_E 1_{\{\eta(x)=-1, \eta(y)=1\}} (f(\eta^{x,y}) - f(\eta)) \; + \; C_A 1_{\{\eta(x)=1, \eta(y)=-1\}} (f(\eta^{x=0,y=0}) - f(\eta)) \\ & + C_C 1_{\{\eta(x)=0, \eta(y)=0\}} (f(\eta^{x=-1,y=1}) - f(\eta)). \end{aligned}$$

In the above formula, $\eta^{x,y}$, $\eta^{x=-1,y=1}$ and $\eta^{x=0,y=0}$ stand for

$$\eta^{x,y}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ \eta(y) & \text{if } z = x, \\ \eta(x) & \text{if } z = y, \end{cases}$$

$$\eta^{x=m,y=k}(z) = \begin{cases} \eta(z) & \text{if } z \neq x, y, \\ m & \text{if } z = x, \\ k & \text{if } z = y, \end{cases}$$

respectively.

Remark 1. *If $C_A > 0, C_C > 0$, the process has a unique conserved quantity $\sum_{x\in\mathbb{T}_N^d} \eta(x)$. On the other hand, if $C_A = C_C = 0$, then the process has two conserved quantities $\sum_{x\in\mathbb{T}_N^d} 1_{\{\eta(x)=1\}}$ and $\sum_{x\in\mathbb{T}_N^d} 1_{\{\eta(x)=-1\}}$.*

3 Main Theorem

In this talk, we consider the case where $C_A > 0$ and $C_C > 0$, so that the process has a unique conserved quantity. This process is reversible with respect to the following one parameter family of translation invariant product measure ν_α.

Definition 1. *For each fixed $\alpha \in [-1, 1]$, let ν_α be a product measure on χ_N^d with marginals given by*

$$\begin{aligned} \nu_\alpha\{\eta(x) = 1\} &= \frac{1 - \Phi(\alpha) + \alpha}{2} \\ \nu_\alpha\{\eta(x) = 0\} &= \Phi(\alpha) \\ \nu_\alpha\{\eta(x) = -1\} &= \frac{1 - \Phi(\alpha) - \alpha}{2}, \end{aligned}$$

for all $x \in \chi_N^d$, where

$$\Phi(\alpha) = \begin{cases} \frac{1-\sqrt{4\beta+\alpha^2-4\beta\alpha^2}}{1-4\beta} & \text{if } \beta \neq \frac{1}{4} \\ \frac{1-\alpha^2}{2} & \text{if } \beta = \frac{1}{4} \end{cases}$$

and $\beta = \frac{C_C}{C_A}$.

The index α stands for the density of particles with charge, namely $E_{\nu_\alpha}[\eta(0)] = \alpha$. We will abuse the same notation ν_α for the product measures on the configuration spaces χ_N^d or $\chi^d := \{-1, 0, 1\}^{\mathbb{Z}^d}$ on the torus or on the infinite lattice.

For a directed bond $b = (x, y)$ and a local functions f, let us define $\nabla_{xy} f$ by

$$\begin{aligned} (\nabla_{xy} f)(\eta) = {} & \sqrt{C_+} 1_{\{\eta(x)=1,\eta(y)=0\}}(f(\eta^{x,y}) - f(\eta)) + \sqrt{C_-} 1_{\{\eta(x)=0,\eta(y)=-1\}}(f(\eta^{x,y}) - f(\eta)) \\ & + \sqrt{C_E} 1_{\{\eta(x)=-1,\eta(y)=1\}}(f(\eta^{x,y}) - f(\eta)) + \sqrt{C_A} 1_{\{\eta(x)=1,\eta(y)=-1\}}(f(\eta^{x=0,y=0}) - f(\eta)) \\ & + \sqrt{C_C} 1_{\{\eta(x)=0,\eta(y)=0\}}(f(\eta^{x=-1,y=1}) - f(\eta)). \end{aligned}$$

Let τ_x be the shift operator acting on local functions f and configurations η as follows:

$$\tau_x f(\eta) = f(\tau_x \eta), \quad (\tau_x \eta)(z) := \eta(z - x), \quad x, z \in \mathbb{Z}^d.$$

For every cylinder function $g : \chi^d \to \mathbb{R}$, consider the formal sum

$$\Gamma_g := \sum_{x \in \mathbb{Z}^d} \tau_x g$$

which does not make sense but for which

$$\nabla \Gamma_g = (\nabla_{0,e_1} \Gamma_g, ..., \nabla_{0,e_d} \Gamma_g)$$

is well defined. We are now in a position to define the diffusion coefficient. For each α, define

$$d(\alpha) = \frac{1}{\chi(\alpha)} \inf_g E_{\nu_\alpha}[(\nabla_{0,e}(\eta(0) + \Gamma_g))^2]$$

where $\inf_g$ is taken over all local functions g and e is a unit vector of arbitrary direction. In this formula $\chi(\alpha)$ stands for the so-called static compressibility which in our case is equal to

$$\chi(\alpha) = E_{\nu_\alpha}[\eta(0)^2] - E_{\nu_\alpha}[\eta(0)]^2 = 1 - \Phi(\alpha) - \alpha^2.$$

For a probability measure μ^N on χ_N^d, we denote $\mathbb{P}_{\mu^N}$ the distribution on the path space $D(\mathbb{R}_+, \chi_N^d)$ of the Markov process $\eta_t = \{\eta_t(x), x \in \mathbb{T}_N^d\}$ with generator $N^2 L_N$, which is accelerated by a factor N^2, and the initial measure μ^N.

Theorem 1. *Let $(\mu^N)_{N \geq 1}$ be a sequence of probability measures on χ_N^d such that the corresponding initial density fields satisfy*

$$\lim_{N \to \infty} \mu^N [| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G(\frac{x}{N}) \eta(x) - \int_{\mathbb{T}^d} G(u) \rho_0(u) du| > \delta] = 0,$$

for every $\delta > 0$, every continuous function $G : \mathbb{T}^d := [0,1)^d \to \mathbb{R}$ and some initial density profile $\rho_0 : \mathbb{T}^d \to [-1,1]$. Then, for every $t > 0$,

$$\limsup_{N \to \infty} \mathbb{P}_{\mu^N} [| \frac{1}{N^d} \sum_{x \in \mathbb{T}_N^d} G(\frac{x}{N}) \eta_t(x) - \int_{\mathbb{T}^d} G(u) \rho(t,u) du| > \delta] = 0,$$

for every $\delta > 0$ and every continuous function $G : \mathbb{T}^d \to \mathbb{R}$, where $\rho(t,u)$ is the unique bounded weak solution of the following nonlinear parabolic equation:

$$\begin{cases} \partial_t \rho(t,u) = \Delta(\tilde{d}(\rho(t,u))) = \sum_{i=1}^d \frac{\partial^2}{\partial u_i^2} \tilde{d}(\rho(t,u)) \\ \rho(0, \cdot) = \rho_0(\cdot), \end{cases}$$

and

$$\tilde{d}(\alpha) = \int_{-1}^{\alpha} d(\gamma) d\gamma.$$

Remark 2. *If we assume* $C_+ + C_- - C_A - 2C_E = 0$*, then our model turns out to be a gradient system. In this case,* $d(\alpha) = -\frac{\Phi'(\alpha)}{2}(C_+ - C_-) + \frac{1}{2}(C_+ + C_-)$ *holds.*

Remark 3. *Generalized exclusion process with* $\kappa = 2$ *is corresponding to our model with* $C_+ = C_- = C_A = C_C = 1$ *and* $C_E = 0$.

