Portfolio management under risk contraints

Lectures given at MITACS-PIMS-UBC Summer School in Risk Management and Risk Sharing

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Chapter 1

Introduction et notations

The aim of these lectures at MITACS-PIMS-UBC Summer School in Risk Management and Risk Sharing is to discuss risk controlled approaches for the pricing and hedging of financial risks.
We will start with the classical dual approach for financial markets, which allows to rewrite super-hedging problems in terms of optimal control problems in standard form. Based on this, we shall then consider hedging and pricing problems under utility or risk minimization criteria. This approach will turn out to be powerful whenever linear (or essentially linear) problems are considered, but not adapted to more general settings with non-linear dynamics (e.g. large investor models, high frequency trading with market impact features, mixed finance/insurance issues).
In the second part of this lecture, we will develop on a new approach for risk control problems based on a stochastic target formulation. We will see how flexible this approach is and how it allows to characterize very easily super-hedging prices in term of suitable Hamilton-Jacobi-Bellman type partial differential equations (PDEs). We will then see how quantile hedging and expected loss pricing problems can be embeded into this framework, for a very large class of financial models.
The third part will be dedicated to optimal management problems under risk constraints. Based on the results of the previous part, we shall see how they can naturally be embeded into optimal control problems with state constraints. Here the state constraint formulation is somehow unusual as it will be given in terms of a corresponding stochastic target problem associated to the risk constraint.

These lectures are organized in small chapters, each of them being focused on a particular aspect.

1 Notations

We first make precise some notations that will be used in all these notes.

In all these lectures notes, we shall consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ supporting a d-dimensional standard Brownian motion W. In the following, $\mathbb{F} = (\mathcal{F}_t)_{0\le t\le T}$ will denote the completed right-continuous filtration generated by W. Here, $T > 0$ is a finite time horizon. If nothing else is specified, we shall assume that $\mathcal{F}_T = \mathcal{F}$.
Given a sub-algebra $\mathcal{G} \subset \mathcal{F}$ and a set $A \subset \mathbb{R}^d$, we write $L^0(A, \mathcal{G})$ for the set of A-valued $\mathcal{G}$-measurable random variables. We similarly write $L^p(A, \mathbb{Q}, \mathcal{G})$, $\mathbb{Q} \sim \mathbb{P}$ and $p \in (0, \infty]$, to denote random variables in $L^0(A, \mathcal{G})$ with finite p-moment under $\mathbb{Q}$, or essentially bounded if $p = \infty$. When A or $\mathcal{G}$ are clearly given by the context, we shall omit them.
For $p \ge 0$, we write $L^p_b(\mathbb{Q}, \mathcal{G})$ to denote the collection of element $G \in L^p(\mathbb{Q}, \mathcal{G})$ such that $G \ge -c$ $\mathbb{Q}$-a.s. for some $c > 0$.
The set predictable processes ψ with values in $\mathbb{R}^d$ satisfying $\mathbb{E}^{\mathbb{Q}}[\int_0^T |\psi_s|^2 ds] < \infty$ is denoted by $L^2_{\mathcal{P}}(\mathbb{Q})$, or simply $L^2_{\mathcal{P}}$ if $\mathbb{Q} = \mathbb{P}$.
If nothing else is specified $\mathbb{E}$ denote the expectation operator under $\mathbb{P}$. Otherwise, we write $\mathbb{E}^{\mathbb{Q}}$ if we want to consider the expectation operator under $\mathbb{Q} \ne \mathbb{P}$.
In the following, inequality between random variables have to be understood in the $\mathbb{P}-$a.s. sense.
We denote by x^i the i-th component of a vector $x \in \mathbb{R}^d$, which will always be viewed as a column vector, with transposed vector x'. We write $|\cdot|$ for the Euclydean norm, and $\mathbb{M}^d$ denotes the set of d-dimensional square matrices. We denote by $\mathbb{S}^d$ the subset of elements of $\mathbb{M}^d$ that are symmetric. For a subset $\mathcal{O}$ of $\mathbb{R}^d$, we denote by $\mathrm{cl}(\mathcal{O})$ its closure, by $\mathrm{int}(\mathcal{O})$ its interior, by $\partial\mathcal{O}$ its boundary, and by $\mathrm{dist}(x, \mathcal{O})$ the Euclidean distance from x to $\mathcal{O}$ with the convention $\mathrm{dist}(x, \emptyset) = \infty$. We denote by $B_r(x)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^d$. If $B = [s, t] \times \mathcal{O}$ for $s \le t$ and $\mathcal{O} \subset \mathbb{R}^d$, we write $\partial_p B := ([s, t) \times \partial\mathcal{O}) \cup (\{t\} \times \mathrm{cl}(\mathcal{O}))$ for its parabolic boundary.
Given a smooth function $\varphi : (t, x) \in \mathbb{R}_+ \times \mathbb{R}^d \mapsto \varphi(t, x) \in \mathbb{R}$, we denote by $\partial_t\varphi$ its derivative with respect to its first variable, and by $D\varphi$ and $D^2\varphi$ its Jacobian

and Hessian matrix with respect to the second one. For $\varphi : (t, x_1, \ldots, x_k) \in \mathbb{R}_+ \times \mathbb{R}^{kd} \mapsto \varphi(t, x) \in \mathbb{R}$, we write $D_{(x_i,x_j)}\varphi$ and $D^2_{(x_i,x_j)}\varphi$ the Jacobian and Hessian matrix associated to the couple (x_i, x_j).

2 Financial market and wealth process

In order to fix ideas and notations, we describe here the typical financial model we have in mind, also more general one will be considered later on.

As usual the financial market will consists in two types of assets. The first one is a risk free asset B, often called cash-account, whose dynamics is given by

$$B_t = 1 + \int_0^t B_s r_s ds = e^{\int_0^t r_s ds} \quad , \ t \geq 0 \ ,$$

where r is a predictable real valued process satisfying

$$\int_0^t |r_s| ds < \infty \quad \text{for all } t \geq 0 \ . \tag{1}$$

For ease of notation, we also introduce the associated stochastic discount factor β:

$$\beta_t := 1/B_t = e^{-\int_0^t r_s ds} \quad , \ t \geq 0 \ .$$

Risky assets (bonds, stocks, derivatives, etc...) are modeled via a d-dimensional process $X = (X^1, \ldots, X^d)$ satisfying

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dW_s$$

where (μ, σ) is a predictable process with valued in $\mathbb{R}^d \times \mathbb{M}^d$ that is bounded on $[0, T]$ $\mathbb{P}-$a.s. Each component X^i of X denotes a given risky asset.
A financial strategy is described by an element of the set $\mathcal{A}$ of d-dimensional predictable processes ϕ satisfying

$$\int_0^t |\phi_s'|^2 ds < \infty \quad \text{for all } t \leq T \ . \tag{2}$$

Each component ϕ_t^i denotes the number of units of asset X^i in the portfolio at time t.
To an initial wealth $y \in \mathbb{R}$ and a strategy $\phi \in \mathcal{A}$, we associate the portfolio process $Y^{y,\phi}$ defined as

$$Y_t^{y,\phi} := y + \int_0^t \phi_s' dX_s + \int_0^t (Y_s^{y,\phi} - \phi_s' X_s) r_s ds \ , \ t \leq T \ . \tag{3}$$

In the following, we say that a strategy ϕ is admissible, and we write $\phi \in \mathcal{A}_b$, if there exists a constant $c > 0$ such that

$$Y_t^{y,\phi} \geq -cB_t \quad \text{for all } t \leq T \ . \tag{4}$$

This condition means that the financial agent has a finite “credit line”, i.e. his wealth can not go too negative. Note that the constant c may depend on the chosen strategy and is not universal. Moreover, since $Y^{y,\phi} = yB + Y^\phi$, see (3), the set $\mathcal{A}_b$ does not depend on the initial endowment y.

For later use observe that $\tilde{Y}^{y,\phi} := \beta Y^{y,\phi}$ solves

$$\tilde{Y}_t^{y,\phi} := y + \int_0^t \phi_s' d\tilde{X}_s \text{ and satisfies } \tilde{Y}_t^{y,\phi} \geq -c \quad \text{for all } t \geq 0$$

for some $c > 0$, where $\tilde{X} := \beta X$ is given by

$$\tilde{X}_t = X_0 + \int_0^t (\tilde{\mu}_s - r_s \tilde{X}_s) ds + \tilde{\sigma}_s dW_s$$

with $\tilde{\mu} := \beta\mu$ and $\tilde{\sigma} := \beta\sigma$. Here, $\tilde{Y}^{y,\phi}$ and $\tilde{X}$ can be interpreted as the discounted values of the wealth and financial assets processes.

Remark 1 When dealing with PDE-oriented approaches, we shall specialize to models of the form (for instance) $r = \rho(X)$, $\mu = \mu(X)$ and $\sigma = \sigma(X)$ where ρ, μ and σ will be considered as deterministic functions. In this case, we shall write $(X_{t,x}(s))_s$ for $(X_s)_s$ to insist on the fact that X takes values x at time t. We will similarly write $(Y^\phi_{t,x,y}(s))_s$ for $(Y^\phi_s)_s$. More general cases where μ and σ depend on ϕ will also be considered. In such a situation, we shall write $(X^\phi_{t,x}(s))_s$ to insist on the dependence of X with respect to the strategy ϕ.

Remark 2 Additional constraints will be imposed later on strategies. This will allow us to consider more general models where some of the components of X will no more be considered as tradable assets but as non-tradable factors (e.g. stochastic volatility in Markovian models).

3 Hedging problem and hedging criteria

The pricing and hedging problem is the following. We are given a random claim $G \in L^0(\mathbb{R}, \mathcal{F}_T)$ that will impact the wealth of an investor at time T. This can

be the payoff of a financial derivative that has been sold at time 0, or any risk related to already engaged positions.
The question is: what is the amount of money required today in order to be able to construct a financial strategy which will allow to reduce this risk in an appropriate way ?
Many approaches can be considered depending on the market and the risk tolerance of the investor.
The first approach consists in trying to make the risk completely disappear. This is the philosophy of the super-hedging point of view: evaluate the risk at its super-hedging price

$$p(G) := \inf\left\{y \in \mathbb{R} \ : \ \exists\, \phi \in \mathcal{A}_b \text{ s.t. } Y_T^{y,\phi} \geq G\right\} .$$

Then, starting from $y = p(G)$, or $y > p(G)$ if the infimum above is not achieved, one can follows a strategy ϕ such that $Y_T^{y,\phi} \geq G$, i.e. the risk is completely covered.
This approach is the most conservative. However, it has two important drawbacks:
1. The associated strategy may not be easy to implement in practice. For instance, it can lead to very large and too quickly varying financial positions. This is typically the case for digital or barrier options for which it can explode near the maturity or the barrier, see e.g. [7] and [19].
2. The computed value may be too large and therefore non-reasonable, see e.g. [10] for an example of stochastic volatility model in which the super-hedging price of a call is just the spot value of the underlying.

In order to answer the first criticism, one can add portfolio constraints in the model, and compute the corresponding super-hedging price under these constraints.

As for the second criticism, we need to relax the $\mathbb{P}-$ a.s. super-hedging criteria. One way to do this, consists in allowing to miss the hedge with a given probability, i.e. compute the so-called quantile hedging price, see [12]:

$$\inf\left\{y \geq -c \ : \ \exists\, \phi \in \mathcal{A}_b \text{ s.t. } \mathbb{P}\left[Y_T^{y,\phi} \geq G\right] \geq p\right\}$$

for some $p \in [0,1)$ and $c \in \mathbb{R}_+$. Here, the constant c is added as a minimum requirement in order to avoid degenerate results.

Another way consists in allowing to miss the hedge with a level of risk, see [13], which leads to problems of the form:

$$\inf \left\{ y \geq -c \; : \; \exists \, \phi \in \mathcal{A}_b \text{ s.t. } \mathbb{E}\left[\ell(Y_T^{y,\phi} - G) \right] \geq l \right\}$$

for some $l \in \text{Image}(\ell)$ and $c \in \mathbb{R}_+$. Here, ℓ is typically a convex non-decreasing function viewed as a loss function. The map $(y, \phi) \mapsto -\mathbb{E}\left[\ell(Y_T^{y,\phi} - G) \right]$ has to be interpreted as a measure of the risk induced by starting with y and following the policy ϕ.

The above questions are equally relevant in portfolio management. Imagine we are given an initial endowment y and we want to manage a financial portfolio according to an expected utility criteria $\xi \mapsto \mathbb{E}[U(\xi)]$, where U is a concave non-decreasing function which models the preferences of the agent. If the agent only considers the standard expected utility maximization problem $\sup\{\mathbb{E}\left[U(Y_T^{y,\phi}) \right], \; \phi \in \mathcal{A}_b\}$, then some specific risks may not be taken into account. They can be incorporated by considering more general formulations of the form:

$$\sup \left\{ \mathbb{E}\left[U(Y_T^{y,\phi}) \right], \; \phi \in \mathcal{A}_b \text{ s.t. } Y_T^{y,\phi} \geq G \right\}$$

or

$$\sup \left\{ \mathbb{E}\left[U(Y_T^{y,\phi}) \right], \; \phi \in \mathcal{A}_b \text{ s.t. } \mathbb{E}\left[\ell(Y_T^{y,\phi} - G) \right] \geq l \right\}$$

which corresponds to adding a new risk constraint to the optimal portfolio management problem.

4 Duality versus stochastic targets

The above problems have been considered in the literature under the angle of the so-called dual approach. It is based on the relation between super-hedgeable claims and probability measures that turn discounted price processes into (local) martingales. This approach allows to appeal to the convex analysis machinery which turns out to be very powerful.

The main drawback of this approach is that it does not allow to consider models where the wealth dynamics in non-linear or in which the financial strategy may have an impact on the price process of financial assets.

We shall see in these lectures how the recent theory of stochastic targets can handle in a direct way such situations.

Part A.

Dual approach to risk based pricing and hedging

Chapter 2

Dual formulation for super-hedging and martingale representation

This first part is dedicated to the so-called dual approach.

1 The complete market case

We first consider the so-called complete market case where *any* risk can be covered.

This corresponds to the situation where σ is invertible with bounded inverse on $[0,T]$ $\mathbb{P}-$a.s. and the risk premium λ defined by

$$\lambda := \tilde{\sigma}^{-1}(\tilde{\mu} - r\tilde{X}) = \sigma^{-1}(\mu - rX)$$

satisfies[1]

$$H := \mathcal{E}\left(-\int_0^{\cdot} \lambda_s' dW_s\right) \text{ is a martingale.} \tag{1}$$

so that $\mathbb{Q} \sim \mathbb{P}$ defined by

$$d\mathbb{Q}/d\mathbb{P} = H_T$$

is the unique element of the set $\mathcal{M}$ of $\mathbb{P}$-equivalent probability measures such that $\tilde{X}$ is a martingale.

[1] This notation means that H solves $H_t = 1 - \int_0^t H_s \lambda_s' dW_s$, $t \leq T$.

We then define the $\mathbb{Q}$-Brownian motion $W^{\mathbb{Q}}$ by

$$W_t^{\mathbb{Q}} = W_t + \int_0^t \lambda_s ds \ ,$$

recall Girsanov's Theorem, so that

$$\tilde{X}_t = X_0 + \int_0^t \tilde{\sigma}_s dW_s^{\mathbb{Q}}$$

and therefore

$$\tilde{Y}_t^{y,\phi} = y + \int_0^t \phi_s' \tilde{\sigma}_s dW_s^{\mathbb{Q}}$$

Remark 1 For $\phi \in \mathcal{A}$, $\tilde{Y}^{y,\phi}$ is a $\mathbb{Q}$-local martingale, i.e. there exists a sequence of stopping times $(\tau_n)_{n\geq 1}$ such that $\tau_n \uparrow \infty$ $\mathbb{P}-$ a.s. and $(\tilde{Y}^{y,\phi}_{\cdot \wedge \tau_n})$ is a $\mathbb{Q}$-martingale for each $n \geq 1$. Since, for $\phi \in \mathcal{A}_b$, $\tilde{Y}^{y,\phi}$ is also bounded from below, a straightforward application of Fatou's Lemma shows that it is indeed a $\mathbb{Q}$-supermatingale.

Under the condition (1), any random variable G such that $\beta_T G \in L_b^1(\mathbb{Q}, \mathcal{F}_T)$ can be written as the time T value of a wealth process. This is a consequence of the martingale representation theorem.

Theorem 1 *Given $G \in L^0$ such that $G \in L^1(\mathbb{Q}, \mathcal{F}_T)$ there exists a predictable process ψ satisfying $\int_0^T |\psi_s|^2 ds < \infty$ such that*

$$\mathbb{E}^{\mathbb{Q}}[G \mid \mathcal{F}_t] = \mathbb{E}^{\mathbb{Q}}[G] + \int_0^t \psi_s' dW_s^{\mathbb{Q}} \ .$$

If $G \in L^2(\mathbb{Q})$, then $\psi \in L^2_{\mathcal{P}}(\mathbb{Q})$.

Otherwise stated, the $\mathbb{Q}$-martingale $(\mathbb{E}^{\mathbb{Q}}[G \mid \mathcal{F}_t])_{t\leq T}$ can be represented in terms of a stochastic integral with respect to $W^{\mathbb{Q}}$.

Corollary 1 *Fix $G \in L^0$ such that $\beta_T G \in L_b^1(\mathbb{Q}, \mathcal{F}_T)$. Then,*

$$p(G) = \mathbb{E}^{\mathbb{Q}}[\beta_T G]$$

and there exists $\phi \in \mathcal{A}_b$ such that

$$V_T^{p(G),\phi} = G \ .$$

If $G \in L^2(\mathbb{Q})$, then $\psi \in L^2_{\mathcal{P}}(\mathbb{Q})$.

Proof. For $y > p(G)$, there exists $\phi \in \mathcal{A}_b$ such that $Y_T^{y,\phi} \geq G$. Since $\tilde{Y}^{y,\phi}$ is a $\mathbb{Q}$-supermatingale, by Remark 1, this implies that $y \geq \mathbb{E}^{\mathbb{Q}}[\beta_T G]$. On the other hand, it follows from Theorem 1 that there exists a predictable process ψ satisfying $\int_0^T |\psi_s|^2 ds < \infty$ such that

$$p(G) + \int_0^t \psi_s' dW_s^{\mathbb{Q}} = \mathbb{E}^{\mathbb{Q}}[\beta_T G \mid \mathcal{F}_t] \ .$$

By taking ϕ defined as $\psi' := \phi'\tilde{\sigma}$, we obtain

$$\tilde{Y}_T^{p(G),\phi} = \beta_T G \ ,$$

where ϕ satisfies $\int_0^T |\phi_s|^2 ds < \infty$, note that $\tilde{\sigma}^{-1}$ is bounded on $[0,T]$ $\mathbb{P}$-a.s., and $\tilde{Y}^{p(G),\phi} = \mathbb{E}^{\mathbb{Q}}[\beta_T G \mid \mathcal{F}_\cdot] \geq -c$ for some $c > 0$. □

2 Incomplete markets and portfolio constraints

In order to take into account the incompleteness of the market and possible portfolio constraints, we shall restrict from now on to admissible strategies $\phi \in \mathcal{A}_b$ such that $\phi \in K$ $dt \times d\mathbb{P}$-a.e., where K is a given convex set of $\mathbb{R}^d$. We denote by $\mathcal{A}_K$ the set of such elements.

Example 1 *Here are some relevant examples:*
1. *Short selling constraints:* $K = [0,\infty)^d$.
2. *"Asset" 1 can not be traded, no constraint on the others:* $K = \{0\} \times \mathbb{R}^{d-1}$.
3. *Bounded positions in any asset:* $K = \prod_{i=1}^{d}[-m_i, M_i]$ *for some* $m_i, M_i \geq 0$.

2.1 The general dual formulation

The aim of this section is to extend the formulation of Corollary 1 to the super-hedging price under constraint:

$$p_K(G) := \inf\{y \in \mathbb{R} \ : \ \exists\, \phi \in \mathcal{A}_K \text{ s.t. } Y_T^{y,\phi} \geq G\} \ .$$

In order to do this, we first need to characterize the set K in term of the support function

$$\zeta \in \mathbb{R}^d \mapsto \delta_K(\zeta) := \sup_{\eta \in K} \eta'\zeta.$$

Proposition 1

$$\eta \in K \iff \inf_{|\zeta|=1} \delta_K(\zeta) - \zeta'\eta \geq 0 \ .$$

Proof. The implication $\Rightarrow$ follows from the definition. Conversely, if $\bar\eta \notin K$, which is convex and closed, then the Hahn-Banach separation theorem, see [18], implies that there exists $\zeta \in \mathbb{R}^d$ such that $\sup_{\eta \in K} \eta'\zeta < \bar\eta'\zeta$. This implies that $\delta_K(\zeta) - \bar\eta'\zeta < 0$, where ζ can always be chosen such that $|\zeta| = 1$ by an obvious normalization. □

In the following, we let $\mathcal{U}_b$ denote the set of $\mathbb{R}^d$-valued predictable processes such that, for some constant $c > 0$, $\sup_{s \leq T}(|\nu_s| + |\delta_K(\nu_s)|) \leq c \ \mathbb{P}-\text{a.s.}$ For $\nu \in \mathcal{U}_b$, we define $\mathbb{Q}^\nu \sim \mathbb{P}$ by

$$d\mathbb{Q}^\nu / d\mathbb{P} := H^\nu_T$$

where

$$H^\nu := \mathcal{E}\left(-\int_0^\cdot (\lambda^\nu_s)' dW_s \right) \quad \text{with} \quad \lambda^\nu := \sigma^{-1}(\mu - rX) - \tilde\sigma^{-1}\nu \ .$$

We also define

$$Z^\nu := \int_0^\cdot \delta_K(\nu_s) ds \ \text{ and the } \mathbb{Q}^\nu \text{ Brownian motion } W^\nu := W + \int_0^\cdot \lambda^\nu_s ds \ .$$

Observe that, for $\nu \in \mathcal{A}_K$,

$$d(\tilde Y^{y,\phi}_t - Z^\nu_t) = \left(\phi'_t \nu_t - \delta_K(\nu_t)\right) dt + \phi'_t \tilde\sigma_t dW^\nu_t \ .$$

In particular, it follows from Proposition 1 that $\tilde Y^{y,\phi} - Z^\nu$ is a $\mathbb{Q}^\nu$-local super-martingale for any $\phi \in \mathcal{A}_K$. Note that, for some $c > 0$, $\tilde Y^{y,\phi} - Z^\nu \geq -c - cT$. Hence, this $\mathbb{Q}^\nu$-local supermartingale is bounded from below and is therefore a $\mathbb{Q}$-super-martingale. This leads to the following first result:

Proposition 2 *Fix $G \in L^0$ such that $\beta_T G \in L^0_b(\mathcal{F}_T)$. Then,*

$$p_K(G) = \inf\{y \in \mathbb{R} \ : \ \phi \in \mathcal{A}_K \ s.t. \ Y^{y,\phi}_T \geq G\} \geq \sup_{\nu \in \mathcal{U}} \mathbb{E}^{\mathbb{Q}^\nu}[\beta_T G - Z^\nu_T] \ .$$

We shall now show that equality actually holds.

Theorem 2 *Fix $G \in L^0$ such that $\beta_T G \in L^0_b(\mathcal{F}_T)$. Then,*

$$p_K(G) = \sup_{\nu \in \mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^\nu}[\beta_T G - Z^\nu_T] \ .$$

Moreover, if $p_K(G) < \infty$, then there exists $\phi \in \mathcal{A}_K$ such that $Y^{p_K(G),\phi}_T \geq G$.

We split the proof of the above result in various Lemma.
Let us now define P as the cadlag adapted process satisfying[2]

$$P_t := \operatorname*{esssup}_{\nu \in \mathcal{U}} J_t^\nu \text{ where } J_t^\nu := \mathbb{E}^{\mathbb{Q}^\nu}[\beta_T G - (Z_T^\nu - Z_t^\nu) \mid \mathcal{F}_t] \ , \ t \leq T$$

Note that the existence of a cadlad process satisfying the above property is not obvious. Here, this follows from arguments developed in [15] and we omit the details.
The key argument for proving Theorem 2 consists in showing that P is a supermatingale under any $\mathbb{Q}^\nu$, $\nu \in \mathcal{U}_b$, see Proposition 4 below.
We first show that the family $\{J_t^\nu \ , \nu \in \mathcal{U}_b\}$ is directed upward in the following sense.

Definition 1 *We say that a family of random variables $\mathcal{E}$ is directed upward is for any $\zeta_1, \zeta_2 \in \mathcal{E}$, there exists $\zeta_3 \in \mathcal{E}$ such that $\zeta_3 \geq \max\{\zeta_1, \zeta_2\}$.*

Proposition 3 *For each t, the family $\{J_t^\nu \ , \nu \in \mathcal{U}_b\}$ is directed upward.*

Proof. Fix $\nu^1, \nu^2 \in \mathcal{U}_b$, and set $\nu^3 = \nu^1 \mathbf{1}_{[0,t)} + \mathbf{1}_{[t,T]} \left(\nu^1 \mathbf{1}_A + \nu^2 \mathbf{1}_{A^c}\right)$, where $A := \{J_t^{\nu^1} \geq J_t^{\nu^2}\}$. Clearly, $J_t^{\nu^3} = \max\{J_t^{\nu^1}, J_t^{\nu^2}\}$. Moreover, if $c > 0$ is such that $\sup_{s \leq T}(|\nu_s^i| + |\delta_K(\nu_s^i)|) \leq c \ \mathbb{P} -$ a.s. for $i = 1, 2$, then the same inequality holds for $i = 3$. Hence, $\nu^3 \in \mathcal{U}_b$. □

In order to prove Proposition 4, we now use the following well-know property of directed upward families, see e.g. [17].

Lemma 1 *If $\mathcal{E}$ is a family directed upward. Then there exists a sequence $(\zeta_n)_{n \geq 1} \subset \mathcal{E}$ such that* $\operatorname{esssup} \mathcal{E} = \lim_{n \to \infty} \uparrow \zeta_n$.

We can now prove the supermartingale property.

Proposition 4 *For all $\nu \in \mathcal{U}_b$, $P - Z^\nu$ is a $\mathbb{Q}^\nu$-supermartingale.*

[2] We recall that $\operatorname{esssup} \mathcal{E}$, for a family $\mathcal{E}$ of random variables, is the smallest random variables which dominates all elements of $\mathcal{E}$, in the a.s. sense.

Proof. Fix $t \geq s$ and $\nu \in \mathcal{U}_b$. Let $(\nu_n)_{n\geq 1}$ be such that $J_t^{\nu^n} \uparrow P_t$ as $n \to \infty$, see Lemma 1 and Proposition 3. For $\nu \in \mathcal{U}_b$, set $\bar{\nu}_n := \nu \mathbf{1}_{[0,t)} + \nu_n \mathbf{1}_{[t,T]}$. Then,

$$\begin{aligned}
\mathbb{E}^{\mathbb{Q}^\nu}[P_t - Z_t^\nu \mid \mathcal{F}_s] &= \mathbb{E}^{\mathbb{Q}^\nu}\left[\lim_{n\to\infty} \uparrow \mathbb{E}^{\mathbb{Q}^{\nu_n}}[\beta_T G - (Z_T^{\nu_n} - Z_t^{\nu_n}) \mid \mathcal{F}_t] - Z_t^\nu \mid \mathcal{F}_s\right] \\
&= \lim_{n\to\infty} \uparrow \mathbb{E}^{\mathbb{Q}^\nu}\left[\mathbb{E}^{\mathbb{Q}^{\nu_n}}[\beta_T G - (Z_T^{\nu_n} - Z_t^{\nu_n}) \mid \mathcal{F}_t] - Z_t^\nu \mid \mathcal{F}_s\right] \\
&= \lim_{n\to\infty} \uparrow \mathbb{E}^{\mathbb{Q}^{\bar{\nu}_n}}\left[\beta_T G - (Z_T^{\bar{\nu}_n} - Z_s^{\bar{\nu}_n}) \mid \mathcal{F}_s\right] - Z_s^\nu \\
&\leq P_s - Z_s^\nu .
\end{aligned}$$

□

Proposition 5 *For each $\nu \in \mathcal{U}_b$, there exists a $\mathbb{Q}^\nu$-martingale M^ν and a non-decreasing process A^ν such that $A_0^\nu = 0$ and $P - Z^\nu = M^\nu - A^\nu$.*

Proof. This follows from the Doob-Meyer decomposition together with the previous proposition. □

In order to conclude the proof, we now apply the martingale representation to M^0 to obtain some predictable process ψ satisfying $\int_0^T |\psi_s|^2 ds < \infty$ such that

$$P_t = P_t - Z_t^0 = P_0 + \int_0^t \psi_s' dW_s^0 - A_t^0 .$$

By taking ϕ such that $\phi' \tilde{\sigma} = \psi'$, we obtain

$$P_t = \tilde{Y}_t^{P_0,\phi} - A_t^0 = \operatorname*{ess\,sup}_{\nu \in \mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^\nu}[\beta_T G - (Z_T^\nu - Z_t^\nu) \mid \mathcal{F}_t] \geq \mathbb{E}^{\mathbb{Q}}[\beta_T G \mid \mathcal{F}_t] \ , \ t \leq T,$$

which implies that $\phi \in \mathcal{A}_b$ and that $Y_T^{P_0,\phi} \geq G$, since $A^0 \geq 0$. To conclude the proof, it remains to shows that $\phi \in K \ dt \times d\mathbb{P}$-a.e. To see this, recall that P can also be decomposed as $P - Z^\nu = M^\nu - A^\nu$. In particular, we must have

$$\begin{aligned}
P - Z^\nu &= P_0 + \int_0^\cdot \psi_s' dW_s^0 - A^0 - Z^\nu \\
&= P_0 + \int_0^\cdot \psi_s' dW_s^\nu + \int_0^\cdot (\psi_s' \tilde{\sigma}_s^{-1} \nu_s - \delta_K(\nu_s)) ds - A^0 \\
&= P_0 + \int_0^\cdot \psi_s' dW_s^\nu + \int_0^\cdot (\phi_s' \nu_s - \delta_K(\nu_s)) ds - A^0
\end{aligned}$$

so that $A^\nu = A^0 - \int_0^\cdot (\phi_s' \nu_s - \delta_K(\nu_s) ds$ which is therefore non-decreasing. It follows that

$$\int_0^\cdot (\phi_s' \nu_s - \delta_K(\nu_s)) ds \leq A_T^0$$

for all $\nu \in \mathcal{U}_b$. By replacing ν by $n\nu$ and by sending $n \to \infty$, we deduce from the above inequality that

$$\int_0^{\cdot} (\phi'_s \nu_s - \delta_K(\nu_s))ds \le 0 \; .$$

Let us now define $\bar\nu$ as $\bar\nu := \arg\min_{|\zeta|=1}(\delta_K(\zeta) - \phi'\zeta)$. Taking $\nu := \bar\nu \mathbf{1}_{\{\delta_K(\bar\nu)-\phi'_s\bar\nu<0\}}$ in the last inequality, shows that $\phi \in K \; dt \times d\mathbb{P}$-a.e, recall Proposition 1. □

2.2 Examples

We conclude this section with three examples of applications. The first one corresponds to a Brownian model with portfolio constraints, the second one to a Black-Scholes model with constraints on the amount of money invested in the asset, the last one to a stochastic volatility model.

Example 2 (Brownian model with portfolio constraint) *Let us consider the case $d = 1$ where $X = X^1$ has the dynamics*

$$X_t = X_0 + \mu t + \sigma W_t \quad t \le T \; ,$$

and $r = 0$. We want to hedge an option of payoff $g(X_T)$ paid at time T under the constraints $K = [-m, M]$ with $M, m \ge 0$. We shall assume here that g is non-decreasing.
In this case, $\delta_K(\zeta) = \zeta^+ M + \zeta^- m$ so that $dom(\delta_K) = \mathbb{R}$. Let us define the function $\hat g$ by $\hat g(x) := \sup_{u\in\mathbb{R}} (g(x+u) - (u^+M + u^-m))$. Then, it follows from Theorem 2 that:

$$\begin{aligned} p_K(G) &= \sup_{\nu\in\mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^\nu}\left[g(X_T) - \int_0^T (\nu_s^+ M + \nu_s^- m)\right] \\ &= \sup_{\nu\in\mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^\nu}\left[g\left(X_0 + \int_0^T \nu_s ds + \sigma W_T^\nu\right) - \int_0^T \delta_K(\nu_s) ds\right] \\ &\le \sup_{\nu\in\mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^\nu}\left[\hat g\left(X_0 + \sigma W_T^\nu\right)\right] \end{aligned}$$

where we used the fact that $g(x) = g(x+u-u) \le \hat g(x-u) + \delta_K(u)$. It follows that

$$p_K(G) \;\; \le \;\; \mathbb{E}^{\mathbb{Q}}\left[\hat g\left(X_0 + \sigma W_T^{\mathbb{Q}}\right)\right] \; .$$

We now observe that, by a formal identification of the law of $W^{\mathbb{Q}}$ under $\mathbb{Q}$ and W^{ν} under $\mathbb{Q}^{\nu}$,

$$p_K(G) \;=\; \sup_{\nu\in\mathcal{U}_b} \mathbb{E}^{\mathbb{Q}}\left[g\left(X_0+\int_0^T \nu_s ds+\sigma W_T^{\mathbb{Q}}\right)-\int_0^T \delta_K(\nu_s)ds\right],$$

see [19] for a rigorous argument. Moreover, any bounded $\mathcal{F}_t$-measurable random variable, with $t<T$, can be written in the form $\int_0^T \nu_s ds$ with $\nu\in\mathcal{U}_b$. Indeed, give $\xi\in L^0(\mathcal{F}_t)$, one has $\int_0^T(\xi/(T-t))\mathbf{1}_{s\geq t}ds=\xi$. Given $\xi\in L^\infty(\mathcal{F}_T)$, one can then approximate it by the sequence $\mathbb{E}\left[\xi \mid \mathcal{F}_{T(1-1/n)}\right]_{n\geq 1}$. It follows that, for g continuous and bounded from below,

$$p_K(G) \;\geq\; \sup_{\xi\in L^\infty(\mathbb{R}_+,\mathcal{F}_T)} \mathbb{E}^{\mathbb{Q}}\left[g\left(X_0+\sigma W_T^{\mathbb{Q}}+\xi\right)-\xi^+M\right].$$

Here, we restrict to non-negative random variable because g is non-decreasing and it should therefore be optimal to restrict to $\nu\geq 0$ or equivalently $\xi\geq 0$. Now, we clearly have

$$\begin{aligned}&\sup_{\xi\in L^\infty(\mathbb{R}_+,\mathcal{F}_T)} \mathbb{E}^{\mathbb{Q}}\left[g\left(X_0+\sigma W_T^{\mathbb{Q}}+\xi\right)-\xi^+M\right]\\ &=\mathbb{E}^{\mathbb{Q}}\left[\sup_{\zeta\in\mathbb{R}_+}\left(g\left(X_0+\sigma W_T^{\mathbb{Q}}+\zeta\right)-\zeta M\right)\right].\end{aligned}$$

This shows that

$$p_K(G) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\hat{g}\left(X_0+\sigma W_T^{\mathbb{Q}}\right)\right],$$

i.e., the price under constraint for the option g is the usual unconstrained price in the Brownian model of the face-lifted payoff $\hat{g}$.

Example 3 (Black-Scholes model with portfolio constraint)
Let us now consider the Black-Scholes model where X is given by

$$dX_t/X_t=\mu dt+\sigma dW_t$$

and $r=0$ for simplicity. In this example, we impose the constraint

$$\psi:=\phi X\in K \;\; dt\times d\mathbb{P}-\text{a.e.}$$

i.e. the amount invested in the risky asset belongs to K. Let $\hat{A}_K$ denote the set of processes $\phi\in\mathcal{A}_b$ such that the above constraint is satisfied.

We shall see how we can reduce the problem of super-hedging a claim $g(X_T)$ to the problem discussed in the previous example.

To do this, first observe that

$$Y_t^{y,\phi} = y + \int_0^t \phi_s dX_s = y + \int_0^t \psi_s dX_s/X_s = y + \int_0^t \psi_s \mu ds + \int_0^t \psi_s \sigma dW_s$$

where $\psi := \phi X$, so that

$$Y_t^{y,\phi} = y + \int_0^t \psi_s d\bar{X}_s$$

with

$$\bar{X}_t := \mu t + \sigma W_t \ .$$

It follows that, at least for g bounded from below,

$$\begin{aligned} \hat{p}_K(g(X_T)) &:= \inf\left\{y \ : \ \exists\, \phi \in \hat{\mathcal{A}}_K \ s.t. \ Y_T^{y,\phi} \geq g(X_T)\right\} \\ &= \inf\left\{y \ : \ \exists\, \psi \in \mathcal{A}_K \ s.t. \ y + \int_0^T \psi_s d\bar{X}_s \geq \bar{g}(\bar{X}_T)\right\} \end{aligned}$$

where $\bar{g}(x) := g(X_0 e^{x-(\sigma^2/2)T})$.
Letting $\bar{p}_K$ be defined as p_K but for the model where the stock price is given by $\bar{X}$, the above arguments show that

$$\hat{p}_K(g(X_T)) = \bar{p}_K(\bar{g}(\bar{X}_T)) \ .$$

In view of the previous example, one can then obtain an explicit formulation for $\hat{p}_K(g(X_T))$.

Example 4 (Stochastic volatility) *In this example, we take $d = 2$ and let (X^1, X^2) be the solution of*

$$\begin{aligned} X_t^1 &= X_0^1 + \int_0^t X_s^1 r ds + \int_0^t X_s^1 \sigma(X_s^2) dW_s^1 \\ X_t^2 &= X_0^2 + \gamma_1 W_t^1 + \gamma_2 W_t^2 \end{aligned}$$

where $\gamma_1, \gamma_2 > 0$, $\sigma \geq \varepsilon$ for some $\varepsilon > 0$ and σ is bounded. We impose the constraint $K := \mathbb{R} \times \{0\}$, i.e. X^2 can not be traded. This corresponds to the simplest stochastic volatility model, in which X^2 should be considered as a factor driving the volatility of X^1, and not as an asset.

In this case, we have $\delta_K(\zeta) = 0$ is $\zeta^1 = 0$ and $\delta_K(\zeta) = \infty$ otherwise. It follows that

$$p_K(g(X_T^1)) = \sup_{\lambda \in \Lambda} \mathbb{E}^{\mathbb{Q}^\lambda} \left[\beta_T g(X_T^1) \right]$$

where Λ denotes the set of real valued predictable processes λ satisfying $\sup_{s \le T} |\lambda_s| \le c$ for some $c > 0$, and $\mathbb{Q}^\lambda$ is defined by

$$\frac{d\mathbb{Q}^\lambda}{d\mathbb{P}} = e^{-\frac{1}{2} \int_0^T (\gamma_2^{-1} \lambda_s)^2 ds + \int_0^T \gamma_2^{-1} \lambda_s dW_s^2} ,$$

which, up to the boundedness imposed on λ, corresponds to the family of all martingale measures for X^1.

We shall come back to this example in Chapter 4 below.

Chapter 3

The pricing equation I: the complete market case

In this chapter, we restrict to the Markovian setting where X is given as the solution of an SDE of the form

$$X_{t,x}(s) = x + \int_t^s r_{t,x}(u) X_{t,x}(u) du + \int_t^s \sigma(X_{t,x}(u)) dW_u^{\mathbb{Q}} , \tag{1}$$

for a risk free interest rate of the form

$$r_{t,x} = \rho(X_{t,x})$$

where ρ, μ and σ are assumed to be Lipschitz continuous, and ρ is such that ρ^- is bounded and $x \mapsto \rho(x)x$ is Lipschitz continuous.

For ease of notations, we shall only consider the case where X can take any values in $\mathbb{R}^d$, also in most financial models we should typically restrict to $(0,\infty)^d$. The arguments being the same in this last case.

The aim of this section is to provide a PDE formulation for the price function of an option of payoff $g(X_{t,x}(T))$ paid at time T, depending on the initial time t and the initial value of X at this time.

In the following, g will be assumed to be continuous with linear growth and uniformly bounded from below.

1 Problem extension and dynamic programming

Motivated by Section 1 of Chapter 2, we now introduce the pricing function associated to the complete market case:

$$(t,x) \in [0,T] \times \mathbb{R}^d \mapsto v(t,x) := \mathbb{E}^{\mathbb{Q}}[\beta_{t,x}(T) g(X_{t,x}(T))]$$

where

$$\beta_{t,x} := e^{-\int_t^{\cdot} \rho(X_{t,x}(s))ds} .$$

The key assertion for deriving a PDE associated to v is the following *dynamic programming* equation which relates the time t value of the price to its time θ value, for any stopping time θ bigger than t. In the following, we shall denote by $\mathcal{T}_{[t,T]}$ the collection of stopping times taking values in $[t,T]$.

Proposition 6 *For all* $\theta \in \mathcal{T}_{[t,T]}$, *we have*

$$v(t,x) = \mathbb{E}^{\mathbb{Q}}\left[\beta_{t,x}(\theta) v(\theta, X_{t,x}(\theta))\right] . \tag{2}$$

Proof. By the flow property of X and the usual tower property, we have

$$v(t,x) \;=\; \mathbb{E}^{\mathbb{Q}}\left[\beta_{t,x}(\theta)\mathbb{E}^{\mathbb{Q}}\left[\beta_{\theta,X_{t,x}(\theta)}(T) g(X_{\theta,X_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta\right]\right] .$$

It then follows from the strong Markov property of X defined by (1) that

$$\begin{aligned} v(\theta, X_{t,x}(\theta)) &= \mathbb{E}^{\mathbb{Q}}\left[\beta_{\theta,X_{t,x}(\theta)}(T) g(X_{\theta,X_{t,x}(\theta)}(T)) \mid (\theta, X_{t,x}(\theta))\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\beta_{\theta,X_{t,x}(\theta)}(T) g(X_{\theta,X_{t,x}(\theta)}(T)) \mid \mathcal{F}_\theta\right] , \end{aligned}$$

hence the required result. □

2 Feynman Kac representation in the smooth case

Using the above proposition, we can now show that, whenever it is smooth enough, v solves the PDE

$$\mathcal{L}^{\mathbb{Q}} v = \rho v \tag{3}$$

on $[0,T) \times \mathbb{R}^d$ with the boundary condition $v(T,\cdot) = g$. Here, $\mathcal{L}^{\mathbb{Q}}$ is the Dynkin operator associated to X under $\mathbb{Q}$:

$$\mathcal{L}^{\mathbb{Q}}\varphi(t,x) \;:=\; \partial_t \varphi(t,x) + \rho(x) x' D\varphi(t,x) + \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma'(x) D^2\varphi(t,x)\right] .$$

2.1 Derivation

Theorem 1 *(Feynman-Kac) Assume that v is continuous on $[0,T] \times \mathbb{R}^d$ and $v \in C^{1,2}([0,T) \times \mathbb{R}^d)$. Then, v is a solution on $[0,T) \times \mathbb{R}^d$ of (3) and satisfies the boundary condition $\lim_{t \nearrow T, z \to x} v(t,z) = g(x)$ on $\mathbb{R}^d$.*

Proof. The boundary condition is a consequence of the continuity assumption on v. It remains to show that v solves (3). We now fix $(t,x) \in [0,T) \times \mathbb{R}^d$. Let θ be the first time when $(s, X_{t,x}(s))_{s \geq t}$ exits a given bounded open neighborhood of (t,x). Set $\theta^h = \theta \wedge (t+h)$ for $h > 0$ small. Using Proposition 6 and Itô's Lemma, we deduce that

$$0 = \mathbb{E}\left[\frac{1}{h}\int_t^{\theta^h} \beta_{t,x}(s)\left(\mathcal{L}^{\mathbb{Q}} v(s, X_{t,x}(s)) - (\rho v)(s, X_{t,x}(s))\right) ds\right] . \tag{4}$$

Now, we observe that $s \mapsto X_{t,x}(s)$ is $\mathbb{P}-$a.s. continuous, so that $|X_{t,x}(s \wedge (t+h)) - x| \to 0$ $\mathbb{P}-$a.s. as $h \to 0$ for each $s \geq t$. Moreover, $\theta > 0$ $\mathbb{P}-$a.s. so that $(\theta^h - t)/h \to 1$ $\mathbb{P}-$a.s. Using the mean value theorem and the continuity of $\mathcal{L}^{\mathbb{Q}} v - \rho v$, we then deduce that

$$\frac{1}{h}\int_t^{\theta^h} \beta_{t,x}(s)\left(\mathcal{L}^{\mathbb{Q}} v(s, X_{t,x}(s)) - (\rho v)(s, X_{t,x}(s))\right) ds$$
$$\to (\mathcal{L}^{\mathbb{Q}} v - \rho v)(t,x) \;\; \mathbb{P}-\text{a.s.}$$

as $h \to 0$. The required result is then obtained by applying the dominated convergence theorem to pass to the limit in (4), observe that $(s, X_{t,x}(s))_{s \geq t}$ is bounded on $[t, \theta]$ by definition of θ. □

2.2 Comparison and uniqueness

In order to show that Theorem 1 provides a full characterization of v, it remains to show that v is the unique solution of (3) within a suitable class of functions. This is a consequence of the following *comparison result*.

Theorem 2 *(Comparison principle) Assume that U and V are continuous on $[0,T] \times \mathbb{R}^d$ and $C^{1,2}$ on $[0,T) \times \mathbb{R}^d$. Assume further that, on $[0,T) \times \mathbb{R}^d$,*

$$\mathcal{L}^{\mathbb{Q}} U \leq \rho U \quad \text{and} \quad \mathcal{L}^{\mathbb{Q}} V \geq \rho V \tag{5}$$

and that $U(T,\cdot) \geq V(T,\cdot)$ on $\mathbb{R}^d$. Finally assume that U and V have polynomial growth. Then, $U \geq V$ on $[0,T] \times \mathbb{R}^d$.

Proof. By possibly replacing U and V by $\tilde{U}(t,x) := e^{\kappa t}U(t,x)$ and $\tilde{V}(t,x) := e^{\kappa t}V(t,x)$ for a large κ, we can assume that $\rho \geq \eta$ on $\mathbb{R}^d$ for some $\eta > 0$. Indeed, $\tilde{U}$ and $\tilde{V}$ would satisfy (5) with ρU and ρV replaced by $(\rho+\kappa)\tilde{U}$ and $(\rho+\kappa)\tilde{V}$, where ρ^- is bounded. Assume now that, for some $(t_0,x_0) \in [0,T]\times\mathbb{R}^d$, we have $U(t_0,x_0) < V(t_0,x_0)$. We shall show that this leads to a contradiction. Fix $\varepsilon > 0$, $\kappa > 0$ and p an integer greater that 1 such that $\limsup_{|x|\to\infty}\sup_{t\leq T}(|U(t,x)|+|V(t,x)|)/(1+|x|^p) = 0$. Then, there is $(\hat{t},\hat{x}) \in [0,T]\times\mathbb{R}^d$ such that, for ε small enough,

$$0 \quad < \quad V(\hat{t},\hat{x}) - U(\hat{t},\hat{x}) - \phi(\hat{t},\hat{x}) = \max_{(t,x)\in[0,T]\times\mathbb{R}^d} \left(V(t,x) - U(t,x) - \phi(t,x)\right) \ ,$$

where

$$\phi(t,x) := \varepsilon e^{-\kappa t}(1+|x|^{2p}) \ .$$

Since $U \geq V$ on $\{T\}\times\mathbb{R}^d$, we must have $\hat{t} < T$. Moreover, the one and second order conditions of optimality imply

$$\partial_t V(\hat{t},\hat{x}) \leq (\partial_t U + \partial_t \phi)(\hat{t},\hat{x}) \ , \ DV(\hat{t},\hat{x}) = (DU + D\phi)(\hat{t},\hat{x})$$

and

$$D^2V(\hat{t},\hat{x}) \quad \leq \quad (D^2U + D^2\phi)(\hat{t},\hat{x})$$

in the sense of matrices. Combined with (5), this leads to

$$\begin{aligned}
\rho(V-U)(\hat{t},\hat{x}) \quad &\leq \quad \mathcal{L}^{\mathbb{Q}}(V-U)(\hat{t},\hat{x}) \\
&\leq \quad \partial_t\phi(\hat{t},\hat{x}) + \rho(\hat{x})\hat{x}'D\phi(\hat{t},\hat{x}) + \mathrm{Tr}\left[\sigma\sigma'(\hat{x})D^2\phi(\hat{t},\hat{x})\right] \\
&\leq \quad \mathcal{L}^{\mathbb{Q}}\phi(\hat{t},\hat{x}) \ .
\end{aligned}$$

Since $x \mapsto \rho(x)x$ and $x \mapsto \sigma(x)$ have linear growth, we can choose $\kappa > 0$ sufficiently large so that

$$\mathcal{L}^{\mathbb{Q}}\phi = -\kappa\phi + \rho x'D\phi + \mathrm{Tr}\left[\sigma\sigma'D^2\phi\right] < 0 \text{ on } [0,T]\times\mathbb{R}^d \ .$$

This contradicts $(V-U)(\hat{t},\hat{x}) > 0$ since $\rho \geq \eta > 0$. □

Corollary 2 *Assume that v is $C^{1,2}([0,T)\times\mathbb{R}^d)\cap C^0([0,T]\times\mathbb{R}^d)$, then it is the unique $C^{1,2}([0,T)\times\mathbb{R}^d)\cap C^0([0,T]\times\mathbb{R}^d)$ solution of (3) satisfying $v(T,\cdot) = g$ in the class of solutions with polynomial growth. If g is bounded from below and Lipschitz continuous, then there exists $\phi \in \mathcal{A}_b$ such that $Y_T^{v(0,X_0),\phi} = g(X_{0,X_0}(T))$ and $\phi = Dv(\cdot, X_{0,X_0})$ on $[0,T)$.*

Proof. Since g has linear growth and ρ is bounded, we deduce from standard estimates that v has linear growth too. The first result then follows from Theorems 1 and 2. Moreover, an application of Itô's Lemma implies that:

$$\begin{aligned} v(0, X_0) + \int_0^T \beta^0(t) Dv(t, X^0(t)) \sigma(X^0(t)) dW_t^{\mathbb{Q}} &= \beta^0(T) v(T, X^0(T)) \\ &= \beta^0(T) g(X^0(T)) \end{aligned}$$

where $X^0 := X_{0,X_0}$ and $\beta^0 := \beta_{0,X_0}$, which is equivalent to $Y_T^{y_0,\phi} = g(X^0(T))$ with $\phi = Dv(\cdot, X^0)$ on $[0, T)$ and $y_0 := v(0, X_0)$. Since g is bounded from below and ρ^- is bounded, we have $\beta^0(T)g(X^0(T))$ bounded from below. Moreover, the fact that g and all the parameters are Lipschitz continuous implies, by standard estimates, that v is Lipschitz continuous in x, uniformly in time. This implies that Dv is bounded so that $\tilde{Y}^{y_0,\phi}$ is a martingale such that $\tilde{Y}^{y_0,\phi}(T)$ is bounded from below. Hence, it is bounded from below on the time interval $[0, T]$. □

2.3 Verification theorem

In practice, the regularity assumptions of the above theorem are very difficult to check and we have to rely on a weaker definition of solutions, like viscosity solutions (see e.g. [8] and below), or to use a verification theorem which essentially consists in showing that, if a smooth solution of (3) exists, then it coincides with v.