Remark 4. *The SOS dynamics describe the evolution of the integer-valued heights of interfaces on the discrete lattice. In the 1-dimensional case, the height difference of SOS dynamics and the configuration of the two-species exclusion process have one-to-one correspondence, see, e.g. [1].*

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Behavior of the massless Gaussian field interacting with the wall

Hironobu Sakagawa *

The massless Gaussian field is one of the probabilistic models of phase separating random interfaces and is represented as a Gibbs random field with long range correlations. The field exhibits many interesting behaviors, especially under the effect of various external potentials (wall, pinning, etc.) and its study has been quite active in recent years (cf. [7] and references therein). In this talk, we discuss the following two topics related to the behavior of the massless Gaussian field interacting with the wall.

- Entropic repulsion of the massless field with self-potentials.
- Confinement of the massless field between two hard walls.

1. The massless Gaussian field

Let $d \geq 2$ and $\Lambda_N = [-N, N]^d \cap \mathbb{Z}^d$. For a configuration $\phi = \{\phi_x\}_{x \in \Lambda_N} \in \mathbb{R}^{\Lambda_N}$, consider the following massless Hamiltonian with quadratic interaction potential:

$$H_N(\phi) := \frac{1}{8d} \sum_{\substack{\{x,y\} \cap \Lambda_N \neq \phi \\ |x-y|=1}} (\phi_x - \phi_y)^2.$$

The corresponding Gibbs measure with 0-boundary conditions is defined by

$$P_N(d\phi) := \frac{1}{Z_N} \exp\{-H_N(\phi)\} \prod_{x \in \Lambda_N} d\phi_x \prod_{x \notin \Lambda_N} \delta_0(d\phi_x), \tag{1}$$

where $d\phi_x$ denotes Lebesgue measure on $\mathbb{R}$ and Z_N is a normalization factor. By summation by parts, this coincides with the law of the centered Gaussian lattice field on $\mathbb{R}^{\Lambda_N}$ whose covariance matrix is given by $(-\Delta_N)^{-1}$, the inverse of a discrete Laplacian on Λ_N with Dirichlet boundary conditions outside Λ_N. The configuration ϕ is interpreted as an effective modelization of a random phase separating interface embedded in $d+1$-dimensional space and the spin ϕ_x at site $x \in \Lambda_N$ denotes its height. This model is called a lattice massless field or a harmonic crystal.

It is well-known that the field has long range correlations under P_N and the following asymptotic behavior of the variance holds:

$$\mathrm{Var}_{P_N}(\phi_0) = (-\Delta_N)^{-1}(0,0) \sim \begin{cases} g_2 \log N & \text{if } d = 2, \\ g_d & \text{if } d \geq 3, \end{cases} \tag{2}$$

as $N \to \infty$, where $g_2 = \frac{2}{\pi}$ and $g_d = (-\Delta)^{-1}(0,0)$ for $d \geq 3$. Δ is a discrete Laplacian on $\mathbb{Z}^d$. Under P_N, the interface is said to be delocalized when $d = 2$ because the variance diverges as $N \to \infty$. While, when $d \geq 3$ the interface is localized because the variance remains finite. If $d \geq 3$, the above variance estimate ensures the existence of the infinite volume limit P_∞. Actually this is given by the law of the centered Gaussian field on $\mathbb{R}^{\mathbb{Z}^d}$ with covariance matrix $(-\Delta)^{-1}$.

*Department of Mathematics, Faculty of Science and Technology, Keio University, 3-14-1 Hiyoshi, Kouhoku-ku, Yokohama 223-8522, JAPAN. *E-mail address*: sakagawa@math.keio.ac.jp

2. Entropic repulsion of the massless field with self-potentials

Entropic repulsion is a problem to study how high an interface is pushed up by a hard wall. The corresponding event is given by

$$\Omega_N^+ := \{\phi; \phi_x \geq 0 \text{ for every } x \in \Lambda_N\}.$$

The representative result is the following:

Theorem 1 ([1], [2]). *Let $d \geq 2$. For every $\varepsilon > 0$ and $\delta > 0$ it holds that*

$$\lim_{N\to\infty} \inf_{x\in\Lambda_{N,\varepsilon}} P_N\Big(\Big| \frac{1}{\sqrt{\log_d(N)}}\phi_x - \sqrt{4g_d}\Big| \leq \delta \;\Big|\; \Omega_N^+\Big) = 1, \tag{3}$$

where $\log_2(N) = (\log N)^2, \log_d(N) = \log N$ *for* $d \geq 3$ *and* $\Lambda_{N,\varepsilon} = \{x \in \Lambda_N; \mathrm{dist}(x, \Lambda_N^c) \geq \varepsilon N\}$.

Namely, the field is pushed up to the level $\sqrt{4g_d}\sqrt{\log_d(N)}$. Compared this result with (2), we see that the hard wall pushes up the interface further at the order of $\sqrt{\log N}$ for every $d \geq 2$ and the interface turns to be delocalized when $d \geq 3$. This is caused by the random fluctuation of the interface naturally arises from the Lebesgue measure $d\phi$ in the Gibbs measure (1), in other words, by entropic effects of the measure. The interface is shifted above to keep enough width of the fluctuation.

We consider the generalization of Theorem 1 to the massless field with self-potentials. For $U : \mathbb{R} \to \mathbb{R}$, let P_N^U be a Gibbs measure corresponding to the Hamiltonian:

$$H_N^U(\phi) := H_N(\phi) + \sum_{x\in\Lambda_N} U(\phi_x).$$

Z_N^U denotes the corresponding partition function. Then, the conditioned measure $P_N(\,\cdot\,|\Omega_N^+)$ in (3) can be considered as a Gibbs measure with the (formal) self-potential $U(r) = \infty \cdot I(r < 0)$. The following result means that entropic repulsion for the massless field occurs even under quite weak repulsive force. We only require that the self-potential is non-increasing and the corresponding Gibbs measure is well-defined.

Theorem 2 ([6]). *Let $d \geq 2$ and $U : \mathbb{R} \to \mathbb{R}$ be an arbitrary non-increasing, non-constant function which satisfies $Z_N^U < \infty$. Then, for every $\varepsilon > 0, \delta > 0$ and $\gamma > 0$, the following holds.*

$$\lim_{N\to\infty} P_N^U\Big(\sharp\Big\{x \in \Lambda_{N,\varepsilon}; \frac{1}{\sqrt{\log_d(N)}}\phi_x \leq (1-\delta)\sqrt{4g_d}\Big\} \geq \gamma|\Lambda_{N,\varepsilon}|\Big) = 0.$$

Also, if U satisfies the condition: there exists $a \in \mathbb{R}$ such that $U(r) = const.$ for every $r \geq a$, then we have

$$\lim_{N\to\infty} P_N^U\Big(\sharp\Big\{x \in \Lambda_{N,\varepsilon}; \frac{1}{\sqrt{\log_d(N)}}\phi_x \geq (1+\delta)\sqrt{4g_d}\Big\} \geq \gamma|\Lambda_{N,\varepsilon}|\Big) = 0.$$

3. Confinement between two hard walls

In the single wall case the interface has a room to move away from a hard wall and it is repelled to the level that it can fluctuate freely without feeling the constraint by the wall. Next we consider the situation where this is not the case, namely, we are interested in the behavior of the interface confined between two hard walls. The corresponding event is given by

$$\mathcal{W}_A(L) := \{\phi; |\phi_x| \leq L \text{ for every } x \in A\}, \; A \subset \mathbb{Z}^d.$$

This problem was originally investigated by Bricmont et.al. [3], see also section 4 of [7]. They showed that under the two walls condition the field turns to be massive and the following large L asymptotics holds. P_∞^L denotes the infinite volume limit of the conditioned measure $P_N(\,\cdot\,|\mathcal{W}_{\Lambda_N}(L))$.

- Free energy :

$$\lim_{N\to\infty} -\frac{1}{N^d}\log P_N(\mathcal{W}_{\Lambda_N}(L)) = \begin{cases} e^{-O(L)}, & \text{if } d = 2, \\ e^{-O(L^2)}, & \text{if } d \geq 3, \end{cases}$$

- Mass (inverse correlation length):

$$\lim_{|x|\to\infty} -\frac{1}{|x|}\log E^{P_\infty^L}[\phi_0\phi_x] = \begin{cases} e^{-O(L)}, \text{ if } d=2, \\ e^{-O(L^2)}, \text{ if } d\geq 3, \end{cases}$$

- Variance :

$$\begin{aligned} &\mathrm{Var}_{P_\infty^L}(\phi_0) = O(L), \text{ if } d=2, \\ &0 \leq \mathrm{Var}_{P_\infty}(\phi_0) - \mathrm{Var}_{P_\infty^L}(\phi_0) \leq e^{-O(L^2)}, \text{ if } d\geq 3. \end{aligned}$$

Our main purpose here is to make refinement of these results. Especially, we can show the precise asymptotic behavior of these quantities in the higher dimensional case $d\geq 3$. We first give the free energy estimate.

Proposition 1 ([4]). *Let $d\geq 3$. For every L large enough, there exists $N_0 = N_0(L)$ such that the following holds for every $N\geq N_0$:*

$$-\frac{1}{N^d}\log P_N(\mathcal{W}_{\Lambda_N}(L)) = e^{-\frac{1}{2g_d}L^2(1+o_L(1))}.$$

Next, we consider the correlation and variance of the field under the two walls condition. For mathematical rigorousness of the proof, we treat a slightly modified model. Let T_N be a d-dimensional lattice torus with size $2N$ (we identify N and $-N$ in Λ_N) and consider the following Hamiltonian:

$$H_{N,m}(\phi) := \frac{1}{8d}\sum_{\substack{\{x,y\}\subset T_N \\ |x-y|=1}} (\phi_x-\phi_y)^2 + \frac{1}{2}m^2\sum_{x\in T_N}\phi_x^2.$$

$P_{N,m}$ is the corresponding Gibbs measure on $\mathbb{R}^{T_N}$ with periodic boundary conditions and $P_{\infty,m}^L$ denotes the infinite volume limit of the conditioned measure $P_{N,m}(\,\cdot\,|\mathcal{W}_{T_N}(L))$. Then we have the following estimates:

Theorem 3 ([5]). *Let $d\geq 3$. For every $\varepsilon>0$, there exists $L_0 = L_0(\varepsilon) > 0$ such that the following holds for every $L\geq L_0$:*

1. $$\liminf_{m\to 0}\liminf_{k\to\infty}\Big\{-\frac{1}{k}\log E^{P_{\infty,m}^L}\big[\phi_0\phi_{[kz]}\big]\Big\} \geq e^{-\frac{1+\varepsilon}{4g_d}L^2},$$
$$\limsup_{m\to 0}\limsup_{k\to\infty}\Big\{-\frac{1}{k}\log E^{P_{\infty,m}^L}\big[\phi_0\phi_{[kz]}\big]\Big\} \leq e^{-\frac{1-\varepsilon}{4g_d}L^2},$$
for every $z\in\mathbb{S}^{d-1} = \{z\in\mathbb{R}^d; |z|=1\}$.

2. $$\liminf_{m\to 0}\Big\{\mathrm{Var}_{P_\infty}(\phi_0) - \mathrm{Var}_{P_{\infty,m}^L}(\phi_0)\Big\} \geq e^{-\frac{1+\varepsilon}{2g_d}L^2},$$
Also, if $d\geq 5$ then we have
$$\limsup_{m\to 0}\Big\{\mathrm{Var}_{P_\infty}(\phi_0) - \mathrm{Var}_{P_{\infty,m}^L}(\phi_0)\Big\} \leq e^{-\frac{1-\varepsilon}{2g_d}L^2}.$$

Remark 1. *Path description of the behavior of the interface between two walls (centering of the interface) is also discussed in [4].*

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Title: *Gibbs measures on combinatorial structures.*

Speaker: *Shankar Bhamidi, UBC and PIMS*

Abstract:

We shall report on some work in progress in understanding various aspects of Gibbs distributions on combinatorial structures and shall describe why such problems are of crucial importance, both in practice and for the rich mathematical theory that one can develop.