Theorem 3 *(Verification) Assume that there exists a $C^{1,2}([0, T) \times \mathbb{R}^d)$ solution φ to (3) with polynomial growth such that*

$$\lim_{t \nearrow T, z \to x} \varphi(t, z) = g(x) \quad \text{on } \mathbb{R}^d . \tag{6}$$

Then, $v = \varphi$.

Proof. Given $n \geq 1$, set

$$\theta_n := \inf\{s \geq t : \ |X_{t,x}(s)| \geq n\} .$$

Note that $X_{t,x}$ is bounded on $[t, \theta_n \wedge T]$. By Itô's Lemma and the fact that φ solves (3), we obtain

$$\varphi(t, x) = \mathbb{E}^{\mathbb{Q}} \left[\beta_{t,x}(\theta_n \wedge T) \varphi(\theta_n \wedge T, X_{t,x}(\theta_n \wedge T)) \right] \tag{7}$$

for each n. Now, observe that $\theta_n \to \infty$ as $n \to \infty$. In view of (6), this implies that

$$\beta_{t,x}(\theta_n \wedge T)\varphi(\theta_n \wedge T, X_{t,x}(\theta_n \wedge T)) \longrightarrow \beta_{t,x}(T)g(X_{t,x}(T)) \;\; \mathbb{P}-\text{a.s.}$$

Moreover, standard estimates, based on the fact that v has polynomial growth, that ρ is bounded from below, and on the Lipschitz continuity of the coefficients, imply that the sequence $(\beta_{t,x}(\theta_n \wedge T)\varphi(\theta_n \wedge T, X_{t,x}(\theta_n \wedge T)))_{n\geq 1}$ is uniformly integrable. We then deduce that $\varphi = v$ by sending $n \to \infty$ in (7) and using the dominated convergence theorem. □

3 Feynman Kac representation in the viscosity sense

Except when σ satisfies the following type of uniform ellipticity condition

$$\exists\, c > 0 \text{ s.t. } \xi'\sigma\sigma'\xi \geq c|\xi|^2 \;\text{ for all } \xi \in \mathbb{R}^d, \tag{8}$$

it is difficult to show (and in general not true) that v is $C^{1,2}$. Still, it can be shown to solve (3) in a weak sense: the viscosity sense. In the subsections below, we explain this notion and show that v is the unique viscosity solution of (3) satisfying $v(T-,\cdot) = g$, in the class of continuous functions with polynomial growth. We refer to [8] for a general overview of the theory of viscosity solutions.

3.1 Viscosity solutions: definition and main properties

Let F be an operator from $[0,T]\times\mathbb{R}^d\times\mathbb{R}\times\mathbb{R}\times\mathbb{R}^d\times\mathbb{S}^d$ into $\mathbb{R}$, where $\mathbb{S}^d$ denotes the set of d-dimensional symmetric matrices. In this section, we will be mostly interested by the case

$$F(t,x,u,q,p,A) = \rho(x)u - q - \rho(x)x'p - \frac{1}{2}\mathrm{Tr}\left[\sigma\sigma'(x)A\right] \;, \tag{9}$$

so that v solves (3) means

$$F(t,x,v(t,x),\partial_t v(t,x), Dv(t,x), D^2v(t,x)) = 0 \;. \tag{10}$$

We say that F is elliptique if it is non increasing with respect to $A \in \mathbb{S}^d$. This is clearly the case for F defined as in (9). In the following, F will always be assumed to be elliptic.

Let us assume for a moment that v is smooth. Let φ be $C^{1,2}$ and $(\hat{t},\hat{x})$ be a (global) minimum point of $v-\varphi$. After possibly adding a constant to φ, one can always assume that $(v-\varphi)(\hat{t},\hat{x})=0$. In this case, the first and second order optimality conditions imply

$$(\partial_t v, Dv)(\hat{t},\hat{x}) = (\partial_t \varphi, D\varphi)(\hat{t},\hat{x}) \text{ and } D^2 v(\hat{t},\hat{x}) \geq D^2\varphi(\hat{t},\hat{x}) \; .$$

Since F is elliptic and $v \geq \varphi$ on the domain with equality at $(\hat{t},\hat{x})$, we deduce that

$$F(\hat{t},\hat{x},\varphi(\hat{t},\hat{x}),\partial_t\varphi(\hat{t},\hat{x}),D\varphi(\hat{t},\hat{x}),D^2\varphi(\hat{t},\hat{x})) \geq 0$$

whenever

$$F(\hat{t},\hat{x},v(\hat{t},\hat{x}),\partial_t v(\hat{t},\hat{x}),Dv(\hat{t},\hat{x}),D^2 v(\hat{t},\hat{x})) = 0 \; .$$

Conversely, if $(\hat{t},\hat{x})$ is a (global) maximum point of $v-\varphi$ then

$$F(\hat{t},\hat{x},\varphi(\hat{t},\hat{x}),\partial_t\varphi(\hat{t},\hat{x}),D\varphi(\hat{t},\hat{x}),D^2\varphi(\hat{t},\hat{x})) \leq 0 \; .$$

This leads to the following notion of viscosity solution.

Definition 2 *Let F be an elliptic operator as defined above. We say that a l.s.c. (resp. u.s.c) function U is a supersolution (resp. subsolution) of* (10) *on* $[0,T)\times\mathbb{R}^d$ *if for all $\varphi \in C^{1,2}$ and $(\hat{t},\hat{x}) \in [0,T)\times\mathbb{R}^d$ such that $0=\min_{[0,T]\times\mathbb{R}^d}(U-\varphi) = (U-\varphi)(\hat{t},\hat{x})$ (resp. $0=\max_{[0,T]\times\mathbb{R}^d}(U-\varphi) = (U-\varphi)(\hat{t},\hat{x})$), we have:*

$$F(\hat{t},\hat{x},\varphi(\hat{t},\hat{x}),\partial_t\varphi(\hat{t},\hat{x}),D\varphi(\hat{t},\hat{x}),D^2\varphi(\hat{t},\hat{x})) \geq 0 \qquad (11)$$
$$(\text{ resp. } \leq 0) \; .$$

We shall say that a locally bounded function is a discontinuous viscosity solution of $F=0$ if U_* and U^* are respectively super- and subsolution, where, for $(t,x)\in[0,T]\times\mathbb{R}^d$,

$$U_*(t,x) = \liminf_{(s,y)\in[0,T)\times\mathbb{R}^d\to(t,x)} U(s,y) \;\text{ and }\; U^*(t,x) = \limsup_{(s,y)\in[0,T)\times\mathbb{R}^d\to(t,x)} U(s,y) \; .$$

If U is continuous, we simply say that it is a viscosity solution.

Note that a smooth solution U is also a viscosity solution, as any point achieves a min (or max) of $U-U$.

Remark 2 If $(\hat{t}, \hat{x}) \in [0, T) \times \mathbb{R}^d$ achieves a minimum of $U - \varphi$ then it achieves a strict minimum of $U - \bar{\varphi}$ where $\bar{\varphi}(t, x) = \varphi(t, x) - |x - \hat{x}|^4 - |t - \hat{t}|^2$. Moreover, if $\bar{\varphi}$ satisfies (11) at $(\hat{t}, \hat{x})$ then φ satisfies the same equation. It is therefore clear that the notion of minimum can be replaced by that of strict minimum. Similarly, we can replace the notion of maximum by the one of strict maximum in the definition of subsolutions.

3.2 Viscosity property

We can now characterize v as a continuous viscosity solution of (3). The continuity of v follows from standard estimates and we omit the proof.

Theorem 4 *The value function v is continuous on $[0, T] \times \mathbb{R}^d$ and is a viscosity solution on $[0, T) \times \mathbb{R}^d$ of* (3)*.*

Proof. We only prove the supersolution property of v. The proof of the subsolution property is symmetric. Let $\varphi \in C^{1,2}$ be such that $0 = \min_{[0,T] \times \mathbb{R}^d}(v - \varphi) = (v - \varphi)(\hat{t}, \hat{x})$ for some $(\hat{t}, \hat{x}) \in [0, T) \times \mathbb{R}^d$. We proceed by contradiction, i.e. we assume that

$$\rho\varphi(\hat{t}, \hat{x}) - \mathcal{L}^{\mathbb{Q}}\varphi(\hat{t}, \hat{x}) < 0$$

and show that this contradicts (2). Indeed, if the above inequality holds at $(\hat{t}, \hat{x})$, then

$$\rho\varphi(t, x) - \mathcal{L}^{\mathbb{Q}}\varphi(t, x) \le 0$$

on a neighborhood of $(\hat{t}, \hat{x})$ of the form $B := B_r(\hat{t}) \times B_r(\hat{x})$, $r \in (\hat{t}, T - \hat{t})$. By Remark 2, we can then assume that there exists $\eta > 0$ such that

$$v \ge \varphi + \eta \quad \text{on } \partial_p B$$

where $\partial_p B$ is the parabolic boundary of B, i.e. $(B_r(\hat{t}) \times \partial B_r(\hat{x})) \cup (\{\hat{t} + r\} \times \mathrm{cl} B_r(\hat{x}))$.

Let θ be the first exit time of $(t, X_{\hat{t},\hat{x}}(t))_{t \geq \hat{t}}$ from B. By Itô's Lemma applied to φ and the above inequalities, we then obtain

$$\begin{aligned} v(\hat{t}, \hat{x}) = \varphi(\hat{t}, \hat{x}) &= \mathbb{E}\left[\beta_{\hat{t},\hat{x}}(\theta)\varphi(\theta, X_{\hat{t},\hat{x}}(\theta))\right] \\ &- \mathbb{E}\left[\int_{\hat{t}}^{\theta} \beta_{\hat{t},\hat{x}}(s)\left(\mathcal{L}^{\mathbb{Q}}\varphi(s, X_{\hat{t},\hat{x}}(s)) - \rho\varphi(s, X_{\hat{t},\hat{x}}(s))\right) ds\right] \\ &\leq \mathbb{E}\left[\beta_{\hat{t},\hat{x}}(\theta)\left(v(\theta, X_{\hat{t},\hat{x}}(\theta)) - \eta\right)\right] \\ &< \mathbb{E}\left[\beta_{\hat{t},\hat{x}}(\theta)v(\theta, X_{\hat{t},\hat{x}}(\theta))\right] , \end{aligned}$$

a contradiction to (2). □

3.3 Uniqueness

An equivalent definition of viscosity solutions

In order to complete the characterization of v, it remains to show that it is the unique solution of (3) satisfying the boundary condition $v(T, \cdot) = g$. For this purpose, we need an alternative definition of viscosity solutions in terms of super- et subjets.

Note first that, if U is l.s.c., $\varphi \in C^{1,2}$ and $(\hat{t}, \hat{x}) \in [0, T) \times \mathbb{R}^d$ is such that $0 = \min_{[0,T]\times\mathbb{R}^d}(U - \varphi) = (U - \varphi)(\hat{t}, \hat{x})$, then a second order Taylor expansion implies

$$\begin{aligned} U(t, x) &\geq U(\hat{t}, \hat{x}) + \varphi(t, x) - \varphi(\hat{t}, \hat{x}) \\ &= U(\hat{t}, \hat{x}) + \partial_t\varphi(\hat{t}, \hat{x})(t - \hat{t}) \\ &+ (x - \hat{x})' D\varphi(\hat{t}, \hat{x}) + \frac{1}{2}(x - \hat{x})' D^2\varphi(\hat{t}, \hat{x})(x - \hat{x}) + o(|t - \hat{t}| + |x - \hat{x}|^2) . \end{aligned}$$

This naturally leads to the notion of *subjet* defined as the set $\mathcal{P}^- U(\hat{t}, \hat{x})$ of points $(q, p, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ satisfying

$$U(t, x) \geq U(\hat{t}, \hat{x}) + q(t - \hat{t}) + (x - \hat{x})'p + \frac{1}{2}(x - \hat{x})'A(x - \hat{x}) + o(|t - \hat{t}| + |x - \hat{x}|^2) .$$

We define similarly the *superjet* $\mathcal{P}^+ U(\hat{t}, \hat{x})$ as the collection of points $(q, p, A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ such that

$$U(t, x) \leq U(\hat{t}, \hat{x}) + q(t - \hat{t}) + (x - \hat{x})'p + \frac{1}{2}(x - \hat{x})'A(x - \hat{x}) + o(|t - \hat{t}| + |x - \hat{x}|^2) .$$

For technical reasons related to Ishii's Lemma, see below, we will also need to consider the "limit" super- and subjets. More precisely, we define $\bar{\mathcal{P}}^+ U(\hat{t}, \hat{x})$

as the set of points $(q,p,A) \in \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ for which there exists a sequence $(t_n, x_n, q_n, p_n, A_n)_n$ of $[0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ such that $(t_n, x_n, q_n, p_n, A_n) \in \mathcal{P}^+ U(t_n, x_n)$ satisfying $(t_n, x_n,\ U(t_n,x_n),\ q_n,\ p_n, A_n) \to (\hat{t}, \hat{x}, U(\hat{t}, \hat{x}), q, p, A)$. The set $\bar{\mathcal{P}}^- U(\hat{t}, \hat{x})$ is defined similarly.

We can now state the alternative definition of viscosity solutions.

Lemma 2 *Assume that F is continuous. A l.s.c. (resp. u.s.c.) function U is a supersolution (resp. subsolution) of* (10) *on* $[0,T) \times \mathbb{R}^d$ *if and only if for all* $(\hat{t}, \hat{x}) \in [0,T) \times \mathbb{R}^d$ *and all* $(\hat{q}, \hat{p}, \hat{A}) \in \bar{\mathcal{P}}^- U(\hat{t}, \hat{x})$ *(resp.* $\bar{\mathcal{P}}^+ U(\hat{t}, \hat{x})$ *)*

$$F(\hat{t}, \hat{x}, U(\hat{t}, \hat{x}), \hat{q}, \hat{p}, \hat{A}) \geq 0 \quad (\text{ resp. } \leq 0)\ . \tag{12}$$

Proof. We only consider the supersolution property. It is clear that the definition of the lemma implies the Definition 2. Indeed, if $(\hat{t}, \hat{x}) \in [0,T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times \mathbb{S}^d$ is a minimum of $U - \varphi$ then $(\partial_t \varphi, D\varphi, D^2\varphi)(\hat{t}, \hat{x}) \in \bar{\mathcal{P}}^- U(\hat{t}, \hat{x})$. It follows that

$$F(\hat{t}, \hat{x}, U(\hat{t}, \hat{x}), \hat{q}, \hat{p}, \hat{A}) \geq 0$$

with $(\hat{q}, \hat{p}, \hat{A}) = (\partial_t \varphi, D\varphi, D^2\varphi)(\hat{t}, \hat{x})$. Since $U \geq \varphi$ and F is elliptic, this implies the required result.
We now prove the converse implication. Fix $(\hat{t}, \hat{x}) \in [0,T) \times \mathbb{R}^d$ and $(\hat{q}, \hat{p}, \hat{A}) \in \bar{\mathcal{P}}^- U(\hat{t}, \hat{x})$. It is clear that, if $(\hat{q}, \hat{p}, \hat{A}) \in \mathcal{P}^- U(\hat{t}, \hat{x})$, then we can find φ locally $C^{1,2}$ such that $(\hat{q}, \hat{p}, \hat{A}) = (\partial_t \varphi, D\varphi, D^2\varphi)(\hat{t}, \hat{x})$, $\varphi = U$ at $(\hat{t}, \hat{x})$ and $U \geq \varphi$ (see e.g. [11] page 225 for an example of construction). We then have

$$F(\hat{t}, \hat{x}, U(\hat{t}, \hat{x}), \hat{q}, \hat{p}, \hat{A}) \geq 0\ .$$

□

Ishii's Lemma and Comparison Theorem

The last ingredient to prove a comparison theorem is the so-called Ishii's Lemma.

Lemma 3 (Ishii's Lemma) *Let U (resp. V) be a l.s.c. supersolution (resp. u.s.c. subsolution) of* (10) *on* $[0,T) \times \mathbb{R}^d$. *Assume that F is continuous and satisfies*

$$F(t, x, u, q, p, A) = F(t, x, u, 0, p, A) - q$$

for all (t,x,u,q,p,A). *Let* $\phi \in C^{1,2,2}([0,T]\times\mathbb{R}^d\times\mathbb{R}^d)$ *and* $(\hat t,\hat x,\hat y)\in[0,T)\times\mathbb{R}^d\times\mathbb{R}^d$ *be such that*

$$\begin{aligned} W(t,x,y) &:= V(t,x)-U(t,y)-\phi(t,x,y) \le W(\hat t,\hat x,\hat y) \\ & \qquad \forall\ (t,x,y)\in[0,T)\times\mathbb{R}^d\times\mathbb{R}^d . \end{aligned}$$

Then, for all $\eta>0$*, there is* $(q_1,p_1,A_1)\in\bar{\mathcal{P}}^+V(\hat t,\hat x)$ *and* $(q_2,p_2,A_2)\in\bar{\mathcal{P}}^-U(\hat t,\hat y)$ *such that*

$$q_1-q_2=\partial_t\phi(\hat t,\hat x,\hat y) \quad , \quad (p_1,p_2)=(D_x\phi,-D_y\phi)(\hat t,\hat x,\hat y)$$

and

$$\begin{pmatrix} A_1 & 0 \\ 0 & -A_2 \end{pmatrix} \le D_{(x,y)}\phi(\hat t,\hat x,\hat y)+\eta\left(D_{(x,y)}\phi(\hat t,\hat x,\hat y)\right)^2 .$$

Proof. The proof is technical and long, we refer to [8] for details. □

We now prove the expected comparison theorem also called maximum principle.

Theorem 5 *(Comparison) Let* U *(resp.* V*) be a l.s.c. supersolution (resp. u.s.c. subsolution) with polynomial growth of* (3) *on* $[0,T)\times\mathbb{R}^d$*. If* $U\ge V$ *on* $\{T\}\times\mathbb{R}^d$*, then* $U\ge V$ *on* $[0,T]\times\mathbb{R}^d$*.*

Proof. We can assume without loss of generality that $\rho>0$ (otherwise we replace U and V by $\tilde U(t,x):=e^{\kappa t}U(t,x)$ and $\tilde V(t,x):=e^{\kappa t}V(t,x)$ for κ large enough). Assume now that there is some point $(t_0,x_0)\in[0,T]\times\mathbb{R}^d$ such that $U(t_0,x_0)<V(t_0,x_0)$. We shall prove that it leads to a contradiction. Let $\varepsilon>0$, $\kappa>0$ and p be an integer greater than 1 such that $\limsup_{|x|\to\infty}\sup_{t\le T}(|U(t,x)|+|V(t,x)|)/(1+|x|^p)=0$. Then there exists $(\hat t,\hat x)\in[0,T]\times\mathbb{R}^d$ such that

$$0 \ < \ V(\hat t,\hat x)-U(\hat t,\hat x)-\phi(\hat t,\hat x,\hat x)=\max_{(t,x)\in[0,T]\times\mathbb{R}^d}\left(V(t,x)-U(t,x)-\phi(t,x,x)\right) ,$$

where

$$\phi(t,x,y):=\varepsilon e^{-\kappa t}(1+|x|^{2p}+|y|^{2p})$$

and ε is chosen small enough. Since $U\ge V$ on $\{T\}\times\mathbb{R}^d$, it is clear that $\hat t<T$. For all $n\ge1$, we can also find $(t_n,x_n,y_n)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ such that

$$0 \ < \ \Gamma_n(t_n,x_n,y_n)=\max_{(t,x,y)\in[0,T]\times\mathbb{R}^d\times\mathbb{R}^d}\Gamma_n(t,x,y) \qquad (13)$$

where

$$\begin{aligned}\Gamma_n(t,x,y) &:= V(t,x)-U(t,y)-\phi(t,x,y)-n|x-y|^2\\ &- \left(|t-\hat{t}|^2+|x-\hat{x}|^4\right).\end{aligned}$$

It is easily checked that, after possibly passing to a subsequence,

$$(t_n,x_n,y_n,\Gamma_n(t_n,x_n,y_n)) \to (\hat{t},\hat{x},\hat{x},\Gamma_0(\hat{t},\hat{x},\hat{x})) \text{ and } n|x_n-y_n|^2 \to 0\,. \quad (14)$$

Moreover, Ishii's Lemma implies that for all $\eta>0$, we can find $(q_1^n,p_1^n,A_1^n)\in \bar{\mathcal{P}}^+V(t_n,x_n)$ and $(q_2^n,p_2^n,A_2^n)\in\bar{\mathcal{P}}^-U(t_n,y_n)$ such that

$$q_1^n-q_2^n=\partial_t\varphi_n(t_n,x_n,y_n) \quad , \quad (p_1,p_2)=(D_x\varphi_n,-D_y\varphi_n)(t_n,x_n,y_n)$$

and

$$\begin{pmatrix} A_1^n & 0\\ 0 & -A_2^n\end{pmatrix} \le D^2_{(x,y)}\varphi_n(t_n,x_n,y_n)+\eta\left(D^2_{(x,y)}\varphi_n(t_n,x_n,y_n)\right)^2\,.$$

where

$$\varphi_n(t,x,y):=\phi(t,x,y)+n|x-y|^2+|t-\hat{t}|^2+|x-\hat{x}|^4\,.$$

In order to obtain the required contradiction, it now suffices to appeal to Lemma 2 and to argue as in the proof of Theorem 2. Using (14), we obtain that for all $\eta>0$

$$\rho(V-U)(\hat{t},\hat{x}) \quad\le\quad \varepsilon_n+\eta C_n+\mathcal{L}^{\mathbb{Q}}\phi(\hat{t},\hat{x},\hat{x})$$

where $\varepsilon_n\to 0$ is independent of η and C_n does neither depend of η. By sending $\eta\to 0$, we deduce that

$$\rho(V-U)(\hat{t},\hat{x}) \quad\le\quad \varepsilon_n+\mathcal{L}^{\mathbb{Q}}\phi(\hat{t},\hat{x},\hat{x})\,.$$

For $\kappa>0$ big enough so that the second term in the right-hand side is strictly negative and n large enough, we get $\rho(V-U)(\hat{t},\hat{x})\le 0$. This contradicts the fact that $(V-U)(\hat{t},\hat{x})>0$ since ρ is assumed to be (strictly) positive. □

Corollary 3 *The value function v is continuous and is the unique viscosity solution on $[0,T)\times\mathbb{R}^d$ of (3) satisfying $\lim_{s\uparrow T,\, y\to x} v(s,y)=g(x)$ in the class of discontinuous viscosity solutions with polynomial growth.*

Chapter 4

The pricing equation II: the incomplete market case

In this section, we provide the pricing equation under portfolio constraints as studied in Section 2 of Chapter 2.

We keep the notations and assumptions on the coefficients of Chapter 3 except that we now assume that ρ is bounded.
We define the value function:

$$(t,x) \mapsto v(t,x) := \sup_{\nu \in \mathcal{U}_b} J(t,x;\nu)$$

where

$$J(t,x;\nu) := \mathbb{E}^{\mathbb{Q}^{\nu}_{t,x}} \left[\beta_{t,x}(T) g(X_{t,x}(T)) - \int_t^T \beta_{t,x}(s) \delta_K(\nu_s) ds \right]$$

and

$$\frac{d\mathbb{Q}^{\nu}_{t,x}}{d\mathbb{P}} = \mathcal{E}^{\nu}_{t,x}(T)$$

with

$$\begin{aligned} \mathcal{E}^{\nu}_{t,x}(s) &= e^{-\frac{1}{2}\int_t^s |\lambda^{\nu}_{t,x}(u)|^2 du - \int_t^s \lambda^{\nu}_{t,x}(u) dW_u} \\ \lambda^{\nu}_{t,x} &:= \sigma(X_{t,x})^{-1}(\mu(X_{t,x}) - \rho(X_{t,x})X_{t,x} - \nu) . \end{aligned}$$

Since ρ is bounded, we have that $\beta_{t,x}$ and $\beta^{-1}_{t,x}$ are bounded. By replacing $\nu \in \mathcal{U}_b$ by $\beta_{t,x}\nu \in \mathcal{U}_b$ and vice-versa, we deduce from Section 2 of Chapter 2 that

$$p_K(G) = v(0, X_0) .$$

Remark 3 One easily checks that $J(\cdot;\nu)$ is l.s.c. for each $\nu \in \mathcal{U}_b$. It follows that v is l.s.c. as well.

1 Dynamic programming principle

The key result for the derivation of a PDE associated to v is the so-called dynamic programming principle. In the following, we denote by $\mathcal{T}^t_{[t,T]}$ the set of elements of $\mathcal{T}_{[t,T]}$ that are independent on $\mathcal{F}_t$.

Theorem 1 *Fix* $(t,x) \in [0,T) \times \mathbb{R}^d$ *and let* $\{\theta^\nu, \nu \in \mathcal{U}_b\} \subset \mathcal{T}^t_{[t,T]}$ *be such that* $X_{t,x}$ *is essentially bounded on* $[t,\theta^\nu]$ *for each* $\nu \in \mathcal{U}_b$. *Then,*

$$v(t,x) = \sup_{\nu \in \mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^\nu_{t,x}} \left[\beta_{t,x}(\theta^\nu) v(\theta^\nu, X_{t,x}(\theta^\nu)) - \int_t^{\theta^\nu} \beta_{t,x}(s)\delta_K(\nu_s)ds \right] .$$

Proof. For ease of notations, we omit, if not necessary, the dependence of θ with respect to ν. Let $\bar{v}(t,x)$ denote the right-hand side term in the above equation. We first show that $v(t,x) \le \bar{v}(t,x)$. To see this, observe that

$$\begin{aligned} J(t,x;\nu) \;\; = \;\; & \mathbb{E}^{\mathbb{Q}^\nu_{t,x}} \left[\beta_{t,x}(\theta) \mathbb{E}^{\mathbb{Q}^\nu_{t,x}} \left[\beta_{\theta,\zeta}(T) g(X_{\theta,\zeta}(T)) - \int_\theta^T \beta_{\theta,\zeta}(s)\delta_K(\nu_s)ds \mid \mathcal{F}_\theta \right] \right. \\ & \left. - \int_t^\theta \beta_{t,x}(s)\delta_K(\nu_s)ds \right] \end{aligned}$$

where $\zeta := X_{t,x}(\theta)$. We now observe that

$$\begin{aligned} & \mathbb{E}^{\mathbb{Q}^\nu_{t,x}} \left[\beta_{\theta,\zeta}(T) g(X_{\theta,\zeta}(T)) - \int_\theta^T \beta_{\theta,\zeta}(s)\delta_K(\nu_s)ds \mid \mathcal{F}_\theta \right] \\ & = \mathbb{E}^{\mathbb{Q}^\nu_{\theta,\zeta}} \left[\beta_{\theta,\zeta}(T) g(X_{\theta,\zeta}(T)) - \int_\theta^T \beta_{\theta,\zeta}(s)\delta_K(\nu_s)ds \mid \mathcal{F}_\theta \right] \le v(\theta,\zeta) \end{aligned}$$

Hence, the fact that $v \le \bar{v}$.

We now prove the converse inequality. To this purpose, given $k \ge 1$, we denote by $\mathcal{U}_{bk}$ the set of elements $\nu \in \mathcal{U}_b$ such that $\sup_{s \le T}(|\nu_s| + |\delta_K(\nu_s)|) \le k \ \mathbb{P}-$a.s. and set $v_k(s,y) := \sup_{\nu \in \mathcal{U}_{bk}} J(s,y;\nu)$. Note that $v_k \uparrow v$ as $k \to \infty$. Then, for fixed $k \ge 1$, one easily checks that $J(\cdot;\nu)$ and v_k are locally uniformly continuous in (t,x), uniformly in $\nu \in \mathcal{U}_{bk}$.

Let $\mathcal{U}^t_{bk}$ denotes the set of elements of $\mathcal{U}_{bk}$ that are independent of $\mathcal{F}_t$. Then, one easily checks, by using the fact that X solves a Brownian SDE, that $v_k(t,x) = \sup_{\nu \in \mathcal{U}^t_{bk}} J(t,x;\nu)$, see [5].

Fix $\varepsilon > 0$. For $(s,y) \in [0,T] \times \mathbb{R}^d$, we can find $\nu^{s,y} \in \mathcal{U}^s_{bk}$ such that

$$J(s,y;\nu^{s,y}) \geq v(s,y) - \varepsilon \;. \tag{1}$$

Let $A \subset \mathbb{R}^d$ be a compact set such that $X_{t,x}$ takes values in A on $[t,\theta^\nu]$ for each $\nu \in \mathcal{U}_b$. It follows from the local uniform continuity of J and v_k that there exists $\eta > 0$ and a finite collection of points $(t_i,x_i)_{i\leq I} \in [0,T] \times A$ such that $\cup_{i\leq I}[t_i-\eta,t_i] \times B_\eta(x_i) \supset A$.

$$|J(\cdot;\nu^{t_i,x_i}) - J(t_i,x_i;\nu^{t_i,x_i}) + |v - v(t_i,x_i)| \leq \varepsilon \quad \text{on } [t_i-\eta,t_i] \times B_\eta(x_i) \;. \tag{2}$$

Combining (1) and (2) leads to

$$J(\cdot;\nu^{t_i,x_i}) \geq v - 3\varepsilon \quad \text{on } [t_i-\eta,t_i] \times B_\eta(x_i) \supset A_i \;, \tag{3}$$

where the A_i can be constructed in such a way that they form a partition of A. Given $\nu \in \mathcal{U}_{bk}$, we now define

$$\bar{\nu} := \nu \mathbf{1}_{[0,\theta)} + \mathbf{1}_{[\theta,T]} \sum_{i\leq I} \nu^{t_i,x_i} \mathbf{1}_{(\theta,X_{t,x}(\theta)\in A_i}$$

Then, using the fact that $\nu^{s,y}$ is independent of $\mathcal{F}_s$, for all $(s,y) \in [0,T] \times \mathbb{R}^d$, we obtain

$$\begin{aligned} J(t,x;\bar{\nu}) &= \mathbb{E}^{\mathbb{Q}^\nu_{t,x}}\Big[\beta_{t,x}(\theta)\mathbb{E}^{\mathbb{Q}^{\bar{\nu}}_{t,x}}\Big[\beta_{\theta,\zeta}(T)g(X_{\theta,\zeta}(T)) - \int_\theta^T \beta_{\theta,\zeta}(s)\delta_K(\bar{\nu}_s)ds \mid \mathcal{F}_\theta\Big] \\ &\qquad - \int_t^\theta \beta_{t,x}(s)\delta_K(\nu_s)ds\Big] \\ &= \mathbb{E}^{\mathbb{Q}^\nu_{t,x}}\left[\beta_{t,x}(\theta)\left(\sum_{i\leq I} J(\theta,\zeta;\nu^{t_i,x_i})\mathbf{1}_{(\theta,\zeta)\in A_i}\right) - \int_t^\theta \beta_{t,x}(s)\delta_K(\nu_s)ds\right] \end{aligned}$$

so that, by (3),

$$\begin{aligned} v_k(t,x) &\geq J(t,x;\bar{\nu}) \\ &\geq \mathbb{E}^{\mathbb{Q}^\nu_{t,x}}\left[\beta_{t,x}(\theta)v_k(\theta,\zeta) - \int_t^\theta \beta_{t,x}(s)\delta_K(\nu_s)ds\right] - 3\varepsilon\mathbb{E}^{\mathbb{Q}^\nu_{t,x}}[\beta_{t,x}(\theta)] \;. \end{aligned}$$

Sending $\varepsilon \to 0$ and using the arbitrariness of ν, then shows that

$$v_k(t,x) \;\geq\; \sup_{\nu\in\mathcal{U}_{bk}} \mathbb{E}^{\mathbb{Q}^\nu_{t,x}}\left[\beta_{t,x}(\theta)v_k(\theta,\zeta) - \int_t^\theta \beta_{t,x}(s)\delta_K(\nu_s)ds\right] \;.$$

The result then follows by sending $k \to \infty$ and by using the monotone convergence theorem. □

Remark 4 In the above proof, we used the approximation v_k in order to reduce to the case where the value function is u.s.c. A more direct approach, based on test functions, could also be adopted, see [5]. In particular, it would allow to provide a weak version of the dynamic programming principle of Theorem 1 even if v was not known to be measurable a-priori. It would then take the form:

$$v(t,x) \leq \sup_{\nu \in \mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^{\nu}_{t,x}} \left[\beta_{t,x}(\theta^{\nu}) v^*(\theta^{\nu}, X_{t,x}(\theta^{\nu}) - \int_t^{\theta^{\nu}} \beta_{t,x}(s) \delta_K(\nu_s) ds \right] ,$$

and

$$v(t,x) \geq \sup_{\nu \in \mathcal{U}_b} \mathbb{E}^{\mathbb{Q}^{\nu}_{t,x}} \left[\beta_{t,x}(\theta^{\nu}) v_*(\theta^{\nu}, X_{t,x}(\theta^{\nu}) - \int_t^{\theta^{\nu}} \beta_{t,x}(s) \delta_K(\nu_s) ds \right] ,$$

for all family of stopping times $\{\theta^{\nu}, \nu \in \mathcal{U}_b\} \subset \mathcal{T}^t_{[t,T]}$ such that $X_{t,x}$ is essentially bounded on $[t, \theta^{\nu}]$, for each $\nu \in \mathcal{U}_b$.
In the above assertion, v^* could be replaced by v if it is known to be measurable, in particular if it is l.s.c. Hence, for $v = v_*$, it coincides with the formulation of Theorem 1.
In view of the arguments below, the later formulation would already be enough to provide a PDE characterization for v_* and v^*.

2 Hamilton-Jacobi-Bellman pricing equation

In this section, we use the dynamic programming principle of Theorem 1 to show that v is a (discontinuous) viscosity solution of

$$\min \left\{ \rho v - \mathcal{L}^{\mathbb{Q}} v \ , \ \min_{|\zeta|=1} \delta_K(\zeta) - \zeta' D v \right\} = 0 \text{ on } [0,T) \times \mathbb{R}^d \ , \qquad (4)$$

and provide a suitable boundary condition at $t = T$, which is related to the face-lifting phenomenon observed in Section 2 of Chapter 2.

2.1 PDE characterization in the domain

We first discuss the supersolution property.

Proposition 7 *The function v is a viscosity supersolution of* (4).

Proof. Fix $\nu = u$ for some $u \in \mathbb{R}^d$ such that $\delta_K(u) < \infty$. Then, it follows from Theorem 1 that

$$\begin{aligned} v(t,x) &\geq \mathbb{E}^{\mathbb{Q}^{\nu}_{t,x}}\left[\beta_{t,x}(\theta^h)v(\theta^h, X_{t,x}(\theta^h)) - \int_t^{\theta^h} \beta_{t,x}(s)\delta_K(u)ds\right] \\ &= \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta^h)\left(\beta_{t,x}(\theta^h)v(\theta^h, X_{t,x}(\theta^h)) - \int_t^{\theta^h} \beta_{t,x}(s)\delta_K(u)ds\right)\right] , \end{aligned}$$

where $\theta^h := \inf\{s \geq t \ : \ |X_{t,x}(s) - x| + |\mathcal{E}^{\nu}_{t,x}(s) - 1| \geq 1\} \wedge (t+h)$. Let φ be a smooth function such that (t,x) achieves a minimum of $v - \varphi$, recall Remark 3. We can always assume that $(v - \varphi)(t,x) = 0$. Thus,

$$\varphi(t,x) \geq \mathbb{E}\left[\mathcal{E}^{\nu}_{t,x}(\theta^h)\left(\beta_{t,x}(\theta^h)\varphi(\theta^h, X_{t,x}(\theta^h)) - \int_t^{\theta^h} \beta_{t,x}(s)\delta_K(u)ds\right)\right] .$$

By following the same arguments as in the proof Theorem 1 and using the arbitrariness of u, we deduce that φ satisfies:

$$(\rho\varphi - \mathcal{L}^{\mathbb{Q}}\varphi)(t,x) + \delta_K(u) - u'D\varphi(t,x) \geq 0 .$$

Since u is abritrary and the set $\{u \in \mathbb{R}^d \ : \ \delta_K(u) < \infty\}$ is a cone which contains 0, this proves the required result. □

Proposition 8 *The function v^* is a viscosity subsolution of* (4).

Proof. Let φ be a smooth function such that (t,x) achieves a strict local maximum of $v^* - \varphi$. We can always assume that $(v - \varphi)(t,x) = 0$. We argue by contradiction and assume that

$$\min\left\{\rho\varphi - \mathcal{L}^{\mathbb{Q}}\varphi \ , \ \min_{|\zeta|=1} \delta_K(\zeta) - \zeta' D\varphi\right\}(t,x) > 0 .$$

Then,

$$\min\left\{\rho\varphi - \mathcal{L}^{\mathbb{Q}}\varphi \ , \ \min_{|\zeta|=1} \delta_K(\zeta) - \zeta' D\varphi\right\} > 0 \ \text{ on } B_\eta(t,x) \tag{5}$$

for some $\eta > 0$ small enough. Let $(t_n, x_n)_n$ be a sequence in $B_\eta(t,x)$ that converges to (t,x) and such that $v(t_n,x_n) \to v^*(t,x)$. Let θ_n be the first exit time of $B_\eta(t,x)$ by $(s, X_{t_n,x_n}(s))_{s\geq t_n}$. Fix $\nu \in \mathcal{U}_b$. Using Itô's Lemma and (5), we then deduce that

$$\varphi(t_n,x_n) \geq \mathbb{E}^{\mathbb{Q}^{\nu}_{t_n,x_n}}\left[\beta_{t_n,x_n}(\theta_n)\varphi(\theta_n, X_{t_n,x_n}(\theta_n)) - \int_{t_n}^{\theta_n} \beta_{t_n,x_n}(s)\delta_K(\nu_s)ds\right] .$$

Moreover, since (t,x) achieves a strict local maximum of $v^*-\varphi$, we have $v-\varphi \leq v^*-\varphi \leq -\xi$ on $\partial_p B_\eta(t,x)$ for some $\xi > 0$. Hence,

$$\begin{aligned}\varphi(t_n,x_n) &\geq \mathbb{E}^{\mathbb{Q}^\nu_{t_n,x_n}}\left[\beta_{t_n,x_n}(\theta_n)v(\theta_n,X_{t_n,x_n}(\theta_n)) - \int_{t_n}^{\theta_n}\beta_{t_n,x_n}(s)\delta_K(\nu_s)ds\right] \\ &\quad +\xi\mathbb{E}^{\mathbb{Q}^\nu_{t_n,x_n}}[\beta_{t_n,x_n}(\theta_n)]\ .\end{aligned}$$

Since ρ is bounded, one easily checks that $\mathbb{E}^{\mathbb{Q}^\nu_{t_n,x_n}}[\beta_{t_n,x_n}(\theta_n)] \geq c$ for some $c > 0$, for all n and $\nu \in \mathcal{U}_b$. We then obtain

$$\begin{aligned}v(t_n,x_n) &\geq \mathbb{E}^{\mathbb{Q}^\nu_{t_n,x_n}}\left[\beta_{t_n,x_n}(\theta_n)v(\theta_n,X_{t_n,x_n}(\theta_n)) - \int_{t_n}^{\theta_n}\beta_{t_n,x_n}(s)\delta_K(\nu_s)ds\right] \\ &\quad +\xi c + v(t_n,x_n) - \varphi(t_n,x_n)\ .\end{aligned}$$

Since $v(t_n,x_n)-\varphi(t_n,x_n) \to 0$ as $n \to \infty$, we obtain a contradiction to Theorem 1 for n large enough. □

2.2 Boundary condition at $t = T$

In order to complete the characterization of v, it remains to provide a terminal condition. We shall show below that

$$v(T-,\cdot) = \hat{g}$$

where $\hat{g}$ is defined as in Chapter 2:

$$\hat{g}(x) := \sup_{\zeta\in\mathbb{R}^d} g(x+\zeta) - \delta_K(\zeta)\ .$$

We split the proof in two separate propositions.

Proposition 9 *For all $x \in \mathbb{R}^d$, $v(T,x) \geq \hat{g}(x)$.*

Proof. Let $(t_n,x_n)_{n\geq 1}$ be a sequence such that $(t_n,x_n) \to (T,x)$ and $v(t_n,x_n) \to v(T,x)$. By the definition of v, we have

$$v(t_n,x_n) \geq \mathbb{E}^{\mathbb{Q}^{\nu^n}_{t_n,x_n}}\left[\beta^n(T)g\left(X^n(T)\right) - \int_{t_n}^T \beta^n(s)\delta_K\left(\nu^n_s\right)ds\right]$$

where $(\beta^n, X^n) := (\beta_{t_n,x_n}, X_{t_n,x_n})$ and $\nu^n_s := \frac{1}{T-t_n}u$, for some $u \in \operatorname{dom}(\delta_K)$. Now observe that $\delta_K(\lambda u) = \lambda\delta_K(u)$ for every $\lambda > 0$, so that

$$\int_{t_n}^T \beta^n(s)\delta_K\left(\nu^n_s\right)ds = \delta_K(u)\frac{1}{T-t_n}\int_{t_n}^T \beta^n(s)ds \underset{n\to\infty}{\longrightarrow} \delta_K(u)\ \mathbb{P}-\text{a.s.}$$

since ρ is bounded. Hence, using the fact that ρ is bounded again and a dominated convergence argument, we obtain

$$\begin{aligned} v(T,x) &= \lim_{n\to\infty} v\left(t_n,x_n\right) \\ &\geq \liminf_{n\to\infty} \mathbb{E}^{\mathbb{Q}^{\nu^n}_{t_n,x_n}}\left[\beta^n(T) g\left(X^n(T)\right)\right] - \delta_K(u). \end{aligned}$$

To conclude the proof, it remains show that $E_n := \mathbb{E}^{\mathbb{Q}^{\nu^n}_{t_n,x_n}}\left[\beta^n(T) g\left(X^n(T)\right)\right] \to g(x+u)$, and use the arbitrariness of $u \in \mathrm{dom}(\delta_K)$.
To see this, first observe that

$$X^n = u\frac{\cdot - t_n}{T-t_n} + x_n + \int_{t_n}^{\cdot} \rho\left(X^n(s)\right) X^n(s) ds + \int_{t_n}^{\cdot} \sigma\left(X^n(s)\right) dW_s^{\nu^n},$$

so that

$$E_n = \mathbb{E}\left[\beta^n(T) g\left(Z^n(T)\right)\right],$$

where Z^n satisfies

$$Z^n = u\frac{\cdot - t_n}{T-t_n} + x_n + \int_{t_n}^{\cdot} \rho\left(Z^n(s)\right) Z^n(s) ds + \int_{t_n}^{\cdot} \sigma\left(Z^n(s)\right) dW_s.$$

Clearly, the sequence $(Z^n(T))_{n\geq 1}$ is bounded in L^2 and converges to $x+u$ $\mathbb{P}-$a.s. Since g is continuous with linear growth, the dominated convergence theorem implies

$$\lim_{n\to\infty} E_n = g(x+u).$$

□

Proposition 10 *Assume that $\hat{g}$ is upper-semicontinuous with linear growth. Assume further that σ is bounded. Then, for all $x \in \mathbb{R}^d$, $v^*(T,x) \leq \hat{g}(x)$.*

Proof. Let $(t_n, x_n)_n$ be a sequence which converges to (T, x_0) and such that $v(t_n, x_n) \to v^*(T, x_0)$. Set $(\beta^n, X^n) = (\beta_{t_n,x_n}, X_{t_n,x_n})$. By definition of v, there is some $\nu^n \in \mathcal{U}_b$ such that

$$v(t_n,x_n) \;\; \leq \;\; \mathbb{E}^{\mathbb{Q}^{\nu^n}}\left[\beta^n(T) g(X^n(T)) - \int_{t_n}^T \beta^n(s)\delta_K(\nu^n_s) ds\right] + n^{-1}\,.$$

Since $\mathrm{dom}(\delta_K)$ is a convex cone and δ_K is 1-homogeneous, we have

$$\begin{aligned} \beta^n(T) g(X^n(T)) \;\; &\leq \;\; \beta^n(T)\hat{g}\left(X^n(T) - \int_{t_n}^T \beta^n(T)^{-1}\beta^n(s)\nu^n_s ds\right) \\ &\quad + \int_{t_n}^T \beta^n(s)\delta_K(\nu^n_s) ds\,. \end{aligned}$$

This implies that

$$v(t_n, x_n) \;\; \le \;\; \mathbb{E}^{\mathbb{Q}^{\nu^n}}\left[\beta^n(T)\hat{g}\left(X^n(T) - \int_{t_n}^T \beta^n(T)^{-1}\beta^n(s)\nu_s^n ds\right)\right] + n^{-1} .$$

In view of the above inequalities and the definition of (t_n, x_n), it remains to show that

$$\limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}^{\nu^n}}\left[\beta^n(T)\hat{g}\left(X^n(T) - \int_{t_n}^T \beta^n(T)^{-1}\beta^n(s)\nu_s^n ds\right)\right] \;\; \le \;\; \hat{g}(T, x_0) \; .(6)$$

From now on, we assume that $\hat{g}$ is uniformly Lipschitz continuous. We shall explain at the end of the proof how to handle cases where it is not true. If $\hat{g}$ is L-Lipschitz, then

$$\begin{aligned}
&\left|\beta^n(T)\hat{g}\left(X^n(T) - \int_{t_n}^T \beta^n(T)^{-1}\beta^n(s)\nu_s^n ds\right) - \beta^n(T)\hat{g}(x_0)\right| \\
&\le L\left|\beta^n(T)X^n(T) - \int_{t_n}^T \beta^n(s)\nu_s^n ds - \beta^n(T)x_0\right| \\
&= L\left|\int_{t_n}^T \beta^n(s)\sigma(X^n(s))dW_s^{\nu^n} + x_n - \beta^n(T)x_0\right|
\end{aligned}$$

where, since σ and ρ are bounded,

$$\mathbb{E}^{\mathbb{Q}^{\nu^n}}\left[\left|\int_{t_n}^T \beta^n(s)\sigma(X^n(s))dW_s^{\nu^n}\right|\right] \le C(T - t_n)^{\frac{1}{2}}$$

for some $C > 0$ independent of n. This proves the required result for $\hat{g}$ Lipschitz. In the case, we $\hat{g}$ is not Lipschitz, then we construct, for each $\varepsilon > 0$, a Lipschitz function Ψ_ε such that $|\hat{g}(x_0) - \Psi_\varepsilon(x_0)| \le \varepsilon$ and $\Psi_\varepsilon \ge \hat{g}$. It follows that, for each ε, we can find some finite $L_\varepsilon > 0$ such that

$$\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}^{\nu_n}}\left[\beta^n(T)\hat{g}\left(X^n(T) - \int_{t_n}^T \beta^n(T)^{-1}\beta^n(s)\nu_s^n ds\right)\right] \\
&\le \limsup_{n\to\infty} \mathbb{E}^{\mathbb{Q}^{\nu_n}}\left[\beta^n(T)\Psi_\varepsilon\left(X^n(T) - \int_{t_n}^T \beta^n(T)^{-1}\beta^n(s)\nu_s^n ds\right)\right] \\
&\le \Psi_\varepsilon(x_0) + \limsup_{n\to\infty} L_\varepsilon \, C \, \left(|x_n - x_0| + (T - t_n)^{1/2}\right) \\
&= \Psi_\varepsilon(x_0) \le \hat{g}(x_0) + \varepsilon
\end{aligned}$$

and the proof is concluded by sending ε to 0.