We shall first look at the space of all unordered trees, labeled trees on N nodes, say $\mathcal{T}_N$. For each tree $\mathbf{t} \in \mathcal{T}_n$ consider the case where the Hamiltonian consists of the number of leaves namely

$$H(\mathbf{t}) = \beta \# \text{ of leaves in } \mathbf{t}$$

for some constant β. Now consider the probability measure on $\mathcal{T}_n$ given by

$$p_N(\mathbf{t}) = \frac{\exp(\beta H(\mathbf{t}))}{Z_N(\beta)}$$

Note that as $\beta \to \infty$ we shall end up with star like trees while as $\beta \to -\infty$ we shall end up with path like trees. We shall identify these regimes of this model (star like, continuum random tree like and path like regimes) and exhibit where the phase transitions occur. We shall also show how these methods can be used to get large deviation results for the uniform random tree model as well as how general theory and work of the speaker and his co-workers (Arnab Sen and Steve Evans at UC Berkeley) implies the convergence of the spectral distribution of the adjacency matrix of a tree $\mathbf{t} \sim p_N(\cdot)$ to a non-degenerate non-random distribution.

We shall then report on some recent work on a related model, the exponential random graph model. Let us first put this problem in the context of recent research and describe why such problems are cruicial. A variety of random graph models have been developed in recent years to study a range of problems on networks, driven by the wide availability of data from many social, telecommunication, biochemical and other networks. The exponential random graph model is a key model, extensively used in the sociology literature. This model seeks to incorporate in random graphs the notion of reciprocity, that is, the larger than expected number of triangles and other small subgraphs. A simple example is the following: Consider the space of all simple graphs on N vertices and call this space $\mathcal{X}$. Let $X \in \mathcal{X}$ be a graph and let $E(X)$ and $T_1(X)$ denote the number of edges and number of triangles in the graph respectively. Fix constants β_0 and β_1 and consider the probability measure $p_N(X)$ given by

$$p_N(X) = \frac{\exp(H(X))}{Z_N(\beta)}$$

where the Hamiltonian is given by

$$H(X) = \beta_0 E(X) + \beta_1 T_1(X)$$

Sampling from these distributions is crucial for parameter estimation hypothesis testing, and more generally for understanding basic features of the network model itself. In practice sampling is typically carried out using Markov chain Monte Carlo, in particular either the Glauber dynamics or the Metropolis-Hasting procedure.

In the talk we characterize the high and low temperature regimes of the exponential random graph model. We establish that in the high temperature regime the mixing time of the Glauber dynamics is $\Theta(n^2 \log n)$, where n is the number of vertices in the graph; in contrast, we show that in the low temperature regime the mixing is exponentially slow for any local Markov chain. Our results, moreover, give a rigorous basis for criticisms made of such models. In the high temperature regime, where sampling with MCMC is possible, we show that any finite collection of edges are asymptotically independent; thus, the model does not possess the desired reciprocity property, and is not appreciably different from the Erdős-Rényi random graph. The last bit is based on joint work done with Guy Bresler and Allan Sly of UC Berkeley.

Phase Transitions for Linear Stochastic Evolutions[1]

Nobuo YOSHIDA[2] (Kyoto University)

We consider a discrete-time stochastic growth model on the d-dimensional lattice. The growth model describes various interesting examples such as oriented site/bond percolation, directed polymers in random environment, time discretizations of binary contact path process. We first investigate the regular/slow growth phase transition in terms of the growth rate of the total populaiton. Then, we explain that the regular/slow growth phase transition is related to the delocalization/localization transition of the spatial distribution of the population.

1 The set-up

Let $A = (A_{x,y})_{x,y \in \mathbb{Z}^d}$ be a random matrix and let $A_1, A_2, ...$ be its independent copies, defined on a probability space $(\Omega, \mathcal{F}, P)$. Here are the set of assumptions we assume for A:

$$0 \le A_{x,y} \in \mathbb{L}^2(P) \text{ for all } x, y \in \mathbb{Z}^d. \tag{1.1}$$

$$A_{x,y} = 0 \text{ a.s. if } |x - y| > r_A \text{ for some non-random } r_A \in \mathbb{N}. \tag{1.2}$$

$$(A_{x+z,y+z})_{x,y \in \mathbb{Z}^d} \stackrel{\text{law}}{=} A \text{ for all } z \in \mathbb{Z}^d. \tag{1.3}$$

$$\text{The columns } \{A_{\cdot,y}\}_{y \in \mathbb{Z}^d} \text{ are independent.} \tag{1.4}$$

$$\begin{array}{l}\text{The set } \{x \in \mathbb{Z}^d \ ; \ \sum_{y \in \mathbb{Z}^d} a_{x+y} a_y \neq 0\} \text{ contains a linear basis of } \mathbb{R}^d, \\ \text{where } a_y = P[A_{0,y}].\end{array} \tag{1.5}$$

We define a Markov chain $N_t = (N_{t,y})_{y \in \mathbb{Z}^d}$, $t \in \mathbb{N}$, with values in $[0, \infty)^{\mathbb{Z}^d}$ by

$$N_{t,y} = \sum_{x \in \mathbb{Z}^d} N_{t-1,x} A_{t,x,y}, \quad t = 1, 2, ... \tag{1.6}$$

Here and in the sequel, we suppose that the initial state N_0 is non-random, $\neq 0$, and $\in \ell^1(\mathbb{Z}^d)$. If we regard $N_t \in [0, \infty)^{\mathbb{Z}^d}$ as a row vector, (1.6) can be interpreted as

$$N_t = N_0 A_1 A_2 \cdots A_t, \quad t = 1, 2, ...$$

2 Results

We look at the growth rate of the "total number" of particles:

$$|N_t| = \sum_{y \in \mathbb{Z}^d} N_{t,y} \quad t = 1, 2, ...$$

which will be kept finite for all t by our assumptions. It is easy to show that $|N_t|/|a|^t$ is a martingale, where

$$|a| = \sum_y a_y, \quad a_y = P[A_{0,y}], \tag{2.1}$$

[1]Talk at "Random processes and systems", February 16–19, Kyoto University.