We conclude this proof by constructing the sequence of functions $(\Psi_\varepsilon)_{\varepsilon>0}$. For $x \in \mathbb{R}^d$, we define

$$G_k(x) \quad = \quad \sup_{z\in\mathbb{R}^d} [\hat{g}(z) - k|z-x|] \quad , \ k \geq 1 \ .$$

Recall that g has linear growth. Clearly, $G_k \geq \hat{g}$ and G_k is k-Lipschitz. Moreover, taking k large enough, it follows from the linear growth and upper-semicontinuity assumptions on $\hat{g}$ that, for all $x \in \mathbb{R}^d$, the maximum is attained in the above definition by some $x_k(x)$. In particular,

$$G_k(x) \quad = \quad \hat{g}(x_k(x)) - k|x_k(x) - x| \ \geq \ \hat{g}(x) \ .$$

Using the linear growth of $\hat{g}$ again, we deduce that $x_k(x) \to x$ as $k \to \infty$ after possibly passing to a subsequence. Since $\hat{g}$ is upper-semicontinuous, this also implies that

$$\hat{g}(x_0) \quad \geq \quad \limsup_{k\to\infty} \hat{g}(x_k(x_0)) \ \geq \ \limsup_{k\to\infty} G_k(x_0) \ \geq \ \hat{g}(x_0) \ .$$

We can then choose k_ε such that $|G_{k_\varepsilon}(x_0) - \hat{g}(x_0)| \leq \varepsilon$ and set $\Psi_\varepsilon := G_{k_\varepsilon}$. $\square$

3 Example: non-hedgeable stochastic volatilty

As an example of application, let us come back to the model of Example 4 of Chapter 2. Note here that the volatility of X^1 is given by $X^1\sigma(X^2)$ which is not bounded. However, the above argument holds. Moreover, X^1 takes values in $(0,\infty)$ but it does not change anything in the above proofs.

In this case, the price function v is therefore a (discontinuous) viscosity solution on $[0,T) \times (0,\infty) \times \mathbb{R}$ of

$$\min\left\{r\varphi - \mathcal{L}^{\mathbb{Q}}\varphi \ , \ \min_{|\zeta|=1} \delta_K(\zeta) - \zeta' D\varphi\right\}(t,x) = 0 \tag{7}$$

where

$$\mathcal{L}^{\mathbb{Q}}\varphi = \partial_t\varphi + rx^1\partial_{x^1}\varphi + \frac{1}{2}\left[(x^1\sigma(x^2))^2\partial^2_{x^1x^1}\varphi + \gamma^2\partial^2_{x^2x^2}\varphi + 2x^1\sigma(x^2)\gamma_1\partial^2_{x^1x^2}\varphi\right]$$

with $\gamma^2 := \gamma_1^2 + \gamma_2^2$. Moreover, it satisfies $v_*(T,x^1,x^2) = v^*(T,x^1,x^2) = \hat{g}(x^1)$.

Since $\delta_K(\zeta) = 0$ is $\zeta^1 = 0$ and $\delta_K(\zeta) = \infty$ is $\zeta^1 \neq 0$, we deduce that

$$v_*(T,x^1,x^2) = v^*(T,x^1,x^2) = \hat{g}(x^1) = g(x)$$

and, using the right-hand side term in (7), that v is a supersolution of $\partial_{x^2}\varphi = 0$ and $-\partial_{x^2}\varphi = 0$ on $[0,T)\times(0,\infty)\times\mathbb{R}$. As for smooth functions, this implies that v does not depend on x^2. We therefore now simply write $v(t,x^1)$. As for smooth function again, this also implies that v is a supersolution on $[0,T)\times(0,\infty)$ of

$$\inf_{x^2} \mathcal{H}_{x^2} = 0 \tag{8}$$

where

$$\mathcal{H}_{x^2}\varphi := r\varphi - \partial_t\varphi - rx^1\partial_{x^1}\varphi - \frac{1}{2}(x^1\sigma(x^2))^2\partial^2_{x^1x^1}\varphi\ ,$$

i.e.

$$r\varphi - \partial_t\varphi - rx^1\partial_{x^1}\varphi - \frac{1}{2}(x^1)^2\left[\underline{\sigma}^2\mathbf{1}_{\partial^2_{x^1x^1}\varphi<0} + \bar{\sigma}^2\mathbf{1}_{\partial^2_{x^1x^1}\varphi\geq 0}\right]\partial^2_{x^1x^1}\varphi = 0\ , \tag{9}$$

where $\bar{\sigma} := \sup_{x^2}\sigma(x^2)$ and $\underline{\sigma} := \inf_{x^2}\sigma(x^2)$. This is the so-called Black-Scholes-Barenblatt equation.

When $\bar{\sigma} < \infty$, σ is continuous and g is continuous with linear growth, it is possible to show that this equation admits a comparison principle in the class of functions with linear growth. In particular, if there exists a smooth solution, say φ, satisfying $\varphi(T-,\cdot) = g$, then $v \geq \varphi$. But on the other hand, (9) and the previous boundary condition imply that

$$\begin{aligned}
\varphi(0,X^1_0) + \int_0^T \beta_s D\varphi(s,X^1_s)dX^1_s &= \beta_T\varphi(T,X^1_T) + \int_0^T \beta_s\mathcal{H}_{X^2_s}\varphi(s,X^1_s)ds \\
&\geq \beta_T\varphi(T,X^1_T) \\
&= \beta_T g(X^1_T)\ ,
\end{aligned}$$

where $X := X_{0,X_0}$ and $\beta = \beta_{0,X_0}$. This shows that $v = \varphi$.

In the limiting case where $\bar{\sigma} = \infty$, then (8) implies that v is concave in x^1. If moreover, $\underline{\sigma} = 0$ and $r = 0$, then it should be non-increasing in time. This implies that $v \geq \bar{g}$, where $\bar{g}$ denotes the concave envelope of g. On the other hand, it is clear that, starting with $\bar{g}(X_0)$ allows to find a super-hedging strategy, which is actually of buy-and-hold type. Hence, $v = \bar{g}$. Note that $\bar{g}(x^1) = x^1$ for $g(x^1) = [x^1 - \kappa]^+$! The same holds for $r \neq 0$ up to passing to discounted quantities.

Chapter 5

Approximate hedging and risk control

In this section, we discuss two approximate hedging technics that were discussed in [12] and [13]. We shall restrict here to the case of complete markets without constraints, because it is *essentially* the only case where explicit formulations can be obtained by standard convex duality technics, and it already provides the general form of the solution. Extensions to incomplete markets are considered in the above mentioned papers. More general models will be discussed in Chapter 7, in a Markovian setting.

1 Quantile hedging

We first discuss the case of a trader who wants to hedge a random payoff $G \in L^0(\mathbb{R}_+) \setminus \{0\}$ from an initial wealth $y > 0$. However, because for instance the hedging price $p(G) = \mathbb{E}^{\mathbb{Q}}[\beta_T G]$ is too high (which can be due to the fact that it was face-lifted in order to avoid explosion of the hedging strategy near the maturity, see Section 2 of Chapter 1), his initial wealth is strictly less than $p(G)$.

1.1 Minimizing the probability of missing the hedge

The first criteria we discuss here is the so-called *quantile hedging* criteria. Namely, we try to find the optimal solution to the problem

$$\inf_{\phi \in \mathcal{A}_+(y)} \mathbb{P}\left[G > Y_T^{y,\phi}\right] \text{ for some } 0 < y < p(G), \tag{1}$$

where $\mathcal{A}_+(y)$ is the restriction of $\mathcal{A}_b$ to strategies leading to non-negative wealth processes.

As shown in [12], this problem can be reduced to a standard test problem in mathematical statistics, which can then be solved by using the Neyman and Pearson's Lemma which we recall below.
To see this, we first note that the problem (1) can be reduced as follows.

Proposition 11 *The following holds:*

$$\sup_{\phi \in \mathcal{A}_+(y)} \mathbb{P}\left[Y_T^{y,\phi} \geq G\right] = \sup\left\{\mathbb{E}\left[\varphi\right] \ , \ \varphi \in L^0(\{0,1\}) \ s.t. \ \mathbb{E}^{\mathbb{Q}}[\beta_T G \varphi] \leq y\right\} \ . \quad (2)$$

Proof. Let us first fix $\phi \in \mathcal{A}_+(y)$. Then, $\varphi := \mathbf{1}_{Y_T^{y,\phi} \geq G}$ satisfies $\mathbb{P}\left[Y_T^{y,\phi} \geq G\right] = \mathbb{E}[\varphi]$ and $G\varphi \leq Y_T^{y,\phi}$ so that $\mathbb{E}^{\mathbb{Q}}[\beta_T G \varphi] \leq \mathbb{E}^{\mathbb{Q}}[\beta_T Y_T^{y,\phi}] \leq y$, see Chapter 2. This shows that the left-hand side term in (2) is smaller than the right-hand side term. Conversely, if $\varphi \in L^0(\{0,1\})$ is such that $\mathbb{E}^{\mathbb{Q}}[\beta_T G\varphi] \leq y$, then it follows from Chapter 2 that there exists $\phi \in \mathcal{A}_b(y)$ such that $Y_T^{y,\phi} \geq G\varphi$. Since $G\varphi \geq 0$, the super-martingale $Y^{y,\phi}$ remains non-negative so that $\phi \in \mathcal{A}_+(y)$. Moreover, $Y_T^{y,\phi} \geq G$ on $\{\varphi = 1\}$. Since $\varphi \in L^0(\{0,1\})$, this implies that $\mathbb{P}\left[Y_T^{y,\phi} \geq G\right] \geq \mathbb{E}\left[\varphi\right]$. □

We next observe that the right-hand side problem in (2) can be interpreted as a statistical test problem:

$$\sup\left\{\mathbb{E}\left[\varphi\right] \ , \ \varphi \in L^0([0,1]) \text{ s.t. } \mathbb{E}^{\mathbb{Q}_G}[\varphi] \leq y/p(G)\right\} \ , \quad (3)$$

where $\mathbb{Q}_G$ is defined by

$$\frac{d\mathbb{Q}_G}{d\mathbb{P}} := \frac{d\mathbb{Q}}{d\mathbb{P}} \frac{\beta_T G}{\mathbb{E}^{\mathbb{Q}}[\beta_T G]},$$

except that we look for a solution of the above test problem in $L^0(\{0,1\})$.

The solution to this problem is given by Neyman and Pearson's Lemma which we now state.

Lemma 4 (Neyman and Pearson) *Let $\mathbb{P}_0$ and $\mathbb{P}_1$ be two probability measures that are absolutely continuous with respect to $\mathbb{P}$. Given $\alpha \in]0,1[$, the solution to the problem*

$$\sup\left\{\mathbb{E}^{\mathbb{P}_1}\left[\xi\right] \ : \ \xi \in L^0([0,1]), \ \mathbb{E}^{\mathbb{P}_0}\left[\xi\right] \leq \alpha\right\} \ ,$$

is given by any random variable of the form

$$\hat{\xi} := \mathbf{1}_{\frac{d\mathbb{P}_1}{d\mathbb{P}} > \hat{a}\frac{d\mathbb{P}_0}{d\mathbb{P}}} + \hat{\gamma}\mathbf{1}_{\frac{d\mathbb{P}_1}{d\mathbb{P}} = \hat{a}\frac{d\mathbb{P}_0}{d\mathbb{P}}}$$

where

$$\hat{a} := \inf\left\{a > 0 \ : \ \mathbb{P}_0\left[\frac{d\mathbb{P}_1}{d\mathbb{P}} > a\frac{d\mathbb{P}_0}{d\mathbb{P}}\right] \leq y\right\}$$

and $\hat{\gamma} \in [0, 1]$ *is such that* $\mathbb{E}^{\mathbb{P}_0}\left[\hat{\xi}\right] = \alpha$.

Remark 5 In the above Lemma, ξ has to be interpreted has a random test of $\text{Hyp}_0 : \mathbb{P}_0$ against $\text{Hyp}_1 : \mathbb{P}_1$. If the state of nature ω is such that $\xi(\omega) = p$, then one accepts Hyp_0 with probability $1 - p$. The quantity $\mathbb{E}^{\mathbb{P}_0}[\xi]$ corresponds to the probability to reject Hyp_0 while Hyp_0 is true (this is the risk of first kind), and $\mathbb{E}^{\mathbb{P}_1}[\xi]$ corresponds to the probability to reject Hyp_0 while Hyp_0 is indeed false (this is called the power of the test).The test $\hat{\xi}$ is called UMP (uniformly most powerful) of size α.

Applying the above lemma to the problem (3) leads to an optimal solution $\hat{\varphi}$ of the following form:

Theorem 1 *Assume that*

$$\hat{c} := \inf\left\{c > 0 \ : \ \mathbb{E}^{\mathbb{Q}}\left[\beta_T G\mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > c\frac{d\mathbb{Q}_G}{d\mathbb{Q}}}\right] \leq y\right\}$$

is such that

$$\mathbb{E}^{\mathbb{Q}}\left[\beta_T G\mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > \hat{c}\frac{d\mathbb{Q}_G}{d\mathbb{Q}}}\right] = y \ .$$

Then, the optimal solution to the problem (1) *is given by the strategy* $\hat{\phi} \in \mathcal{A}_+(y)$ *satisfying*

$$Y_T^{y,\hat{\phi}} = G\hat{\varphi}$$

where

$$\hat{\varphi} = \mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > \hat{c}\frac{d\mathbb{Q}_G}{d\mathbb{Q}}} \ .$$

In most applications, $\hat{c} > 0$ is such that $\mathbb{E}^{\mathbb{Q}}\left[\beta_T G\mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > c\frac{d\mathbb{Q}_G}{d\mathbb{Q}}}\right] = y$, recall that $y < p(G) = \mathbb{E}^{\mathbb{Q}}[\beta_T G]$, so that the optimal strategy $\hat{\phi}$ satisfies

$$Y_T^{y,\hat{\phi}} = G\mathbf{1}_{\hat{A}}$$

for $\hat{A} := \{d\mathbb{P}/d\mathbb{Q} > \hat{c}\; d\mathbb{Q}_G/d\mathbb{Q}\}$. It means that the optimal solution consists in hedging a digital type option which pays G on $\hat{A}$ and 0 otherwise.
Such a behavior is certainly not nice in practice since it may lead, as in general for discontinuous payoffs, to an explosion of the number of assets to have in the portfolio near to the maturity.
Note that, right from the beginning, one could criticize the criteria which is only concerned with the probability of not missing the hedge but does not take into account of the sizes of the potential losses.

In the case where $\hat{c} > 0$ only satisfies $\mathbb{E}^{\mathbb{Q}}\left[\beta_T G \mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > c \frac{d\mathbb{Q}_G}{d\mathbb{Q}}}\right] < y$, the above Theorem does not apply. However, the same reasoning can still be applied for the optimal *success ratio* problem

$$\sup_{\phi \in \mathcal{A}_+(y)} \mathbb{E}\left[\frac{Y_T^{y,\phi}}{G} \wedge 1\right] \quad \text{for some } 0 < y < p(G), \tag{4}$$

with the convention $z/0 = \infty$ for $z \in \mathbb{R}$.

Theorem 2 *The optimal solution to the problem* (4) *is given by the strategy* $\hat{\phi} \in \mathcal{A}_+(y)$ *satisfying*

$$Y_T^{y,\hat{\phi}} = G\hat{\varphi}$$

where

$$\hat{\varphi} = \mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > \hat{c} \frac{d\mathbb{Q}_G}{d\mathbb{Q}}} + \hat{\gamma} \mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} = \hat{c} \frac{d\mathbb{Q}_G}{d\mathbb{Q}}}$$

with

$$\hat{c} := \inf\left\{c > 0 \;:\; \mathbb{E}^{\mathbb{Q}}\left[\beta_T G \mathbf{1}_{\frac{d\mathbb{P}}{d\mathbb{Q}} > c \frac{d\mathbb{Q}_G}{d\mathbb{Q}}}\right] \le y\right\}$$

and where $\hat{\gamma} \in [0,1]$ *is such that* $\mathbb{E}^{\mathbb{Q}}[\beta_T G \hat{\varphi}] = y$.

Note that, when $\hat{\gamma} = 0$, then it coincides with the solution of the quantile hedging problem.

1.2 Quantile hedging price

The quantile hedging price of the payoff G is the minimal initial wealth that allows to hedge the option with a given probability of success, namely

$$p(G;\alpha) := \inf\left\{y \ge 0 \;:\; \exists\, \phi \in \mathcal{A}_+(y) \text{ s.t. } \mathbb{P}[Y_T^{y,\phi} \ge G] \ge \alpha\right\}, \text{ for } \alpha \in [0,1]\,.$$

Clearly, $p(G;1) = p(G)$ and $p(G;0) = 0$. For $\alpha \in (0,1)$, it can be computed thanks to the results of the previous section. Indeed, given $0 < y < p(G)$, one can find $\alpha(y) \in (0,1)$ such that

$$\alpha(y) = \sup_{\phi \in \mathcal{A}_+(y)} \mathbb{P}\left[Y_T^{y,\phi} \geq G\right] .$$

Then, by definition,

$$p(G;\alpha) = \inf\{y \geq 0 \ : \ \alpha(y) \geq \alpha\} .$$

We shall see in Chapter 7 how the quantile hedging price can be directly related to a PDE, without having to invert the value function of an optimization problem, as suggested here.

2 Hedging under expected loss constraints

2.1 Minimizing the expected shortfall

In order to better take into account the amount of possible losses, we now consider a risk control criteria of the form $\ell((G - V_T^{x,\phi})^+)$, i.e. we try to minimize

$$\inf_{\phi \in \mathcal{A}_+(y)} \mathbb{E}\left[\ell((G - Y_T^{y,\phi})^+)\right] \text{ for some } 0 < y < p(G) , \tag{5}$$

where, as above, $\mathcal{A}_+(y)$ is the restriction of $\mathcal{A}_b$ to strategies leading to non-negative wealth processes. Here, the *loss function* ℓ is C^1 strictly convex, increasing and defined on $\mathbb{R}_+$, $\ell(0) = 0$, and such that $\nabla\ell(+\infty) = \infty$, $\nabla\ell(0+) = 0$. We note $I := (\nabla\ell)^{-1}$, the inverse of the derivative of ℓ. As above, we assume that $G \in L^0(\mathbb{R}_+) \setminus \{0\}$

Theorem 3 *There exists a solution $\hat{\phi} \in \mathcal{A}_+(y)$ to the problem* (5)*. It satisfies*

$$Y_T^{y,\hat{\phi}} = \hat{\varphi}(\hat{c})G$$

where, for $c > 0$,

$$\hat{\varphi}(c) := \mathbf{1}_{G>0}\left(1 - \frac{I(c\beta_T d\mathbb{Q}/d\mathbb{P})}{G} \wedge 1\right) ,$$

and $\hat{c} > 0$ is the unique positive solution of

$$\mathbb{E}^{\mathbb{Q}}\left[\beta_T \hat{\varphi}(c) G\right] = y .$$

Proof. 1. First of all, one can observe that

$$\mathbb{E}\left[\ell((G - Y_T^{y,\phi})^+)\right] = \mathbb{E}\left[\ell(G(1-\varphi^\phi))\right]$$

where $\varphi^\phi := [(Y_T^{y,\phi}/G) \wedge 1]\mathbf{1}_{G>0}$ satisfies $\mathbb{E}^{\mathbb{Q}}\left[\beta_T \varphi^\phi G\right] \le y$. Conversely, if $\varphi \in L^0([0,1])$ satisfies the above constraint, then φG can be reached by a financial portfolio starting from y whose discounted value is a $\mathbb{Q}$-martingale and therefore remains non negative, since $G \ge 0$, see Section 1 in Chapter 2. The above problem is thus equivalent to

$$\inf_{\varphi \in L^0([0,1])} \mathbb{E}\left[\ell((1-\varphi)G)\right] \text{ under the constraint } \mathbb{E}^{\mathbb{Q}}\left[\beta_T \varphi G\right] \le y. \tag{6}$$

2. We now check that existence holds in the above problem by using the following technical lemma which we state without proof.

Lemma 5 (Komlos Lemma) *Let $(\zeta_n)_n$ be a sequence of random variables that are uniformly bounded in $L^1(\mathbb{P})$. Then, there exists a sequence $(\bar\zeta_n)_n$ and a random variable $\bar\zeta$ in $L^1(\mathbb{P})$ such that $\bar\zeta_n \to \bar\zeta$ $\mathbb{P}-a.s.$ and*

$$\bar\zeta_n \in conv(\zeta_k,\ k \ge n) \quad \mathbb{P}-a.s.$$

for all $n \ge 1$, where conv denotes the convex envelope.

Since $\varphi \mapsto \mathbb{E}\left[\ell(G(1-\varphi))\right]$ is convex, one deduces from the preceding Lemma that there exists a minimizing sequence $(\varphi_n)_n$ which converges $\mathbb{P}-$a.s. to some $\hat\varphi$ in $L^0([0,1])$. One concludes by using Fatou's Lemma and the fact that $\ell \ge 0$.

3. We now check that $\hat\varphi$ has the form given in the Theorem. Given $\varphi \in L^0([0,1])$ and $\varepsilon \in [0,1]$, let us set

$$\varphi_\varepsilon := \varepsilon\varphi + (1-\varepsilon)\hat\varphi$$

and

$$F_\varphi(\varepsilon) := \mathbb{E}\left[\ell((1-\varphi_\varepsilon)G)\right] .$$

Recall that ℓ is convex so that its derivative is non-decreasing. Using a monotone convergence argument, one then easily checks that the right-derivative $\nabla F_\varphi(0+)$ of F_φ at 0 exists and satisfies

$$\nabla F_\varphi(0+) = \mathbb{E}\left[\nabla\ell((1-\hat\varphi)G)(\hat\varphi - \varphi)G\right] .$$

Since F_φ is convex, because ℓ is convex, $\hat{\varphi}$ should satisfy the first order optimality condition $\nabla F_\varphi(0+) \geq 0$ for any $\varphi \in L^0([0,1])$. This amounts to say that $\hat{\varphi}$ satisfies

$$\mathbb{E}^{\mathbb{Q}_{\hat{\varphi}}}\left[\hat{\varphi}\right] \geq \mathbb{E}^{\mathbb{Q}_{\hat{\varphi}}}\left[\varphi\right] \tag{7}$$

for any $\varphi \in L^0([0,1])$ such that, recall (6),

$$\mathbb{E}^{\mathbb{Q}_G}\left[\varphi\right] \leq \frac{y}{p(G)} =: \alpha \tag{8}$$

where $\mathbb{Q}_{\hat{\varphi}}$ and $\mathbb{Q}_G$ are the probability measures associated to the densities

$$\begin{aligned} \frac{d\mathbb{Q}_{\hat{\varphi}}}{d\mathbb{P}} &= \nabla\ell((1-\hat{\varphi})G)G/\mathbb{E}\left[\nabla\ell((1-\hat{\varphi})G)G\right] \\ \frac{d\mathbb{Q}_G}{d\mathbb{P}} &= \frac{d\mathbb{Q}}{d\mathbb{P}}\beta_T G/\mathbb{E}^{\mathbb{Q}}\left[\beta_T G\right] . \end{aligned}$$

As in the previous section, this can be interpreted as a random test: test the hypothesis $\mathbb{Q}_{\hat{\varphi}}$ against $\mathbb{Q}_G$ with a level α. It then follows from Neyman and Pearson's Lemma, see above, that the optimal test $\hat{\varphi}$ takes the value 0 if $d\mathbb{Q}_{\hat{\varphi}}/d\mathbb{P} < c\, d\mathbb{Q}_G/d\mathbb{P}$ and the value 1 if $d\mathbb{Q}_{\hat{\varphi}}/d\mathbb{P} > c\, d\mathbb{Q}_G/d\mathbb{P}$, for a given positive constant c which depends on the size of the test. First note that one should have $\hat{\varphi} < 1$ on $\{G > 0\}$ since $\nabla\ell(0) = 0$ and therefore $d\mathbb{Q}_{\hat{\varphi}}/d\mathbb{P} = 0 < d\mathbb{Q}_G/d\mathbb{P}$ when $\hat{\varphi} = 1$ and $G > 0$. This implies that $d\mathbb{Q}_{\hat{\varphi}}/d\mathbb{Q}_G \leq c$ on $\{G > 0\}$. On $\{G = 0\}$, one has $d\mathbb{Q}_{\hat{\varphi}}/d\mathbb{P} = d\mathbb{Q}_G/d\mathbb{P} = 0$, and we set $\hat{\varphi} = 1$, see step 3 below. This leads to the definition of $\hat{\varphi}$ given in the Theorem.
3. In order to justify that we can take $\hat{\varphi} = 1$ on $\{G = 0\}$, it suffices to check that $\hat{c} > 0$ is such that $\mathbb{E}^{\mathbb{Q}_G}[\hat{\varphi}(\hat{c})] = y/p(G)$. To see this recall that $\nabla\ell$ is increasing, continuous and satisfies $\nabla\ell(+\infty) = \infty$ as well as $\nabla\ell(0+) = 0$, by assumption. It follows that I is increasing, continuous and satisfies $\nabla I(+\infty) = \infty$, $\nabla I(0+) = 0$. This implies that $\hat{\varphi}((0,\infty)) = [0, \mathbf{1}_{G>0})$ $\mathbb{P}-$a.s. and that $c \in (0,\infty) \mapsto \hat{\varphi}(c)$ is $\mathbb{P}-$a.s. continuous. Using the monotone convergence theorem, we then deduce that $c \in (0,\infty) \mapsto k(c) := \mathbb{E}^{\mathbb{Q}}[\beta_T G\hat{\varphi}(c)]$ is continuous and satisfies $k((0,\infty)) \supset (0, \mathbb{E}^{\mathbb{Q}}[\beta_T G\mathbf{1}_{G>0}]) = (0, p(G))$. The uniqueness of $\hat{c}$ follows from the fact that I is strictly increasing and that $y/p(G) < 1$ so that $\mathbb{P}\left[I(\hat{c}\beta_T d\mathbb{Q}/d\mathbb{P}) < G\right] > 0$. □

2.2 Expected shortfall price

As for the quantile hedging approach, one can define an expected shortfall price:

$$\inf\left\{y \geq 0 \ :\ \exists\, \phi \in \mathcal{A}_+(y) \text{ s.t. } \mathbb{E}[\ell((G - Y_T^{y,\phi})^+)] \leq l\right\},\ \text{for } l \in \ell(\mathbb{R}_+) .$$

It can be deduced from the result of Theorem 3 by following the arguments of Section 1.2 above. As for the quantile hedging price, we shall see in Chapter 7 how it can be directly related to a PDE.

Part B.

The stochastic target approach

Chapter 6

Super-hedging problems

1 Model and problem formulation

In this part, we consider a more general model in which the trading strategy ϕ may have an impact on the wealth process, namely the dynamics of the risky assets is given by

$$X^{\phi}_{t,x}(s) = x + \int_t^s \mu(X^{\phi}_{t,x}(u), \phi_u)du + \int_t^s \sigma(X^{\phi}_{t,x}(u), \phi_u)dW_u \ , \tag{1}$$

where W is the Brownian motion under the original probability measure $\mathbb{P}$.
As in the previous chapter, the risk free interest rate is a function ρ which depends only on x.
It follows that the wealth dynamics is given by

$$Y^{\phi}_{t,x,y}(s) = y + \int_t^s \mu_Y(Z^{\phi}_{t,x,y}(u), \phi_u)du + \int_t^s \sigma_Y(X^{\phi}_{t,x}(u), \phi_u)dW_u \ , \tag{2}$$

where

$$\mu_Y(x, y, a) := a'\mu(x, a) + (y - a'x)\rho(x) \text{ and } \sigma_Y(x, a) := a'\sigma(x, a) \ .$$

The aim of this Chapter is to provide a PDE characterization of the hedging price under constraint without appealing to the dual formulation of Chapter 2, which will anyway not be correct for the above model whenever μ and σ depends in a non-trivial way of the strategy ϕ.
We recall that the associated value function is given by

$$v(t, x) := \inf \left\{ y \in \mathbb{R} \ : \ \exists \, \phi \in \mathcal{A}_K \text{ s.t. } Y^{\phi}_{t,x,y}(T) \geq g(X^{\phi}_{t,x}(T)) \right\}$$

where the payoff function is assumed to be continuous, with linear growth and uniformly bounded from below.

We assume in all this part that μ, σ and ρ are locally Lipschitz continuous, that (1) admits a unique strong solution for any $\phi \in \mathcal{A}_K$, that there exists a unique solution $\psi(x,p)$ to the root problem

$$\sigma_Y(x,a) = p'\sigma(x,a) \text{ for some } a \in \mathbb{R}^d,$$

for any $(x,p) \in \mathbb{R}^d \times \mathbb{R}^d$, and that

$$(x,p) \in \mathbb{R}^d \times \mathbb{R}^d \mapsto \psi(x,p) \text{ is locally Lipschitz.} \tag{3}$$

In order to prove the supersolution property stated below, we shall also assume

$$\limsup_{|a|\to\infty} \inf_{(x,p)\in A} |\sigma_Y(x,a) - p'\sigma(x,a)| = \infty \tag{4}$$

for all compact set $A \subset \mathbb{R}^d \times \mathbb{R}^d$.

The results given below could be obtained in much more general situations, however this would require a substantially more technical analysis, see [4].

2 Geometric dynamic programming principle

The main tool for providing a PDE characterization of v is the *geometric dynamic programming principle* of Soner and Touzi [21], see also [20] and [6] for an extension to American type options.

Theorem 1 *Fix $(t,x,y) \in [0,T] \times \mathbb{R}^{d+1}$. Let $(\theta^\phi,\ \phi \in \mathcal{A}_K)$ denote a family of stopping times in $\mathcal{T}^t_{[t,T]}$. Then the following holds:*
(DP1): *If $y > v(t,x)$, then there exists $\phi \in \mathcal{A}_K$ such that*

$$Y^\phi_{t,x,y}(\theta^\phi) \geq v(\theta^\phi, X^\phi_{t,x}(\theta^\phi)) \ .$$

(DP2): *If $y < v(t,x)$, then*

$$\mathbb{P}\left[Y^\phi_{t,x,y}(\theta^\phi) > v(\theta^\phi, X^\phi_{t,x}(\theta^\phi))\right] < 1 \ \ \forall\, \phi \in \mathcal{A}_K \ .$$

We shall not provide a rigorous proof of this result and refer to [21] and the remarks in [6]. We only explain the main argument.

If $y > v(t,x)$, then, by definition of v, there exists $\phi \in \mathcal{A}_K$ such that $Y^{\phi}_{t,x,y}(T) \geq g(X^{\phi}_{t,x}(T))$. On the other hand, if $Y^{\phi}_{t,x,y}(\theta^{\phi}) < v(\theta^{\phi}, X^{\phi}_{t,x}(\theta^{\phi}))$ on a set of non zero measure, then starting from time θ^{ϕ} is in not always possible to find a strategy $\tilde{\phi}$ such that $Y^{\tilde{\phi}}_{\theta^{\phi}, X^{\phi}_{t,x}(\theta^{\phi}), Y^{\phi}_{t,x,y}(\theta^{\phi})}(T) \geq g(X^{\tilde{\phi}}_{\theta^{\phi}, X^{\phi}_{t,x}(\theta^{\phi})}(T))$. This, combined with the flow property, contradicts the fact that $Y^{\phi}_{t,x,y}(T) \geq g(X^{\phi}_{t,x}(T))$ $\mathbb{P}-$a.s.
On the other hand, if $Y^{\phi}_{t,x,y}(\theta^{\phi}) > v(\theta^{\phi}, X^{\phi}_{t,x}(\theta^{\phi}))$ $\mathbb{P}-$a.s., then starting from the time θ^{ϕ}, one can construct a strategy which allows to super-hedge the claim. This should imply that $y \geq v(t,x)$.

3 Derivation of the pricing equation

3.1 PDE characterization

Before to provide the rigorous characterization of v, let us explain the main idea. Assume that v is smooth and that (DP1) above holds with $y = v(t,x)$, which would be the case if the infimum in the definition of v was achieved. Then, one can find $\phi \in \mathcal{A}_K$ such that

$$Y^{\phi}_{t,x,y}(\theta) \geq v(\theta, X^{\phi}_{t,x}(\theta)) ,$$

for any stopping time $\theta \in \mathcal{T}^t_{[t,T]}$. Applying this formally for $\theta = t+$, this implies that

$$Y^{\phi}_{t,x,y}(t+) \geq v(t+, X^{\phi}_{t,x}(t+)) ,$$

so that, by Itô's Lemma,

$$\mu_Y(x,y,\phi_t)dt + \sigma_Y(x,\phi_t)dW_t \geq \mathcal{L}^{\phi_t} v(t,x)dt + Dv(t,x)'\sigma(x,\phi_t)dW_t$$

where, for $a \in \mathbb{R}^d$ and a smooth function φ,

$$\mathcal{L}^a\varphi := \partial_t\varphi + \mu(\cdot,a)'D\varphi + \frac{1}{2}\mathrm{Tr}[\sigma\sigma'(x,a)D^2\varphi] .$$

Since the dW term behaves like $\sqrt{dt}\varepsilon$ for a standard Gaussian random variable ε, this necessarily implies that

$$\sigma_Y(x,\phi_t) = Dv(t,x)'\sigma(x,\phi_t)$$

so that

$$\phi_t = \psi(x, Dv(t,x))$$

by the definition of ψ above. Coming back to the previous inequality and recalling that $y = v(t, x)$ then leads to

$$\mathcal{G}v(t, x) \geq 0$$

where, for a smooth function φ,

$$\mathcal{G}\varphi := \mu_Y(\cdot, \varphi, \psi(\cdot, D\varphi)) - \mathcal{L}^{\psi(\cdot, D\varphi)}\varphi \,.$$

Moreover, ϕ_t, and therefore $\psi(x, Dv(t, x))$, should take values in K. Recalling Proposition 1 in Section 2 of Chapter 2, this implies that

$$\mathcal{H}v(t, x) \geq 0$$

where

$$\mathcal{H}\varphi := \inf_{|\zeta|=1} \left(\delta_K(\zeta) - \zeta'\psi(\cdot, Dv)\right) \geq 0$$

for a smooth function φ.

The optimality included in the definition of v defined as an infimum should actually show that one of the above inequalities is sharp, i.e. v solves

$$\min\left\{\mathcal{G}\varphi\ ,\ \mathcal{H}\varphi\right\} = 0 \text{ on } [0, T) \times \mathbb{R}^d \,. \tag{5}$$

This can be checked by using the second part (DP2) of the geometric dynamic programming principle.

Theorem 2 *Assume that v is locally bounded. Then, v_* and v^* are respectively viscosity super- and subsolutions of* (5)*.*

The proof is divided in two parts.

Viscosity supersolution property

Before to prove the supersolution property of Theorem 2, we formulate the following remark to which we shall appeal in the proof.

Remark 1 Fix $(x, p) \in A \subset \mathbb{R}^d \times \mathbb{R}^d$, with A compact. Assume that there exists $\varepsilon > 0$ such that

$$\inf_{|\zeta|=1} \left(\delta_K(\zeta) - \zeta'\psi\right) \leq -\varepsilon \ \text{ on } A \,. \tag{6}$$

Then, there exists $c_\varepsilon > 0$ such that

$$\inf_{a\in K} |a - \psi| \geq c_\varepsilon \text{ on } A \,. \tag{7}$$

This follows from the fact that $\inf_{a\in K} |a-\psi| = 0$ implies $\psi \in K$, since K is closed, which, together with Proposition 1 in Section 2 of Chapter 2, would imply that $\inf_{|\zeta|=1} (\delta_K(\zeta) - \zeta'\psi) \geq 0$.

Moreover, (4) and (7) implies that there exists $k_\varepsilon > 0$ such that

$$\inf_{a\in K} |\sigma_Y(x,a) - p'\sigma(x,a)| \geq k_\varepsilon \text{ for } (x,p) \in A \,. \tag{8}$$

Otherwise, we would find (a,x,p) in a compact subset of $K \times A$ such that $\sigma_Y(x,a) - p'\sigma(x,a) = 0$, which would imply $a = \psi(x,p)$, a contradiction.

We can now provide the proof of the supersolution property.

1. Fix $(t_0,x_0) \in [0,T) \times \mathbb{R}^d$ and let φ be a smooth function such that

$$\text{(strict)} \min_{[0,T]\times\mathbb{R}^d} (v_* - \varphi) = (v_* - \varphi)(t_0,x_0) = 0 \,. \tag{9}$$

Assume to the contrary that $\min\{\mathcal{G}\varphi, \mathcal{H}\varphi\}(t_0,x_0) < 0$. Then, by continuity of the operators, there exists $r, \varepsilon > 0$ such that $B_0 := B_r(t_0,x_0) \subset [0,T) \times \mathbb{R}^d$ and

$$\min\{\mathcal{G}(\varphi+\zeta) \;,\; \mathcal{H}\varphi\} \leq -2\varepsilon \text{ for } |\zeta| \leq r \text{ on } B_r(t_0,x_0) \,.$$

Recalling Remark 1 and the very definition of $\mathcal{H}$, this implies that

$$\begin{aligned} \mu_Y(x,y,a) - \mathcal{L}^a\varphi(t,x) \leq -\varepsilon \;\forall\; (t,x,y,a) \in B_0 \times \mathbb{R} \times K \\ \text{s.t. } |\sigma_Y(x,a) - Dv(t,x)'\sigma(x,a)| \leq k_\varepsilon \text{ and } |y - \varphi(t,x)| \leq r \end{aligned} \tag{10}$$

for some $k_\varepsilon > 0$.

For later use, observe that, by (9) and the definition of φ,

$$\zeta := \min_{\partial_p B_\varepsilon(t_0,x_0)} (v_* - \varphi) > 0 \,, \tag{11}$$

where $\partial_p B_\varepsilon(t_0,x_0)$ denotes the parabolic boundary of $B_\varepsilon(t_0,x_0)$.

2. Let $(t_n,x_n)_{n\geq 1}$ be a sequence in B_0 which converges to (t_0,x_0) and such that $v(t_n,x_n) \to v_*(t_0,x_0)$. Set $y_n = v(t_n,x_n) + n^{-1}$ and observe that

$$\gamma_n := y_n - \varphi(t_n,x_n) \to 0 \,. \tag{12}$$

For each $n \geq 1$, we have $y_n > v(t_n, x_n)$. It thus follows from (DP1) of Theorem 1, that there exists some $\phi^n \in \mathcal{A}_K$ such that

$$Y^n(t \wedge \theta^n) \geq v(t \wedge \theta^n, X^n(t \wedge \theta^n)) \quad \text{for } t \geq t_n, \tag{13}$$

where

$$Z^n = (X^n, Y^n) := \left(X^{\phi^n}_{t_n,x_n}, Y^{\phi^n}_{t_n,x_n,y_n}\right) \text{ and } \theta^n := \theta^o_n \wedge \theta^1_n ,$$

with

$$\begin{aligned} \theta^o_n &:= \left\{s \geq t_n : (s, X^{\phi^n}_{t_n,x_n}(s)) \notin B_0\right\} \\ \theta^1_n &:= \left\{s \geq t_n : |Y^{\phi^n}_{t_n,x_n,y_n}(s) - \varphi(s, X^{\phi^n}_{t_n,x_n}(s))| \geq r\right\} . \end{aligned}$$

Let us define

$$A_n := \left\{s \in [t_n, \theta_n] \ : \ \mu_Y\left(Z^n(s), \phi^n_s\right) - \mathcal{L}^{\phi^n_s}\varphi\left(s, X^n(s)\right) > -\varepsilon\right\} , \tag{14}$$

and observe that (10) implies that the process

$$\delta^n_s := \sigma_Y(X^n(s), \phi^n_s) - D\varphi(s, X^n(s))'\sigma_X(X^n(s), \phi^n_s)$$

satisfies

$$|\delta^n_s| > k_\varepsilon \text{ for } s \in A_n. \tag{15}$$

3. Using (13), the definition of ζ in (11) and the definition of θ_n, we then obtain

$$\begin{aligned} Y^n(t \wedge \theta_n) &\geq \varphi\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) + \left(\zeta \mathbf{1}_{\{\theta^o_n = \theta_n\}} + r\mathbf{1}_{\{\theta^o_n > \theta_n\}}\right) \mathbf{1}_{\{t=\theta_n\}} \\ &\geq \varphi\left(t \wedge \theta_n, X^n(t \wedge \theta_n)\right) + (\zeta \wedge r)\, \mathbf{1}_{\{t=\theta_n\}} , \quad t \geq t_n . \end{aligned}$$

Since φ is smooth, it follows from Itô's Lemma, (10), (12) and the definition of δ^n that

$$-\left(\zeta \wedge r\right) \mathbf{1}_{\{t<\theta_n\}} \leq K^n_t , \tag{16}$$

where

$$K^n_t := \gamma_n - (\zeta \wedge \varepsilon) + \int_{t_n}^{t\wedge\theta_n} b^n_s ds + \int_{t_n}^{t\wedge\theta_n} \delta^n_s dW_s ,$$

with

$$b^n_s := \left[\mu_Y\left(Z^n(s), \phi^n_s\right) - \mathcal{L}^{\phi^n_s}\varphi\left(s, X^n(s)\right)\right] \mathbf{1}_{A_n}(s) .$$

Let M^n be the exponential local martingale defined by $M^n_{t_n} = 1$ and, for $s \geq t_n$,

$$dM^n_s \quad = \quad -M^n_s b^n_s |\delta^n_s|^{-2} \delta^n_s dW_s \,,$$

which is well defined by (15) and the Lipschitz continuity of the coefficients. By Itô's formula and (16), we see that $M^n K^n$ is a local martingale which is bounded from below by the submartingale $-(\zeta \wedge r)\, M^n$. Then, $M^n K^n$ is a supermartingale, and it follows from (16) that

$$0 \leq \mathbb{E}\left[M^n_{\theta_n} K^n_{\theta_n}\right] \leq \gamma_n - (\zeta \wedge r) < 0 \,,$$

for n large enough, recall (12), which leads to a contradiction. □

Viscosity subsolution property

1. Fix $(t_0, x_0) \in [0, T) \times \mathbb{R}^d$ and let φ be a smooth function such that

$$\text{(strict)} \max_{[0,T]\times\mathbb{R}^d} (v^* - \varphi) = (v^* - \varphi)(t_0, x_0) = 0 \,. \tag{17}$$

We assume to the contrary that

$$\min\{\mathcal{G}\varphi \;,\; \mathcal{H}\varphi\}(t_0, x_0) \geq 2\eta \tag{18}$$

for some $\eta > 0$, and work towards a contradiction.
Under the above assumption, we may find $r > 0$ such that

$$\mu_Y(\cdot, \varphi + \zeta, \psi(\cdot, D\varphi)) - \mathcal{L}^{\psi(\cdot, D\varphi)}\varphi > \eta \text{ for } |\zeta| \leq r \text{ on } B_0 := B_r(t_0, x_0) \,. \tag{19}$$

For later use note that, by (17) and the definition of φ,

$$-\zeta := \max_{\partial_p B_r(t_0,x_0)} (v^* - \varphi) < 0 \,. \tag{20}$$

Moreover, we can find a sequence $(t_n, x_n)_{n\geq 1}$ in B_0 which converges to (t_0, x_0) and such that $v(t_n, x_n) \to v^*(t_0, x_0)$. Set $y_n = v(t_n, x_n) - n^{-1}$ and observe that

$$\gamma_n := y_n - \varphi(t_n, x_n) \to 0 \,. \tag{21}$$

2. We now let $Z^n := (X^n, Y^n)$ denote the solution of (1)-(2) associated to the Markovian control $\hat{\phi}^n := \psi(\cdot, D\varphi(\cdot, X^n))$ and the initial condition $Z^n(t_n) =$

(x_n, y_n), recall that ψ is assumed to be locally Lipschitz. We next define the stopping times

$$\begin{aligned}\theta_n^o &:= \inf\{s \geq t_n \ : \ (s, X^n(s)) \notin B_0\}\,,\\ \theta_n &:= \inf\{s \geq t_n \ : \ |Y^n(s) - \varphi(s, X^n(s))| \geq r\} \wedge \theta_n^o\,.\end{aligned}$$

Note that, by definition of $\hat{\phi}^n$ and (19), $Y^n - \varphi(\cdot, X^n)$ is non-decreasing on $[t_n, \theta_n]$, so that

$$Y^n(\theta_n) - \varphi(\theta_n, X^n(\theta_n)) \geq y_n - \varphi(t_n, x_n) = \gamma_n > -r \tag{22}$$

for n large enough, recall (21). Since $\varphi \geq v^* \geq v$, it follows that

$$\begin{aligned}Y^n(\theta_n) - v\left(\theta_n, X^n(\theta_n)\right) &\geq \mathbf{1}_{\{\theta_n < \theta_n^o\}}\left\{Y^n(\theta_n) - \varphi\left(\theta_n, X^n(\theta_n)\right)\right\}\\ &\qquad + \mathbf{1}_{\{\theta_n = \theta_n^o\}}\left\{Y^n(\theta_n^o) - v^*\left(\theta_n^o, X^n(\theta_n^o)\right)\right\}\\ &= r\mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}}\left\{Y^n(\theta_n^o) - v^*\left(\theta_n^o, X^n(\theta_n^o)\right)\right\}\\ &\geq r\mathbf{1}_{\{\theta_n < \theta_n^o\}} + \mathbf{1}_{\{\theta_n = \theta_n^o\}}\left\{Y^n(\theta_n^o) + \zeta - \varphi\left(\theta_n^o, X^n(\theta_n^o)\right)\right\}\\ &\geq r \wedge \zeta + \mathbf{1}_{\{\theta_n = \theta_n^o\}}\left\{Y^n(\theta_n^o) - \varphi\left(\theta^o, X^n(\theta_n^o)\right)\right\}\ .\end{aligned}$$

In view of (22), this leads to

$$Y^n(\theta_n) - v\left(\theta_n, X^n(\theta^n)\right) \quad \geq \quad (r \wedge \zeta)/2$$

for n large enough, since $\gamma_n \to 0$. Recalling that $y_n = v(t_n, x_n) - n^{-1} < v(t_n, x_n)$, this is clearly in contradiction with (DP2) of Theorem 1. □

3.2 Boundary condition as $t = T$

Note that by construction $v(T, \cdot) = g$. However, it follows from the previous sections that v satisfies $\mathcal{H}v \geq 0$, in the viscosity sense, which implies that Dv is constrained on $[0, T)$. This constraint should propagate up to T. Hence, v should solve

$$\min\{\varphi - g\ ,\ \mathcal{H}\varphi\} = 0 \text{ on } \{T\} \times \mathbb{R}^d\,. \tag{23}$$

We shall see below that this boundary condition is naturally related to the face-lifting phenomenon observed in Chapters 2 and 4.

Theorem 3 *Assume that v is locally bounded. Then, v_* and v^* are respectively super- and subsolution of* (23).

Proof. The proofs follow from similar arguments as in the previous section, up to the standard trick which consists in adding a term of the form $\pm\sqrt{T-t+\alpha}$ to the test function φ, so that, for t close to T and $\alpha > 0$ small enough, it satisfies

$$\pm\mathcal{G}(\varphi \pm \sqrt{T-\cdot+\alpha}) \geq 0 \; .$$

We only explain the argument for the subsolution property. The supersolution property is proved by using the same trick combined with the arguments used to prove the supersolution property in Section 3.1.
Let $x_0 \in \mathbb{R}^d$ and φ be a smooth function such that

$$(\text{strict}) \max_{[0,T]\times\mathbb{R}^d}(v^* - \varphi) = (v^* - \varphi)(T, x_0) = 0 \, .$$

Assume that

$$\min\{v^* - g \; , \; \mathcal{H}\varphi\}(T, x_0) \geq 4\eta \; . \tag{24}$$

Set $\tilde{\varphi}(t,x) := \varphi(t,x) + \sqrt{T-t+\alpha} - \sqrt{\alpha}$. Since $\partial_t\tilde{\varphi}(t,x) \to -\infty$ as $t \to T$ and $\alpha \to 0$, we deduce that, for $r, \alpha > 0$ small enough,

$$\begin{aligned} \min\left\{\tilde{\varphi} - g \; , \; \mu_Y\left(\cdot, \tilde{\varphi} + \zeta, \psi(\cdot, D\tilde{\varphi})\right) - \mathcal{L}^{\psi(\cdot, D\tilde{\varphi})}\tilde{\varphi} \; , \; \mathcal{H}\tilde{\varphi}\right\} \geq \eta \\ \text{for } |\zeta| \leq r \text{ on } B_0 := [T-r, T] \times B_r(x_0) \; . \end{aligned} \tag{25}$$

Also observe that, since $(v^* - \tilde{\varphi})(T, x_0) = 0$ and (T, x_0) achieves a strict maximum, we can choose $r > 0$ so that

$$v^*(t,x) \leq \tilde{\varphi}(t,x) - \varepsilon/2 \quad \text{for all} \;\; (t,x) \in [T-r, T] \times \partial B_r(x_0) \; , \tag{26}$$

which, together with $v(T, \cdot) = g$ and (25), leads to

$$v(t,x) - \tilde{\varphi}(t,x) \leq -\zeta \;\; \text{for all} \;\; (t,x) \in \partial_p B_0 \tag{27}$$

for some $r, \varepsilon, \zeta > 0$ small enough but so that the above inequalities still hold.
By following the arguments of the previous section, we deduce that (25) and (27) lead to a contradiction of (GDP2).

□

4 Extension to more general dynamics

It should be noted that the above proofs and results do not depend on the specific form of μ_Y and σ_Y defined in Section 1, but only on the general assumptions we made.