[2][e-mail] `nobuo@math.kyoto-u.ac.jp`, [URL] `http://www.math.kyoto-u.ac.jp/~nobuo`

so that $|a|^t$ can be considered as the mean growth rate of $|N_t|$. We first investigate whether the limit:

$$|\overline{N}_\infty| \stackrel{\text{def}}{=} \lim_{t\to\infty} |N_t|/|a|^t \tag{2.2}$$

vanishes almost surely or not. Our results on the positivity of (2.2) can be summarized as follows (cf. [Yo08a]):

i) If $d \geq 3$ and the matrix A_t is not "too random", then, $|\overline{N}_\infty| > 0$ with positive probability.

ii) In any dimension d, if the matrix A_t is "random enough", then, $|\overline{N}_\infty| = 0$, almost surely. Moreover, the convergence is exponentially fast.

iii) For $d = 1, 2$, $|\overline{N}_\infty| = 0$, almost surely, under mild assumptions on A_t. Moreover, the convergence is exponentially fast for $d = 1$.

We will refer i) as *regular growth phase*, and ii)—iii) as *slow growth phase*. In the regular growth phase, $|N_t|$ grows as fast as its mean growth rate with positive probability, whereas in the slow growth phase, the growth of $|N_t|$ is slower than its mean growth rate almost surely. There is a close connection between the growth rate of $|N_t|$ and the spatial distribution of the particles:

$$\rho_t(x) = \frac{N_{t,x}}{|N_t|}\mathbf{1}_{\{|N_t|>0\}}, \quad x \in \mathbb{Z}^d \tag{2.3}$$

as $t \nearrow \infty$. The connection is roughly as follows (with some technical assumptions disregarded).

(iv) The regular growth implies that, conditionally on the event $\{|\overline{N}_\infty| > 0\}$, the spatial distribution (2.3) has a Gaussian scaling limit togeter with the *delocalization property*:

$$\lim_{t\to\infty} \sup_{x\in\mathbb{Z}^d} \rho_t(x) = 0, \ \text{ in probability.} \tag{2.4}$$

cf. [Na08].

(v) In contrast to (iv) above, the slow growth triggers the *path localization* (cf. [Yo08b]). In the slow growth phase, there exists $c \in (0, 1)$ such that,

$$\{|N_t| > 0 \text{ for all } t \in \mathbb{N}\} = \left\{ \sup_{x\in\mathbb{Z}^d} \rho_t(x) \geq c, \ i.o. \right\} \quad \text{a.s.} \tag{2.5}$$

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Heat transport: a weak coupling approach
Stefano Olla, joint work with Carlangelo Liverani

Let us consider a region $\Lambda \subset \mathbb{Z}^d$, set $N = |\Lambda|$, the number of sites in Λ. At each site we have a ν-dimensional, $\nu \geq 2$, nonlinear oscillator and we assume that such oscillators interact weakly via a non-liner potential. The Hamiltonian is given by $(q_i, p_i)_{i\in\Lambda} \in \mathbb{R}^{2\nu N}$

$$H_\varepsilon^\Lambda := \sum_{i\in\Lambda} \frac{1}{2}\|p_i\|^2 + \sum_{i\in\Lambda} U(q_i) + \varepsilon \sum_{|i-j|=1} V(q_i - q_j),$$

where $U, V \in \mathcal{C}^\infty(\mathbb{R}^\nu, \mathbb{R})$. We assume the potential U is a strictly convex with $U(0) = 0$ and $\nabla U(0) = 0$, moreover it is radially symmetric ($U(q) = \bar{U}(\|q\|)$ with $c^{-1} \leq \bar{U}'' \leq c$ for some finite positive constant c. We assume $V'(q)^2 \leq CU(q)$. For simplicity of notations, we choose $\nu = 2$.

In addition to the hamiltonian dynamics, we consider random forces that conserve the kinetic energy of each atom, given by independent diffusions on the spheres $\|p_i\|^2 = cost$). In order to define such diffusions, consider the vector fields

$$X_i := p_i^1 \partial_{p_i^2} - p_i^2 \partial_{p_i^1} =: Jp_i \cdot \partial_{p_i},$$

and the second order operator

$$S = \sum_{i\in\Lambda} X_i^2$$

The generator of the process is given by

$$L_{\varepsilon,\Lambda} =: A_\varepsilon + \sigma^2 S$$

where $A_\varepsilon = \{H_\varepsilon^\Lambda, \cdot\}$, the usual Hamiltonian operator.

The single particles energies are

$$\mathcal{E}_i^\varepsilon(q,p) = \frac{1}{2}\|p_i\|^2 + U(q_i) + \frac{1}{2d}\varepsilon \sum_{|i-j|=1} V(q_i - q_j).$$

The time evolution of these energies is given by:

$$\frac{d\mathcal{E}_i^\varepsilon}{dt} = \varepsilon \sum_{|i-k|=1} j_{i,k} \tag{1}$$

where the energy currents are defined by

$$j_{i,k} = \frac{1}{2d}\nabla V(q_i - q_k) \cdot (p_i + p_k) \tag{2}$$

Note that $j_{i,k} = -j_{k,i}$ and that this is a function only of q_i, p_i, q_k, p_k.

If $\varepsilon = 0$ the dynamics is given by non-interacting oscillators, and consequently the energy of each oscillator is a conserved quantity. So for $\varepsilon = 0$ there is a family of equilibrium measure parametrized by the vector $\underline{a} = (a_i)_{i\in\Lambda}$ of the energy of each oscillator. This is given by $\mu_{\underline{a}}^\Lambda$, the Liouville measure associated to the Hamiltonian flow H_0^Λ on the surface

$$\Sigma_{\underline{a}} := \{q, p \ : \ a_i = \mathcal{E}_i^0(q,p) = \frac{1}{2}\|p_i\|^2 + U(q_i)\} = \bigotimes_{i\in\Lambda} \Sigma_{a_i}.$$

This is also called microcanonical measure. Clearly, letting μ_a be the Liouville measure on the 3 dimensional surface Σ_a, we have $\mu_{\underline{a}}^\Lambda = \otimes_{i\in\Lambda}\mu_{a_i}$. By the symmetry between p and $-p$ it follows that $\mu_{\underline{a}}(j_{i,k}) = 0$ for each $\underline{a}$.

We are interested in the stochastic process of the time rescaled energies

$$\boldsymbol{\mathcal{E}}_i^\varepsilon(t) = \mathcal{E}_i^\varepsilon(q(\varepsilon^{-2}t), p(\varepsilon^{-2}t)).$$

In order to define the parameters of the macroscopic evolution, consider the dynamics of 2 non-interacting oscillators ($\varepsilon = 0$), each starting with the micro-canonical distribution with corresponding energy a_1 and a_2. Let us denote by $\mathbb{E}_{a_1,a_2}(\cdot)$ the corresponding expectation in this equilibrium measure. This permits us to define the following positive function on $\mathbb{R}_+^2$

$$\gamma^2(a_1, a_2) = \int_0^\infty \mathbb{E}_{a_1,a_2}\left(j_{1,2}(\omega_t) j_{1,2}(\omega_0)\right)\, dt \tag{3}$$

We prove that $\gamma^2 \in \mathcal{C}^\infty(\mathbb{R}_+^2)$. Notice that γ^2 is a symmetric function of a_1, a_2. Correspondingly we define the *macroscopic* current by the antisymmetric function

$$\alpha(a_1, a_2) = \sigma^2(\partial_{a_1} - \partial_{a_2})\gamma^2(a_1, a_2). \tag{4}$$

Here is our main result:

Theorem 1. *In the limit $\varepsilon \to 0$, the process $\{\boldsymbol{\mathcal{E}}_i^\varepsilon\}_{i\in\Lambda}$ converges, in law, to the stochastic process $\{\boldsymbol{\mathcal{E}}_i\}_{i\in\Lambda}$ determined by the stochastic differential equations*

$$d\boldsymbol{\mathcal{E}}_i = \sum_{|i-k|=1} \alpha(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k) dt + \sum_{|i-k|=1} \sigma\gamma(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k) dB_{\{i,k\}} \tag{5}$$

where $B_{\{i,k\}} = -B_{\{k,i\}}$ are independent standard Brownian motions.

Notice that the generator of this diffusion on $\mathbb{R}_+^\Lambda$ is given by

$$\mathcal{L} = \sum_{i\in\Lambda} \sum_{|k-i|=1} \left(\sigma^2\gamma(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k)^2(\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k})^2 + \alpha(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k)(\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k})\right) \tag{6}$$

and, for any inverse temperature $\beta > 0$, the product probability measure

$$\prod_{i\in\Lambda} \beta e^{-\beta\boldsymbol{\mathcal{E}}_k} \tag{7}$$

is stationary and reversible for the diffusion generated by (6).

By (4) we can rewrite the generator as

$$\mathcal{L} = \sigma^2 \sum_{\substack{i,k\in\Lambda,\\ |k-i|=1}} (\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k})\gamma^2(\boldsymbol{\mathcal{E}}_i, \boldsymbol{\mathcal{E}}_k)(\partial_{\boldsymbol{\mathcal{E}}_i} - \partial_{\boldsymbol{\mathcal{E}}_k}) \tag{8}$$

The process (5) is close the the one studied by Varadhan in [1]. In this paper Varadhan proves an *hydrodynamic limit*, i. e. that under certain condition on the initial distribution, for any test function G on $\mathbb{R}^d$ we have the convergence

$$\lim_{N\to\infty} \frac{1}{N^d} \sum_i G(i/N)\boldsymbol{\mathcal{E}}_i(N^2 t) = \int G(y)u(y,t)dy \tag{9}$$

where $u(y, t)$ is the solution of a nonlinear heat equation

$$\partial_t u = \nabla D(u)\nabla u \tag{10}$$

Yet our case it is not covered by such result (due to the degenacy at zero of the diffusion coefficients and the non strict convexity of the potential of the invariant measure). In any case the extension of Varadhan's work to the present case would allow to obtain the *heat equation* in the present setting via a diffusive limit. We plan to work on such an extension in the future.

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Self-avoiding walk and the renormalisation group

David Brydges (University of British Columbia)

Self-avoiding walks on $\mathbb{Z}^d$ are simple-random walk paths without self-interstections. Self-avoiding walks of the same length are declared to be equally likely. The basic question is how far on average is their endpoint from the origin? The lace expansion has answered this in dimensions 5 and higher. For $d = 2$, the scaling limit is conjectured to be $\mathrm{SLE}_{8/3}$ modulo reparametrisation. For $d = 3$, there are only numerical results.

I will describe some parts of work in progress with Gordon Slade for the case $d = 4$. Our immediate goal is to prove that the critical two-point function (Green function) for a spread-out model of self-avoiding walks on $\mathbb{Z}^4$ decays like $|x|^{-2}$ at large distances, as it does for simple random walk.

Critical behavior and limit theorems for long-range oriented percolation in high dimensions

Akira Sakai[1]

1. Motivation Oriented percolation on $\mathbb{Z}^d \times \mathbb{Z}_+$ is a stochastic-geometrical model defined as follows:

- Each bond $b = ((u,n),(v,n+1))$, where $u, v \in \mathbb{Z}^d$ and $n \in \mathbb{Z}_+$, is either occupied or vacant. Let $\underline{b} = (u,n)$ and $\overline{b} = (v,n+1)$.
- We say that (x,n) is connected to (y,l), denoted $(x,n) \to (y,l)$, if $(x,n) = (y,l)$ or if there is a sequence of $l-n$ occupied bonds $b_1, b_2, \dots, b_{l-n}$ such that $\underline{b}_1 = (x,n)$, $\overline{b}_{l-n} = (y,l)$ and $\overline{b}_i = \underline{b}_{i+1}$ for all $i \geq 1$.
- Each bond $((u,n),(v,n+1))$ is occupied with probability $pD(v-u)$, independently of the other bonds, where D is a $\mathbb{Z}^d$-symmetric probability distribution, hence $p \geq 0$ is the expected number of occupied bonds per vertex. We denote by $\mathbb{P}_p$ the associated probability measure, and by $\mathbb{E}_p$ its expectation.

Let

$$\varphi_p(x,n) = \mathbb{P}_p\big((o,0) \to (x,n)\big), \qquad Z_p(k;n) = \sum_{x \in \mathbb{Z}^d} e^{ik\cdot x}\varphi_p(x,n), \qquad \hat{\varphi}_p(k,z) = \sum_{n=0}^{\infty} z^n Z_p(k;n).$$

For $\mathcal{C}_n = \{x \in \mathbb{Z}^d : (o,0) \to (x,n)\}$ and $\mathcal{C} = \bigcup_{n=0}^{\infty} \mathcal{C}_n$, we have $\mathbb{E}_p[|\mathcal{C}_n|] = Z_p(0;n)$ and $\mathbb{E}_p[|\mathcal{C}|] = \hat{\varphi}_p(0,1)$. We denote the radius of convergence of $\hat{\varphi}_p(0,z)$ by $|z| = m_p$.

We consider the following long-range model: Let $\alpha > 0$ and suppose that $h : \mathbb{R}^d \to [0,\infty)$ is a rotation-invariant bounded function satisfying the asymptotic behavior $h(x) \sim C_h|x|^{-d-\alpha}$ for some constant $C_h \in (0,\infty)$. We define

$$D(x) = \frac{h(x/L)}{\sum_{y \in \mathbb{Z}^d} h(y/L)} \qquad (x \in \mathbb{Z}^d),$$

where $L < \infty$ is the spread-out parameter which controls the breadth of the potential. Notice that the variance of D does not exist when $\alpha \leq 2$.