This implies that much more general dynamics could be considered. In particular, we could set

$$\mu_Y(x,y,a) := y\left(a'[x]^{-1}\mu(x,a) + (1-a'\mathbf{1})\rho(x)\right) \text{ and } \sigma_Y(x,a) := ya'[x]^{-1}\sigma(x,a) ,$$

with $\mathbf{1} = (1,\ldots,1)$ and $[x]$ denoting the diagonal matrix with i-th diagonal component given by x^i. In this case, the dynamics of Y is given by

$$\begin{aligned} dY^{\phi}_{t,x,y}(s) &= Y^{\phi}_{t,x,y}(s)\phi'_s[X^{\phi}_{t,x}(s)]^{-1}dX^{\phi}_{t,x}(s) \\ &\quad + \left(Y^{\phi}_{t,x,y}(s) - Y^{\phi}_{t,x,y}(s)\phi'_s\mathbf{1}\right)\rho(X_{t,x}(s))ds . \end{aligned} \tag{28}$$

This corresponds to a model where ϕ^i denotes the proportion of the wealth invested in the i-th risky asset. In this case, we have to put restrictions on the coefficient μ and σ in order to ensure that X has positive components whenever the initial condition belongs to $(0,\infty)^d$, and the viscosity solution properties have to be stated on $(0,\infty)^d$ instead of $\mathbb{R}^d$.

5 Examples in the Black and Scholes model

a. If $\sigma(x) = x\sigma$ and $\mu(x) = x\mu$ where $\sigma > 0$ and μ is a real constant, then $\psi(x,p) = p$. Moreover, if $K = \mathbb{R}$, then $\delta_K(\zeta) = \infty$ for $\zeta \neq 0$ and therefore if $|\zeta| = 1$. It follows that v is a discontinuous viscosity solution of

$$\begin{aligned} 0 &= D\varphi x\mu + (\varphi - D\varphi x)\rho - \partial_t\varphi - x\mu D\varphi - \frac{1}{2}x^2\sigma^2 D^2\varphi \\ &= \rho\varphi - \partial_t\varphi - \rho x D\varphi - \frac{1}{2}x^2\sigma^2 D^2\varphi \end{aligned}$$

which is (3) of Chapter 3. Moreover, the boundary condition at $t = T$ is simply given by g.

b. If $K \neq \mathbb{R}$, then the same computations lead to (4) of Chapter 4. As for the boundary condition at $t = T$, we obtain

$$\min\left\{\varphi(T,\cdot) - g \ , \ \inf_{|\zeta|=1} \delta_K(\zeta) - \zeta D\varphi(T,\cdot)\right\} = 0 .$$

in the discontinuous viscosity sense. One can show that $\hat{g}$ defined in Chapter 2 is the minimal supersolution of this equation, and that the above characterization of v actually implies that $v^*(T,\cdot) = v_*(T,\cdot) = \hat{g}$ as demonstrated in Chapter 4.

c. Let us now consider the case of the Black and Scholes model with Y defined as in (28), i.e. where ϕ represents the proportion of the wealth invested in each asset. Then, for $g \geq 0$ so that $v > 0$, the PDE (5) reads:

$$\min\left\{\rho\varphi - \partial_t\varphi - \rho x D\varphi - \frac{1}{2}x^2\sigma^2 D^2\varphi \; , \; \inf_{|\zeta|=1} \delta_K(\zeta)\varphi - \zeta x D\varphi\right\} = 0$$

on $[0,T) \times (0,\infty)$, and the boundary condition is given by

$$\min\left\{\varphi(T,\cdot) - g \; , \; \inf_{|\zeta|=1} \delta_K(\zeta)\varphi(T,\cdot) - \zeta x D\varphi(T,\cdot)\right\} = 0$$

on $(0,\infty)$. Under suitable assumptions, one can show that the smaller supersolution of the above equation is given by

$$\breve{g}(x) := \sup_{\zeta\in K} e^{-\delta_K(\zeta)} g(xe^{\zeta})$$

and that $v(T-,\cdot)$ actually coincides with $\breve{g}$, see e.g. [22].

Chapter 7

Approximate hedging with controlled risk

We now turn to quantile and shortfall based pricing problems. More precisely, we let G be a given real valued measurable function on $\mathbb{R}^d \times \mathbb{R}_+$, satisfying

$$y \in \mathbb{R}_+ \mapsto \Psi(x, y) \text{ is non-decreasing for all } x \in \mathbb{R}^d ,$$

and define

$$v(t, x, p) := \min \left\{ y \geq 0 \ : \ \exists\, \phi \in \mathcal{A}_K \ \text{ s.t. } \mathbb{E}\left[\Psi(X^{\phi}_{t,x,y}(T), Y^{\phi}_{t,x,y}(T)) \right] \geq p \right\} .$$

For $\Psi(x, y) = \mathbf{1}_{y \geq g(x)}$, this corresponds to the quantile hedging problem discussed in Section 1 of Chapter 5. For $\Psi(x, y) = -\ell((g(x)-y)^+)$, this corresponds to the expected loss pricing rule of Section 2 of Chapter 5.

The aim of this chapter is to show how such problems can be embedded into the class of general stochastic target problems as discussed in Chapter 6.

In the rest of this chapter, we shall often write $Z^{\phi}_{t,x,y}$ for $(X^{\phi}_{t,x}, Y^{\phi}_{t,x,y})$.

1 Problem reduction

The key point is the following observation made in [4].

Proposition 1 *Fix* $(t, x) \in [0, T] \times \mathbb{R}^d$ *and assume that* $\Psi(Z^{\phi}_{t,x,y}(T)) \in L^2$ *for any* $\phi \in \mathcal{A}_K$ *and* $y \geq 0$. *Then,*

$$v(t, x, p) = \min \left\{ y \geq 0 \ : \ \exists\, (\phi, \alpha) \in \mathcal{A}_K \times L^2_{\mathcal{P}} \ \ s.t. \ \Psi(Z^{\phi}_{t,x,y}(T)) \geq P^{\alpha}_{t,p}(T) \right\} \tag{1}$$

where

$$P^{\alpha}_{t,p} := p + \int_0^{\cdot} \alpha'_s dW_s \ .$$

Moreover, the above terms are also equal to

$$\min \left\{ y \geq 0 \ : \ \exists\, (\phi, \alpha) \in \mathcal{A}_K \times L^2_{\mathcal{P}} \ \ s.t. \ Y^{\phi}_{t,x,y}(T) \geq \Psi^{-1}(X^{\phi}_{t,x}(T), P^{\alpha}_{t,p}(T)) \right\}$$

where Ψ^{-1} *denotes the right inverse of* Ψ *in the y-variable.*

Proof. Let $\bar{v}(t, x, p)$ denote the right-hand side in (1). Then, for $y > \bar{v}(t, x, p)$, there exists $(\phi, \alpha) \in \mathcal{A}_K \times L^2_{\mathcal{P}}$ such that $\Psi(Z^{\phi}_{t,x,y}(T)) \geq P^{\alpha}_{t,p}(T)$. Since $P^{\alpha}_{t,p}$ is a martingale, taking expectation leads to $\mathbb{E}\left[\Psi(Z^{\phi}_{t,x,y}(T))\right] \geq p$. This implies that $\bar{v}(t, x, p) \geq v(t, x, p)$. Conversely, if $y > v(t, x, p)$ then there exists $\phi \in \mathcal{A}_K$ such that $M_0 := \mathbb{E}\left[\Psi(Z^{\phi}_{t,x,y}(T))\right] \geq p$. Let us define the martingale $M := \mathbb{E}\left[\Psi(Z^{\phi}_{t,x,y}(T)) \mid \mathcal{F}_{\cdot}\right]$. It follows from the martingale representation theorem, see Theorem 1 of Chapter 2 or [14], that there exists $\alpha \in L^2_{\mathcal{P}}$ such that $M = P^{\alpha}_{t,M_0}$, with P defined as in the proposition. In particular, $\Psi(Z^{\phi}_{t,x,y}(T)) = P^{\alpha}_{t,M_0}(T) \geq P^{\alpha}_{t,p}(T)$, since $M_0 \geq p$. This proves that $\bar{v}(t, x, p) \leq v(t, x, p)$ and concludes the proof. □

Otherwise stated, it suffices to consider an augmented system (X, Y, P) with an augmented control (ϕ, α) and apply the technics introduced in Chapter 6 above. In particular, the geometric dynamic programming of Chapter 6 applies here.

Theorem 1 *Fix* $(t, x, y, p) \in [0, T] \times \mathbb{R}^{d+1} \times \mathbb{R}$. *Let* $(\theta^{\phi,\alpha}, \ (\phi, \alpha) \in \mathcal{A}_K \times L^2_{\mathcal{P}})$ *denote a family of stopping times in* $\mathcal{T}^t_{[t,T]}$. *Then the following holds:*
(DP1): *If* $y > v(t, x, p)$, *then there exists* $(\phi, \alpha) \in \mathcal{A}_K \times L^2_{\mathcal{P}}$ *such that*

$$Y^{\phi}_{t,x,y}(\theta^{\phi,\alpha}) \geq v(\theta^{\phi,\alpha}, X^{\phi}_{t,x}(\theta^{\phi,\alpha}), P^{\alpha}_{t,p}(\theta^{\phi,\alpha})) \ .$$

(DP2): *If* $y < v(t, x)$, *then*

$$\mathbb{P}\left[Y^{\phi}_{t,x,y}(\theta^{\phi,\alpha}) > v(\theta^{\phi,\alpha}, X^{\phi}_{t,x}(\theta^{\phi,\alpha}), P^{\alpha}_{t,p}(\theta^{\phi,\alpha}))\right] < 1 \ \ \forall\, (\phi, \alpha) \in \mathcal{A}_K \times L^2_{\mathcal{P}} \ .$$

2 Pricing equation

In view of Theorem 1, one can now apply the same arguments as in Section 3.1 of Chapter 6. The only difference is that we now have to take into account a new control α and a new state process P.

2.1 In the domain

Before to state the PDE characterization in the domain, let us first introduce the notations corresponding to our stochastic target problem.

First, we assume that the equation

$$\sigma_Y(x,a) = p'\sigma(x,a) + qb' \text{ for some } a \in \mathbb{R}^d$$

admits a unique solution $\psi(x,p,q,b)$ which is locally Lipschitz continuous. The Dynkin operator associated to (X,P) for the value of the control (a,b) is denoted by

$$\mathcal{L}^{a,b}\varphi := \mathcal{L}^a\varphi + D^2_{xp}\varphi'\sigma(\cdot,a)b + \frac{1}{2}b^2 D^2_p\varphi$$

where $\mathcal{L}^a$ is defined as in the previous chapter, $D^2_{xp}\varphi$ stands for the second order cross derivatives $(\partial^2\varphi/\partial_{x^i p})_{i\le d}$ and $D^2_p\varphi$ is the second derivative with respect to p.

We then consider the counterparts of operators $\mathcal{G}$ and $\mathcal{H}$ associated to $a = \psi(\cdot,b)$, for b given:

$$\mathcal{G}^b\varphi := \mu_Y(\cdot,\psi^b_\varphi) - \mathcal{L}^{\psi^b_\varphi,b}\varphi \text{ and } \mathcal{H}^b\varphi := \inf_{|\zeta|=1}(\delta_K(\zeta) - \zeta'\psi^b_\varphi)$$
$$\text{with } \psi^b_\varphi := \psi(\cdot, D_x\varphi, D_p\varphi, b)\ .$$

In the following, we set

$$\mathcal{O} := \{p \in \mathbb{R}\ :\ 0 < v(t,x,p) < \infty \text{ for all } (t,x) \in [0,T]\times\mathbb{R}^d\}$$

and we assume that

$$\mathcal{O} \text{ is non-empty, convex and closed.}$$

Note that the convexity is obvious, and is indeed not an assumption, whenever $\mathcal{O}$ is non-empty.

In what follows v_* and v^* are defined as the semicontinuous envelopes of v in the three variables (t,x,p) when approximated by a sequence $(t_n,x_n,p_n) \in [0,T)\times\mathbb{R}^d\times\text{int}(\mathcal{O})$

Theorem 2 *The following holds:*
(i) *If K is compact, then v_* is a viscosity supersolution of*

$$\max\left\{\sup_{b\in\mathbb{R}^d}\min\left\{\mathcal{G}^b\varphi\ ,\ \mathcal{H}^b\varphi\right\}\ ,\ -|D_p\varphi|\right\}=0\ \ on\ [0,T)\times\mathbb{R}^d\times\mathcal{O}\ . \tag{2}$$

(ii) *v^* is a viscosity subsolution of*

$$\sup_{b\in\mathbb{R}^d}\min\left\{\varphi\ ,\ \mathcal{G}^b\varphi\ ,\ \mathcal{H}^b\varphi\right\}=0\ \ on\ [0,T)\times\mathbb{R}^d\times\mathcal{O}\ . \tag{3}$$

Proof. We do not provide the entire proof because, thanks to Theorem 1, it follows exactly the line of arguments of Section 3.1 of Chapter 6. We only explain an additional technical point which should be taken into account in order to derive the supersolution property. Namely, in Section 3.1 of Chapter 6 we used the assumption (4) in order to deduce (8) from (6). Here the problem comes from the new control b that is a-priori not bounded. However, if $D_p\varphi\neq 0$ and $a\in K$ solves $\sigma_Y(x,a)=D_x\varphi'\sigma(x,a)+D_p\varphi b'$, then the fact that K is compact along with the regularity assumptions on σ_Y and σ imply that b has to belong to a compact set. As a conclusion, the proof of the supersolution property can be reproduced without difficulty when $D_p\varphi\neq 0$. When $D_p\varphi=0$, the viscosity supersolution property is satisfied by construction. As for the subsolution property, nothing changes except that we need to have $y_n=v(t_n,x_n)-n^{-1}\geq 0$ in the proof of Section 3.1 of Chapter 6, see just before (21), since the initial wealth should be non-negative in the definition of our criteria. In order to ensure this, we need to have $v^*>0$ at the point where the maximum of the difference with the test function is achieved. □

2.2 Boundary condition at $t=T$

By similar arguments, the boundary condition of Theorem 3 of Chapter 6 extends to this context.

Theorem 3 *Assume that Ψ^{-1} is continuous on $\mathbb{R}^d\times\mathcal{O}$. Then the following holds:*
(i) *If K is compact, then v_* is a viscosity supersolution of*

$$\max\left\{\min\left\{\varphi-\Psi^{-1}\ ,\ \sup_{b\in\mathbb{R}^d}\mathcal{H}^b\varphi\right\}\ ,\ -|D_p\varphi|\right\}=0\ \ on\ \{T\}\times\mathbb{R}^d\times\mathcal{O}\ . \tag{4}$$

(ii) v^* *is a viscosity supersolution of*

$$\min\left\{\varphi\ ,\ \varphi-\Psi^{-1}\ ,\ \sup_{b\in\mathbb{R}^d}\mathcal{H}^b\varphi\right\}=0\ \text{on}\ \{T\}\times\mathbb{R}^d\times\mathcal{O}\ . \quad (5)$$

In our context of financial mathematics, one can usually say a little bit more on the boundary condition whenever their exists a well-behaved martingale measure. To see this, let $H_{t,x}$ be defined by

$$H_{t,x}(s)=1-\int_t^s H_{t,x}(u)\lambda(X_{t,x}(u))dW_u\ \text{ with }\lambda(x):=\sigma^{-1}(\mu(x)-\rho(x)x)\ ,$$

where we implicitly assume that σ is invertible and that H is well-defined as a martingale for any initial conditions (t,x).

Proposition 2 *Assume that $\mathcal{O}$ is compact. Fix $(x,p)\in\mathbb{R}^d\times\mathcal{O}$ and assume that for all sequence $(t_n,x_n,p_n)_n\subset[0,T)\times\mathbb{R}^d\times\mathrm{int}(\mathcal{O})$ that converges to (T,x,p), and for all sequence $(\phi_n)_n\subset\mathcal{A}_K$, we have*

$$\begin{aligned}&\mathbb{E}\left[|H_{t_n,x_n}(T)\beta_{t_n,x_n}(T)\widehat{\Psi}^{-1}(X^{\phi_n}_{t_n,x_n}(T),p)-\widehat{\Psi}^{-1}(x,p)|\right]\to 0\\ \textit{and}\quad &\mathbb{E}\left[|H_{t_n,x_n}(T)\beta_{t_n,x_n}(T)\nabla^+\widehat{\Psi}^{-1}(X^{\phi_n}_{t_n,x_n}(T),p)-\nabla^+\widehat{\Psi}^{-1}(x,p)|\right]\to 0\ ,\end{aligned}$$

where $\widehat{\Psi}^{-1}$ denotes the convex envelope of Ψ^{-1} with respect to p, and $\nabla^+\widehat{\Psi}^{-1}$ its right-derivative with respect to p. Then,

$$v_*(T,x,p)\geq\widehat{\Psi}^{-1}(x,p)\ .$$

When $\widehat{\Psi}^{-1}$ and $\nabla^+\widehat{\Psi}^{-1}$ are continuous with polynomial growth in x, μ and σ are uniformly Lipschitz in x, and λ is bounded, then the above assumptions are trivially satisfied.

Note that, if (5) holds on $\{T\}\times\mathbb{R}^d\times\mathcal{O}$, v^* can be shown to be convex and if v_* is strictly increasing in p, then this implies that v_* and v^* are super- and subsolutions of

$$\min\left\{\varphi\ ,\ \varphi-\widehat{\Psi}^{-1}\ ,\ \sup_{b\in\mathbb{R}^d}\mathcal{H}^b\varphi\right\}=0\ \text{on}\ \{T\}\times\mathbb{R}^d\times\mathcal{O}\ ,$$

using the fact that $v_*\geq 0$ by construction. In the limiting case where $K=\mathbb{R}^d$, and therefore $\mathcal{H}^b\varphi\equiv\infty$, then the boundary condition simply reads

$$\varphi(T,\cdot)=\widehat{\Psi}^{-1}\vee 0\ .$$

Otherwise stated, a first face-lift of the natural terminal condition Ψ^{-1} is due to the additional state process P. When $K \neq \mathbb{R}^d$, then an additional face-lift is required as explained in Chapter 6. We shall provide two examples in Sections 3 and 4 below.

We conclude this section with the proof of the above proposition[1].

Proof of Proposition 2. Let us set $(X^n, Y^n, P^n, \beta^n, H^n) := (X^{\phi_n}_{t_n,x_n}, Y^{\phi_n}_{t_n,x_n,y_n}, P^{\alpha_n}_{t_n,p_n}, \beta_{t_n,x_n}(T), H_{t_n,x_n})$ for $y_n := v(t_n, x_n, p_n) + 1/n$ and $(\phi_n, \alpha_n) \in \mathcal{A}_K \times L^2_{\mathcal{P}}$ such that

$$Y^n(T) \geq \Psi^{-1}(X^n(T), P^n(T)) \ .$$

Then, by the supermartingale property of $H^n\beta^nY^n$, one has

$$y_n \geq \mathbb{E}\left[H^n(T)\beta^n(T)\Psi^{-1}(X^n(T), P^n(T))\right]$$

and, by choosing $(t_n, x_n, p_n)_n$ such that $v(t_n, x_n, p_n) \to v_*(T, x, p)$, we obtain

$$\begin{aligned} v_*(T,x,p) &\geq \liminf_{n\to\infty} \mathbb{E}\left[H^n(T)\beta^n(T)\Psi^{-1}(X^n(T), P^n(T))\right] \\ &= \widehat{\Psi}^{-1}(x,p) + \liminf_{n\to\infty} \delta_n \ , \end{aligned}$$

where

$$\delta_n := \mathbb{E}\left[H^n(T)\beta^n(T)\Psi^{-1}(X^n(T), P^n(T))\right] - \widehat{\Psi}^{-1}(x,p) \ .$$

It remains to show that $\liminf_n \delta_n \geq 0$. To see this, first observe that $\Psi^{-1} \geq \widehat{\Psi}^{-1}$ so that

$$\delta_n \geq \mathbb{E}\left[H^n(T)\beta^n\widehat{\Psi}^{-1}(X^n(T), P^n(T)) - \widehat{\Psi}^{-1}(x,p)\right] \ .$$

Moreover, by convexity of $\widehat{\Psi}^{-1}$, we have

$$\widehat{\Psi}^{-1}(X^n(T), P^n(T)) \geq \widehat{\Psi}^{-1}(X^n(T), p) + \nabla^+\widehat{\Psi}^{-1}(X^n(T), p)(P^n(T) - p)$$

so that

$$\begin{aligned} \delta_n \ \geq\ & \mathbb{E}\left[\nabla^+\widehat{\Psi}^{-1}(x,p)P^n(T) - H^n(T)\beta^n(T)\nabla^+\widehat{\Psi}^{-1}(X^n(T),p)p\right] \\ & -\mathbb{E}\left[\left|H^n(T)\beta^n(T)\widehat{\Psi}^{-1}(X^n(T),p) - \widehat{\Psi}^{-1}(x,p)\right|\right] \\ & -|\mathcal{O}|_\infty\mathbb{E}\left[\left|H^n(T)\beta^n(T)\nabla^+\widehat{\Psi}^{-1}(X^n(T),p) - \nabla^+\widehat{\Psi}^{-1}(x,p)\right|\right] \ , \end{aligned}$$

[1] There is a slight error in the proof of the corresponding result in [4], see their Proposition 3.2 in which P is $\mathbb{P}$-martingale and not a $\mathbb{Q}$-martingale. We take this opportunity to correct it and we thank Nizar Touzi for the discussions we had on this point. A rigorous version is given in Moreau [16]

where $|\mathcal{O}|_\infty := \max\{|q|,\ q \in \mathcal{O}\} < \infty$. Since P^n is a martingale (under $\mathbb{P}$), this implies

$$\begin{aligned}\delta_n \quad \geq \quad & -\mathbb{E}\left[|\nabla^+\widehat{\Psi}^{-1}(x,p)p_n - H^n(T)\beta^n(T)\nabla^+\widehat{\Psi}^{-1}(X^n(T),p)p|\right] \\ & -\mathbb{E}\left[|H^n(T)\beta^n(T)\widehat{\Psi}^{-1}(X^n(T),p) - \widehat{\Psi}^{-1}(x,p)|\right] \\ & -|\mathcal{O}|_\infty\mathbb{E}\left[|H^n(T)\beta^n(T)\nabla^+\widehat{\Psi}^{-1}(X^n(T),p) - \nabla^+\widehat{\Psi}^{-1}(x,p)|\right] \ ,\end{aligned}$$

and the required result follows from the assumptions of the proposition. □

2.3 Discussion of the boundary condition on $\partial\mathcal{O}$

Since $\mathcal{O}$ is convex, it takes the form $[m, M]$ with $M, -m \in (-\infty, \infty]$. Obviously the boundary condition is meaningful only when $M < \infty$ or $m < \infty$.
In order to recover a minimum of structure, we impose the following conditions:

$$\Psi(y,x) \geq M \Longrightarrow y \geq g(x) \quad \text{and} \quad \Psi(0,x) \geq m \ \ , \forall\ (x,y) \in \mathbb{R}^d \times \mathbb{R} \qquad (6)$$

for some continuous function g.

For $\Psi(x,y) = \mathbf{1}_{y \geq g(x)}$, which corresponds to the quantile hedging problem, this holds for $M = 1$ and $m = 0$. For $\Psi(x,y) = -\ell((g(x)-y)^+)$, which corresponds to the expected loss pricing rule for ℓ convex non-decreasing, this holds with $M = -\ell(0)$ and $m = -\ell(\infty) = -\infty$.

When M is finite, (6) implies that $v(t,x,M)$ coincides with the super-hedging price of $g(X_{t,x}(T))$. When m is finite, (6) implies that $v(t,x,m) = 0$. Moreover, it is clear that v is non-decreasing in the p-variable. It follows that

$$v_*(t,x,M) \leq v^*(t,x,M) \leq v(t,x,M) \text{ and } v^*(t,x,m) \geq v_*(t,x,m) \geq v(t,x,m)\ .$$

However, equality may fail in the above inequalities.

This point being highly technical, we shall not discuss it further here. We refer to [4] for natural conditions, which are typically satisfied in financial applications, under which equality holds. In particular, it is the case in the two examples of application below.

3 Example 1: Quantile hedging and Follmer-Leukert's formula

3.1 Supersolution characterization of the quantile hedging price

In this section, we specialize the discussion to the quantile hedging problem of Föllmer and Leukert [12], which we already discussed in Chapter 5. We consider the non-constrained case $K = \mathbb{R}$ and we restrict to the one dimensional Black and Scholes model for ease of notations, see [4] for a more general setting.
It means that

$$\mu(x,a) = x\mu \text{ and } \sigma(x,a) = x\sigma \tag{7}$$

where μ and $\sigma > 0$ are now constants. We fix $\rho = 0$ for simplicity.

Then, the coefficients of the wealth process Y are given by

$$\mu_Y(x,y,a) = ax\mu \ , \ \sigma_Y(x,a) = ax\sigma \ . \tag{8}$$

Finally, we take

$$\Psi(x,y) = \mathbf{1}_{\{y-g(x)\geq 0\}} \quad \text{for some Lipschitz function} \quad g : \mathbb{R} \longrightarrow \mathbb{R}_+. \tag{9}$$

The stochastic target problem $v(t,x,p)$ corresponds to the problem of super-hedging the contingent claim $g(X_{t,x}(T))$ with probability p.