It has been known that this model exhibits a phase transition: there is a $p_c = p_c(d,\alpha,L) \geq 1$ such that

$$\chi_p := \hat{\varphi}_p(0,1) \begin{cases} < \infty & (p < p_c), \\ = \infty & (p \geq p_c), \end{cases} \qquad \Theta_p := \mathbb{P}_p(|\mathcal{C}| = \infty) \begin{cases} = 0 & (p \leq p_c), \\ > 0 & (p > p_c). \end{cases}$$

It is of great interest to investigate critical behavior and limit theorems for the observables around $p = p_c$:

$$\chi_p \underset{p\uparrow p_c}{\approx} (p_c - p)^{-\gamma}, \qquad \Theta_p \underset{p\downarrow p_c}{\approx} (p - p_c)^{\beta}, \qquad m_p - m_{p_c} \underset{p\uparrow p_c}{\approx} (p_c - p)^{\tau},$$

$$\mathbb{P}_{p_c}(|\mathcal{C}| \geq n) \underset{n\uparrow\infty}{\approx} n^{-1/\delta}, \qquad Z_{p_c}(0;n) \underset{n\uparrow\infty}{\approx} n^{\eta}.$$

More precisely, we are interested in the existence (in which sense?) of the critical exponents $\gamma, \beta, \delta, \tau, \eta$, the dependence of their values on d, α, L and a limit distribution representing the critical system.

[1] Creative Research Initiative "Sousei", Hokkaoid University. http://www.math.sci.hokudai.ac.jp/∼sakai/

2. Results The first result is about the infrared bound on the Fourier-Laplace transform of the two-point function φ_p:

Theorem 1 ([4]). *For $d > d_c \equiv 2(\alpha \wedge 2)$, there is an $L_0 = L_0(d, \alpha) < \infty$ such that the following holds for all $L \geq L_0$: there is a $C = C(d, \alpha, L) < \infty$ such that, for any $p \in (0, p_c)$, $k \in [-\pi, \pi]^d$, $m < m_p$ and $\theta \in [-\pi, \pi]$,*

$$|\hat{\varphi}_p(k, me^{i\theta})| \leq \frac{C}{1 - \hat{D}(k) + p(m_p - m) + |\theta|}.$$

It is expected that the above infrared bound does not hold for $d < d_c$ (there is some evidence to support this expectation [6]).

Corollary 2 ([1, 2, 3, 4]). *For $d > d_c$, there is an $L_0 = L_0(d, \alpha) < \infty$ such that $m_{p_c} = 1$ and the following hold for all $L \geq L_0$:*

$$\chi_p \underset{p\uparrow p_c}{\asymp} (p_c - p)^{-1}, \qquad \Theta_p \underset{p\downarrow p_c}{\asymp} p - p_c, \qquad m_p - 1 \underset{p\uparrow p_c}{\asymp} p_c - p, \qquad \mathbb{P}_{p_c}(|\mathcal{C}| \geq n) \underset{n\uparrow\infty}{\asymp} n^{-1/2}.$$

Here, $f \asymp g$ means that f/g is bounded away from zero and infinity. In this sense, $\gamma = \beta = \tau = 1$ and $\delta = 2$.

The following theorem is the main result:

Theorem 3 ([4, 5]). *For $k \in \mathbb{R}^d$ and $n \in \mathbb{N}$, we let*

$$k_n = k \times \begin{cases} n^{-\frac{1}{\alpha\wedge 2}} & (\alpha \neq 2), \\ (n \log n)^{-1/2} & (\alpha = 2). \end{cases}$$

For $d > d_c$, there is an $L_0 = L_0(d, \alpha) < \infty$ such that the following hold for all $L \geq L_0$: for any d, α, L and $p \in (0, p_c]$, there are constants $C_1, C_2 \in (0, \infty)$ such that

$$Z_p(0; n) \sim C_1 m_p^{-n}, \qquad \frac{Z_p(k_n; n)}{Z_p(0; n)} \sim \exp(-C_2|k|^{\alpha\wedge 2}) \qquad (k \in \mathbb{R}^d).$$

In particular, $\eta = 0$ in a stronger sense.

The key ingredient for the proof of Theorem 3 is a new fractional-moment method for the lace-expansion coefficients [5], by which we can overcome difficulties due to the heavy tail of D.

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Non-Equilibrium Dynamics of Determinantal Processes with Infinite Particles

Hideki Tanemura, Chiba University
(*joint work with Makoto Katori, Chuo University*)

We denote by $\mathfrak{M}$ the space of nonnegative integer-valued Radon measures on $\mathbb{R}$, which is a Polish space with the vague topology. Any element ξ of $\mathfrak{M}$ can be represented as $\xi(\cdot) = \sum_{j\in\Lambda} \delta_{x_j}(\cdot)$ with a sequence of points in $\mathbb{R}$, $\boldsymbol{x} = (x_j)_{j\in\Lambda}$ satisfying $\xi(K) = \sharp\{x_j : x_j \in K\} < \infty$ for any compact subset $K \subset \mathbb{R}$. The index set $\Lambda = \mathbb{N} \equiv \{1, 2, \dots\}$ or a finite set. We call an element ξ of $\mathfrak{M}$ an unlabeled configuration, and a sequence $\boldsymbol{x}$ a labeled configuration. As a generalization of a notion of determinantal (Fermion) point process on $\mathbb{R}$ for a probability measure on $\mathfrak{M}$ [8, 7], we give the following definition for $\mathfrak{M}$-valued processes.

Definition 1 *An $\mathfrak{M}$-valued process $(\mathbb{P}, \Xi(t), t \in [0,\infty))$ is said to be determinantal with the correlation kernel $\mathbb{K}$, if for any $M \geq 1$, any sequence $(N_m)_{m=1}^M$ of positive integers, any time sequence $0 < t_1 < \cdots < t_M < \infty$, the $(N_1, \dots, N_M)$-multitime correlation function is given by a determinant,*

$$\rho\Big(t_1, \xi^{(1)}; \dots ; t_M, \xi^{(M)}\Big) = \det_{\substack{1\leq j\leq N_m, 1\leq k\leq N_n \\ 1\leq m,n\leq M}} \Big[\mathbb{K}(t_m, x_j^{(m)}; t_n, x_k^{(n)})\Big],$$

where $\xi^{(m)}(\cdot) = \sum_{j=1}^{N_m} \delta_{x_j^{(m)}}(\cdot), 1 \leq m \leq M$.

The process $\Xi(t) = \sum_{j=1}^{N} \delta_{X_j(t)}$ with the SDEs

$$dX_j(t) = dB_j(t) + \sum_{1\leq k\leq N, k\neq j} \frac{dt}{X_j(t) - X_k(t)}, \quad 1 \leq j \leq N, \quad t \in [0,\infty), \tag{0.1}$$

where $B_j(t)$'s are independent one-dimensional standard Brownian motions, starting from its equilibrium measure $\mu_{N,\sigma^2}^{\mathrm{GUE}}$ is determinantal [2]. We call the process $\Xi(t)$ Dyson's model. In this talk we first show that, for any fixed configuration $\xi^N \in \mathfrak{M}$ with $\xi(\mathbb{R}) = N$, Dyson's model starting from ξ^N is determinantal and its correlation kernel $\mathbb{K}^{\xi^N}$ is given by using the *multiple Hermite polynomials*. For $\xi \in \mathfrak{M}$, when $\lim_{L\to\infty} \mathbb{K}^{\xi\cap[-L,L]}$ converges to a locally integrable function, the limit is written as $\mathbb{K}^{\xi}$ and an $\mathfrak{M}$-valued process is defined such that it is determinantal with the correlation kernel $\mathbb{K}^{\xi}$. In this case, we say that the process $(\mathbb{P}_\xi, \Xi(t), t \in [0,\infty))$ is *well defined with the correlation kernel* $\mathbb{K}^{\xi}$. In case $\xi(\mathbb{R}) = \infty$, the process $(\mathbb{P}_\xi, \Xi(t), t \in [0,\infty))$ is Dyson's model with infinite particles. We give sufficient conditions so that the process $(\mathbb{P}_\xi, \Xi(t), t \in [0,\infty))$ is well defined, in which the correlation kernel is generally expressed using a double integral with the heat kernels of an *entire function* represented by an infinite product. The class of configurations satisfying the conditions, denoted by $\mathfrak{Y}$, is large enough to carry the Poisson point processes, Gibbs states with regular conditions, as well as $\mu_{\sin}$, the determinantal point process with the sine kernel. We also show that the process $(\mathbf{P}_{\sin}, \Xi(t), t \in [0,\infty))$, which is obtained from the limit of equilibrium Dyson's model with finite particles [5], is given by

$$\mathbf{P}_{\sin}(\cdot) = \int_{\mathfrak{M}} \mu_{\sin}(d\xi)\mathbb{P}_{\xi}(\cdot) \tag{0.2}$$

and that this infinite-dimensional reversible process is Markovian.

We also discuss generalizations of the above results to the other determinantal processes.

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Interacting Brownian motions with 2D Coulomb potentials

Hirofumi Osada (Kyushu University)

Interacting Brownian motions (IBMs) are infinitely numerous Brownian particles moving in Euclidean spaces with the effect of interaction potentials and self potentials.

In this talk I consider the case that the interacting potentials are 2D Coulomb potentials (logarithmic potentials).

I first present general theorems for the construction of IBMs with 2D Coulomb potentials as diffusions. Second, I present a representation theorem of IBMs as a solution of infinitely dimensional stochastic differential equations.

As an application, I give two examples. One is the Dyson model in infinite dimensions and the other is the Ginibre IBMs. Both models are translation and rotation invariant in space, and as such, are prototypes of dimensions $d = 1, 2$, respectively. Their equilibrium states are thermodynamical limits of the spectrum of random matrices called GOE, GUE and GSE (Dyson model) and the Ginibre ensemble (Ginibre IBMs).

The dynamical properties of these diffusions are very different from one of the IBMs with Ruelle class interacting potentials (the Gibbsian case) because of the strong, long range effect of the logarithmic potentials. I talk about a conjecture on this.

Kinetic limit of the weakly nonlinear Schrödinger equation
with random initial data

Herbert Spohn
Zentrum Mathematik, Technische Universität München

Abstract: We are interested in the nonlinear Schrödinger equation on the d-dimensional lattice $\mathbb{Z}^d$. This is a hyperbolic evolution equation for the complex-valued wave field $\psi : \mathbb{Z}^d \to \mathbb{C}$ and reads

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\psi(x,t) = \sum_{y\in\mathbb{Z}^d} \alpha(x-y)\psi(y,t) + \lambda|\psi(x,t)|^2\psi(x,t)\,. \tag{1}$$

α are the finite range hopping amplitudes, $\alpha : \mathbb{Z}^d \to \mathbb{R}$, $\alpha(x) = \alpha(-x)$. λ is the strength of the nonlinearity, $\lambda \geq 0$. We work on the lattice, since the equilibrium measure for (1) is ultraviolet divergent. (1) is of Hamiltonian type. If the canonical variables are introduced through $\psi(x) = (q_x + ip_x)/\sqrt{2}$, then the Hamiltonian for (1) reads

$$H(\psi) = \sum_{x,y\in\mathbb{Z}^d} \alpha(x-y)\psi(x)^*\psi(y) + \tfrac{1}{2}\lambda \sum_{x\in\mathbb{Z}^d} |\psi(x)|^4 \,. \tag{2}$$

One could also consider other nonlinear wave equations, but the nonlinear Schrödinger equation is invariant under the interchange of q and p which is of advantage in the proof.

(1) is solved as initial value problem. The initial data are random. Usually they have infinite energy, more precisely a bounded energy per unit volume. A standard choice would be to take ψ as a Gaussian random field with $\mathbb{E}(\psi(x)) = 0$, $\mathbb{E}(\psi(x)\psi(y)) = 0$, $\mathbb{E}(\psi(x)^*\psi(y)) = C(x-y)$ with $\widehat{C}(k) \geq 0$ and C a function of rapid decay.

A standard approximation is to consider λ small and to study the first time scale on which the nonlinearity becomes effective, which would be times of order λ^{-2} in our context. Thus, if we denote the time t covariance by $C(x-y,t) = \mathbb{E}(\psi(x,t)^*\psi(y,t))$ then one has to study the limit of

$$C(x-y, \lambda^{-2}t) \quad \text{for } \lambda \to 0 \,. \tag{3}$$

The limit covariance is expected to be governed by a kinetic equation with a cubic nonlinear collision operator. For this reason (3) is called "kinetic limit".

The kinetic limit seems to be of perturbative nature. Still it is surprisingly difficult to prove, despite considerable efforts in the past. We will explain on recent progress towards a proof, which is worked out jointly with Jani Lukkarinen, University of Helsinki. For a more details we refer to arXiv:math-ph/0807.5072.