Note that the above assumptions ensure that $v(\cdot,1)$ is continuous and is given by $v(t,x,1) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[g(X_{t,x}(T))\right]$ where $\mathbb{Q}_{t,x}$ is the $\mathbb{P}$-equivalent martingale measure defined by

$$d\mathbb{Q}_{t,x}/d\mathbb{P} = \exp\left(-\frac{T}{2}|\lambda|^2 - \lambda W_T\right) \ , \ \lambda := \mu/\sigma \ .$$

For later use, let us denote by $W^{\mathbb{Q}_{t,x}} := W - W_t + \lambda(\cdot - t)$ the $\mathbb{Q}_{t,x}$-Brownian motion defined on $[t,T]$.

In Chapter 5, we have solved the quantile hedging problem by means of the Neyman and Pearson's lemma from mathematical statistics. We shall see here how we can recover this result in the Markovian setting.

Note that, in this particular model, we have

$$\psi(x,p,q,b) = p + bq/(x\sigma) \ \text{ and } \ \delta_K = \infty \text{ on } \mathbb{R}\setminus\{0\} \ ,$$

so that Theorem 2 implies that v_* should be a viscosity supersolution on $[0,T) \times (0,\infty) \times (0,1)$ of

$$-\partial_t \varphi - \frac{\sigma^2 x^2}{2} D_{xx}\varphi - \inf_{b \in \mathbb{R}} \left(-b D_p \varphi \lambda + x\sigma b D_{xp}\varphi + \frac{b^2}{2} D_{pp}\varphi \right) \geq 0 \ . \qquad (10)$$

Here, the conditions of Theorem 2 are not satisfied because K is not compact, and we have omitted the condition $D_p\varphi \neq 0$. However, the above holds for test functions such that $D_{pp}\varphi(t_0, x_0, p_0) > 0$ at the point where the minimum is achieved. The reason for this is that it allows to recover compactness on the set of b's on which the above infimum is taken. This provides the required continuity on the operator associated to the above PDE in a neighborhood of $D_{pp}\varphi(t_0, x_0, p_0)$. Using this continuity, the proof of Theorem 2 in Chapter 6 can be reproduced without difficulty.

Moreover, the conditions of Proposition 2 trivially hold with $\Psi^{-1}(x,p) = g(x)\mathbf{1}_{p>0}$, whose convex envelope in the p-variable is given by $\widehat{\Psi}^{-1}(x,p) = pg(x)$. It follows that

$$v_*(T,x,p) \geq pg(x) \ . \qquad (11)$$

3.2 Formal explicit resolution

The key idea for solving (10)-(11) is to introduce the Legendre-Fenchel dual function of v_* with respect to the $p-$variable in order to remove the non-linearity in (10):

$$w(t,x,q) := \sup_{p \in \mathbb{R}} \{pq - v_*(t,x,p)\} \ , \qquad (t,x,q) \in [0,T] \times (0,\infty) \times \mathbb{R} \ . \quad (12)$$

Note that

$$w(\cdot,q) = \infty \ \text{ for } q < 0 \quad \text{and} \quad w(\cdot,q) = \sup_{p \in [0,1]} \{pq - v_*(\cdot,p)\} \ \text{ for } q > 0 \ , \quad (13)$$

since

$$v_* \geq 0, \ v_*(\cdot,p) = 0 \ \text{ for } p < 0 \text{ and } v_*(\cdot,p) = \infty \text{ for } p > 1, \qquad (14)$$

by construction. One can actually show, see [4], that

$$v_*(t,x,1) = v(t,x,1) \text{ and } v_*(t,x,0) = 0 \ , \qquad (15)$$

recall the discussion of Section 2.3.

Using the PDE characterization of v_* above, we shall prove below that w is an upper-semicontinuous viscosity subsolution on $[0,T)\times(0,\infty)\times(0,\infty)$ of

$$-\partial_t w - \frac{x^2\sigma^2}{2} D_{xx} w - \frac{\lambda^2 q^2}{2} D_{qq} w - x\sigma\lambda D_{xq} w \quad \leq \quad 0 \tag{16}$$

with the boundary condition

$$w(T,x,q) \leq (q - g(x))^+ \ . \tag{17}$$

Recalling the Feynman-Kac representation and comparison results of Theorems 4 and 5 of Chapter 3, this implies that

$$w(t,x,q) \leq \bar{w}(t,x,q) := \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\left(Q_{t,x,q}(T) - g\left(X_{t,x}(T)\right)\right)^+\right] \ , \tag{18}$$

on $[0,T]\times(0,\infty)\times(0,\infty)$, where the process $Q_{t,x,q}$ is defined by the dynamics

$$\frac{dQ(s)}{Q(s)} \quad = \quad \lambda dW_s^{\mathbb{Q}_{t,x}} \ \ , \ Q_{t,x,q}(t) = q \in (0,\infty) \ . \tag{19}$$

Given the explicit representation of $\bar{w}$, we can now provide a lower bound to v_* by using (13).
Clearly the function $\bar{w}$ is convex in q and there is a unique solution $\bar{q}$ to the equation

$$\begin{aligned}
\frac{\partial \bar{w}}{\partial q}(t,x,\bar{q}) \quad &= \quad \mathbb{E}^{\mathbb{Q}_{t,x}}\left[Q_{t,x,1}(T)\mathbf{1}_{\{Q_{t,x,\bar{q}}(T)\geq g(X_{t,x}(T))\}}\right] \\
&= \quad \mathbb{P}\left[Q_{t,x,\bar{q}}(T) \geq g\left(X_{t,x}(T)\right)\right] \\
&= \quad p \ ,
\end{aligned} \tag{20}$$

where we have used the fact that $d\mathbb{P}/d\mathbb{Q}_{t,x} = Q_{t,x,1}(T)$. It follows that the value function of the quantile hedging problem v admits the lower bound

$$\begin{aligned}
v(t,x,p) \quad &\geq \quad p\bar{q} - \bar{w}\left(t,x,\bar{q}\right) \\
&= \quad \bar{q}\left[p - \mathbb{E}^{\mathbb{Q}_{t,x}}\left[Q_{t,x,1}(T)\mathbf{1}_{\{\bar{q}Q_{t,x,1}(T)\geq g(X_{t,x}(T))\}}\right]\right] \\
&+ \quad \mathbb{E}^{\mathbb{Q}_{t,x}}\left[g\left(X_{t,x}(T)\right)\mathbf{1}_{\{\bar{q}Q_{t,x,1}(T)\geq g(X_{t,x}(T))\}}\right] \\
&= \quad \mathbb{E}^{\mathbb{Q}_{t,x}}\left[g\left(X_{t,x}(T)\right)\mathbf{1}_{\{\bar{q}Q_{t,x,1}(T)\geq g(X_{t,x}(T))\}}\right] =: \bar{y} \ .
\end{aligned}$$

On the other hand, it follows from the martingale representation theorem, see Corollary 1 of Chapter 2, that we can find $\phi \in \mathcal{A}_b$ such that

$$Y_{t,x,\bar{y}}^{\phi}(T) \geq g\left(X_{t,x}(T)\right)\mathbf{1}_{\{\bar{q}Q_{t,x,1}(T)\geq g(X_{t,x}(T))\}} \ .$$

Since, $\mathbb{P}\left[\bar{q}Q_{t,x,1}(T) \geq g\left(X_{t,x}(T)\right)\right] = p$ by (20), this implies that $v(t,x,p) = \bar{y}$, which corresponds exactly to the solution found in Chapter 5.

3.3 Rigorous PDE characterization of the Fenchel-Legendre transform

To conclude our argument, it remains to prove that w is a viscosity subsolution of (16)-(17).

First note that the fact that w is upper-semicontinuous on $[0,T]\times(0,\infty)\times(0,\infty)$ follows from the lower-semicontinuity of v_* and the representation in the right-hand side of (13), which allows to reduce the computation of the sup to the compact set $[0,1]$. Moreover, the boundary condition (17) is an immediate consequence of (11) and (14).

We now turn to the PDE characterization inside the domain. Let φ be a smooth function with bounded derivatives and $(t_0,x_0,q_0)\in[0,T)\times(0,\infty)\times(0,\infty)$ be a local maximizer of $w-\varphi$ such that $(w-\varphi)(t_0,x_0,q_0)=0$.

a. We first show that we can reduce to the case where the map $q\mapsto\varphi(\cdot,q)$ is strictly convex. Indeed, since w is convex, we necessarily have $D_{qq}\varphi(t_0,x_0,q_0)\ge 0$. Given $\varepsilon,\eta>0$, we now define $\varphi_{\varepsilon,\eta}$ by $\varphi_{\varepsilon,\eta}(t,x,q):=\varphi(t,x,q)+\varepsilon|q-q_0|^2+\eta|q-q_0|^2(|q-q_0|^2+|t-t_0|^2+|x-x_0|^2)$. Note that (t_0,x_0,q_0) is still a local maximizer of $w-\varphi_{\varepsilon,\eta}$. Since $D_{qq}\varphi(t_0,x_0,q_0)\ge 0$, we have $D_{qq}\varphi_{\varepsilon,\eta}(t_0,x_0,q_0)\ge 2\varepsilon>0$. Since φ has bounded derivatives, we can then choose η large enough so that $D_{qq}\varphi_{\varepsilon,\eta}>0$. We next observe that, if $\varphi_{\varepsilon,\eta}$ satisfies (16) at (t_0,x_0,q_0) for all $\varepsilon>0$, then (16) holds for φ at this point too. This is due to the fact that the derivatives up to order two of $\varphi_{\varepsilon,\eta}$ at (t_0,x_0,q_0) converge to the corresponding derivatives of φ as $\varepsilon\to 0$.

b. From now on, we thus assume that the map $q\mapsto\varphi(\cdot,q)$ is strictly convex. Let $\tilde\varphi$ be the Fenchel transform of φ with respect to q, i.e.

$$\tilde\varphi(t,x,p):=\sup_{q\in\mathbb{R}}\{pq-\varphi(t,x,q)\}\,.$$

Since φ is strictly convex in q and smooth on its domain, $\tilde\varphi$ is strictly convex in p and smooth on its domain, see e.g. [18]. Moreover, we have

$$\begin{aligned}\varphi(t,x,q) &= \sup_{p\in\mathbb{R}}\{pq-\tilde\varphi(t,x,p)\}\\ &= J(t,x,q)q-\tilde\varphi(t,x,J(t,x,q)) \;\text{ on } (0,T)\times(0,\infty)\times(0,\infty) \qquad (21)\end{aligned}$$

where $q\mapsto J(\cdot,q)$ denotes the inverse of $p\mapsto D_p\tilde\varphi(\cdot,p)$, recall that $\tilde\varphi$ is strictly convex in p.

We now deduce from the assumption $q_0 > 0$ and (13) that we can find $p_0 \in [0,1]$ such that $v(t_0, x_0, q_0) = p_0 q_0 - v_*(t_0, x_0, p_0)$ which, by using the very definition of (t_0, x_0, p_0, q_0) and w, implies that

$$(t_0, x_0, p_0) \text{ is a local minimizer of } v_* - \tilde{\varphi} \text{ such that } (v_* - \tilde{\varphi})(t_0, x_0, p_0) = 0 \quad (22)$$

and

$$\varphi(t_0, x_0, q_0) = \sup_{p \in \mathbb{R}} \{p q_0 - \tilde{\varphi}(t_0, x_0, p)\} = p_0 q_0 - \tilde{\varphi}(t_0, x_0, p_0) \quad (23)$$
$$\text{with } p_0 = J(t_0, x_0, q_0)$$

where the last equality follows from (21) and the strict convexity of the map $p \mapsto p q_0 - \tilde{\varphi}(t_0, x_0, p)$ in the domain of $\tilde{\varphi}$.

We conclude the proof by discussing three alternative cases depending on the value of p_0.
1. If $p_0 \in (0,1)$, then (22) implies that $\tilde{\varphi}$ satisfies (10) at (t_0, x_0, p_0) and the required result follows by exploiting the link between the derivatives of $\tilde{\varphi}$ and the derivatives of its p-Fenchel transform φ, which can be deduced from (21).
2. If $p_0 = 1$, then the first boundary condition in (15) and (22) imply that (t_0, x_0) is a local minimizer of $v_*(\cdot, 1) - \tilde{\varphi}(\cdot, 1) = v(\cdot, 1) - \tilde{\varphi}(\cdot, 1)$ such that $(v(\cdot, 1) - \tilde{\varphi}(\cdot, 1))(t_0, x_0) = 0$. This implies that $\tilde{\varphi}(\cdot, 1)$ satisfies (2) of Chapter 3 at (t_0, x_0), so that $\tilde{\varphi}$ satisfies (10) for $b = 0$ at (t_0, x_0, p_0). We can then conclude as in 1. above.
3. If $p_0 = 0$, then the second boundary condition in (15) and (22) imply that (t_0, x_0) is a local minimizer of $v_*(\cdot, 0) - \tilde{\varphi}(\cdot, 0) = 0 - \tilde{\varphi}(\cdot, 0)$ such that $0 - \tilde{\varphi}(\cdot, 0)(t_0, x_0) = 0$. In particular, (t_0, x_0) is a local maximum point for $\tilde{\varphi}(\cdot, 0)$ so that $(\partial_t \tilde{\varphi}, D_x \tilde{\varphi})(t_0, x_0, 0) = 0$ and $D_{xx} \tilde{\varphi}(t_0, x_0, 0) \leq 0$. This implies that $\tilde{\varphi}(\cdot, 0)$ satisfies (10) at (t_0, x_0, p_0), for $b = 0$. We can then argue as in the first case. □

4 Example 2: Expected shortfall

Let us now consider the same model as above but with a risk constraint expressed through a quadratic loss function as in Section 2 of Chapter 5 (more general loss functions could obviously be considered, up to more tricky computations). This corresponds to

$$\Psi(x, y) = -((g(x) - y)^+)^2 ,$$

so that

$$\Psi^{-1}(x,p) = (g(x) - \sqrt{-p})^+ \quad \text{for } p \le 0 \ .$$

As in the previous section, we obtain that, for any test function φ and $(t,x,p) \in [0,T) \times (0,\infty) \times (-\infty,0)$ that achieves a minimum of $v_* - \varphi$ and such that $D_{pp}\varphi(t,x,p) > 0$, one has

$$-\partial_t\varphi - \frac{\sigma^2 x^2}{2} D_{xx}\varphi - \inf_{b\in\mathbb{R}} \left(-bD_p\varphi\lambda + x\sigma b D_{xp}\varphi + \frac{b^2}{2} D_{pp}\varphi \right) \ge 0 \ . \qquad (24)$$

By the same arguments as above, one can also show that the Fenchel-Legendre transform

$$w(t,x,q) := \sup_{p\in\mathbb{R}} \{pq - v_*(t,x,p)\} = \sup_{p\in(-\infty,0]} \{pq - v_*(t,x,p)\} \ , \qquad (25)$$

satisfies (16) on $[0,T)\times(0,\infty)\times(0,\infty)$, in the viscosity sense. As for the terminal condition, we obtain $v_*(T,x,p) \ge (g(x) - \sqrt{-p})^+$, so that

$$w(T,x,q) = \left((4q)^{-1} - g(x)\right) \mathbf{1}_{\{(2q)^{-1} \le g(x)\}} + (-qg(x)^2) \mathbf{1}_{\{(2q)^{-1} > g(x)\}} =: W(x,q) \ .$$

It follows that

$$w(t,x,q) \ge \bar{w}(t,x,q) := \mathbb{E}^{\mathbb{Q}_{t,x}} \left[W(X_{t,x}(T), Q_{t,x,q}(T)) \right] \ ,$$

and therefore

$$v(t,x,p) \quad \ge \quad \sup_{q>0} \left(qp - \mathbb{E}^{\mathbb{Q}_{t,x}} \left[W(X_{t,x}(T), Q_{t,x,q}(T)) \right] \right) .$$

Direct computations combined with the identity $d\mathbb{P}/d\mathbb{Q}_{t,x} = Q_{t,x,1}(T)$ then show that the optimum in the right-hand side term is achieved by $\bar{q} > 0$ such that

$$\begin{aligned} -p &= -\partial_q \bar{w}(t,x,\bar{q}) \\ &= \mathbb{E}\left[(2Q_{t,x,\bar{q}}(T))^{-2} \wedge g(X_{t,x}(T))^2 \right] \ . \end{aligned}$$

Combining the above assertions implies that

$$v(t,x,p) \quad \ge \quad \mathbb{E}^{\mathbb{Q}_{t,x}} \left[\left(g\left(X_{t,x}(T)\right) - (2Q_{t,x,\bar{q}}(T))^{-1} \right)^+ \right] =: \bar{y} \ .$$

On the other hand, it follows from the martingale representation theorem, see Corollary 1 of Chapter 2, that we can find $\phi \in \mathcal{A}_b$ such that

$$Y^{\phi}_{t,x,\bar{y}}(T) = \left(g\left(X_{t,x}(T)\right) - (2Q_{t,x,\bar{q}}(T))^{-1} \right)^+$$

which, by the above identity, satisfies

$$\mathbb{E}[(g\left(X_{t,x}(T)\right) - Y^{\phi}_{t,x,y(t,x,p)})^{+})^2] = \mathbb{E}\left[(2Q_{t,x,\bar{q}}(T))^{-2} \wedge g(X_{t,x}(T))^2\right] = p\ .$$

This shows that

$$v(t,x,p) = \mathbb{E}^{\mathbb{Q}_{t,x}}\left[\left(g\left(X_{t,x}(T)\right) - (2Q_{t,x,\bar{q}}(T))^{-1}\right)^{+}\right]\ ,$$

which is the result obtained in Section 2 of Chapter 5.

Chapter 8

Optimal portfolio management under risk constraints

1 Problem formulation

2 Hamilton-Jacobi-Bellman equation

3 Example:

Part C.

Exercices

Exercise 1 We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} := (\mathcal{F}_t)_{t\le T}$ be the filtration, satisfying the usual conditions, induced by a $\mathbb{P}$-Brownian motion W. We assume that $\mathcal{F}_T = \mathcal{F}$.
Let us consider the Black-and-Scholes one dimensional model with interest rate equal to 0, i.e. $r \equiv 0$, in which the dynamics of the risky asset is given by

$$X_t = X_0 e^{(\mu-\sigma^2/2)t+\sigma W_t} \ , \ \ t \le T \ ,$$

where W is a Brownian motion under $\mathbb{P}$, $\mu \in \mathbb{R}$ and $X_0, \sigma > 0$.

The aim of this exercise is to study the super-hedging problem under constraints on the proportion of the wealth invested in X. Namely, we fix $m < M$, and say that a predictable process is admissible if it takes values in $[m, M]$ $dt \times d\mathbb{P}$-a.e. on $[0, T]$. We denote by $\mathcal{A}$ the collection of such processes. The wealth process $Y^{y,\phi}$ associated to the initial wealth $y > 0$ and the strategy $\phi \in \mathcal{A}$ has the dynamics

$$Y_t^{y,\phi} = y + \int_0^t \frac{\phi_s Y_s^{y,\phi}}{X_s} dX_s \ , \ t \le T \ .$$

1. Justify (in words) the above dynamics.

2. Show that

$$Y_t^{y,\phi} = y + \int_0^t \phi_s Y_s^{y,\phi} \mu ds + \int_0^t \phi_s Y_s^{y,\phi} \sigma dW_s \ , \ t \le T \ .$$

From now on, we fix a bounded random variable $G \in L^0(\mathcal{F}_T)$ satisfying $G > 0$ $\mathbb{P}-$a.s. The super-hedging price is defined as

$$p(G) := \inf\{y > 0 \ : \ \exists\, \phi \in \mathcal{A} \text{ s.t. } Y_T^{y,\phi} \ge G\} \ .$$

We set

$$\delta(\zeta) = \zeta^+ M - \zeta^- m \text{ with } \zeta^+ = \zeta \mathbf{1}_{\zeta>0} \text{ and } \zeta^- = -\zeta \mathbf{1}_{\zeta<0} \text{ for } \zeta \in \mathbb{R} \ .$$

We denote by $\mathcal{U}$ the set of predictable processes ν such that $|\nu| \le c$ $dt \times d\mathbb{P}$-a.e. on $[0, T]$ for some $c > 0$ which depends on ν. We finally define

$$\mathcal{E}^\nu := e^{-\int_0^\cdot \delta(\nu_s)ds} e^{-\frac{1}{2}\int_0^\cdot |\lambda_s^\nu|^2 ds - \int_0^\cdot \lambda_s^\nu dW_s} \ ,$$

where

$$\lambda^\nu := (\mu - \nu)/\sigma \ .$$

3. Show that for any $(\phi, \nu) \in \mathcal{A} \times \mathcal{U}$,

$$Y^{y,\phi}\mathcal{E}^{\nu} = y + \int_0^{\cdot} Y_s^{y,\phi}\mathcal{E}_s^{\nu} \left(\phi_s \sigma - \lambda_s^{\nu}\right) dW_s + \int_0^{\cdot} Y_s^{y,\phi}\mathcal{E}_s^{\nu} \left(\phi_s \nu_s - \delta(\nu_s)\right) ds \ .$$

4. Using the definitions of δ and $\mathcal{A}$, show that $Y^{y,\phi}\mathcal{E}^{\nu}$ is a $\mathbb{P}$ super-martingale for any $(\phi, \nu) \in \mathcal{A} \times \mathcal{U}$.

5. Let $y > 0$ and $\phi \in \mathcal{A}$. Show that, if $Y_T^{y,\phi} \geq G$, then

$$y \geq \bar{p}(G) := \sup_{\nu \in \mathcal{U}} \mathbb{E}\left[\mathcal{E}_T^{\nu} G\right] \ .$$

6. Show that this implies that $p(G) \geq \bar{p}(G)$.

We now aim at proving the converse inequality. We first assume that there exists a cadlag adapted process P such that

$$P_t = \operatorname*{esssup}_{\nu \in \mathcal{U}} J_t^{\nu} \ \text{ for all } t \leq T \ ,$$

where

$$J_t^{\nu} := \mathbb{E}\left[\mathcal{E}_T^{\nu} G \mid \mathcal{F}_t\right] / \mathcal{E}_t^{\nu} \text{ for } \nu \in \mathcal{U} \text{ and } t \leq T \ .$$

7. Show that the family $\{J_t^{\nu},\ \nu \in \mathcal{U}\}$ is directed upward for all $t \leq T$.

8. Show that for any $\nu^1, \nu^2 \in \mathcal{U}$ and $s \leq t \leq T$, there exists $\nu^3 \in \mathcal{U}$ such that

$$\frac{\mathcal{E}_t^{\nu^1}}{\mathcal{E}_s^{\nu^1}} \frac{\mathcal{E}_T^{\nu^2}}{\mathcal{E}_t^{\nu^2}} = \frac{\mathcal{E}_T^{\nu^3}}{\mathcal{E}_s^{\nu^3}} \ .$$

9. Deduce that $\mathcal{E}^{\nu} P$ is a $\mathbb{P}$-supermatingale for any $\nu \in \mathcal{U}$.

In view of the last question, and the multiplicative Doob-Meyer decomposition, it follows that, we can find a family of martingales $\{M^{\nu},\ \nu \in \mathcal{U}\}$ and a non-increasing process $\{A^{\nu},\ \nu \in \mathcal{U}\}$ such that

$$\mathcal{E}^{\nu} P = M^{\nu} A^{\nu} \ , \ A^{\nu} > 0 \ , \ M^{\nu} > 0 \text{ and } A_0^{\nu} = 1 \text{ for all } \nu \in \mathcal{U} \ . \tag{1}$$

In the following, we denote by 0 a process ν such that $\nu = 0$ $dt \times d\mathbb{P}$-a.e. on $[0, T]$.

10. Show that $M_T^0 \geq \mathcal{E}_T^0 P_T \geq \mathcal{E}_T^0 G > 0$.

11. Deduce that there exists a predictable process, $\mathbb{P}-$a.s. square integrable, ψ^0 such that

$$M_T^0 = M_0^0 + \int_0^T M_s^0 \psi_s^0 dW_s \geq \mathcal{E}_T^0 G \; .$$

12. Show that $M^0/\mathcal{E}^0$ can be rewritten as

$$M^0/\mathcal{E}^0 = M_0^0 + \int_0^{\cdot} \frac{M_s^0}{\mathcal{E}_s^0} \left(\lambda_s^0 + \psi_s^0\right) dW + \int_0^{\cdot} \frac{M_s^0}{\mathcal{E}_s^0} \left(\lambda_s^0 \psi_s^0 + |\lambda_s^0|^2\right) ds$$

13. Deduce that

$$Y^0 := M^0/\mathcal{E}^0 = Y^{M_0,\phi^0} \text{ and } Y_T^{M_0,\phi^0} \geq G \; ,$$

for some $\mathbb{P}-$a.s. square integrable predictable process ϕ^0.

14. Deduce from the equality $M^\nu = \mathcal{E}^\nu Y^0 A^0/A^\nu$ that

$$\int_0^{\cdot} F_s^\nu \left(\phi^0 \nu_s - \delta(\nu_s)\right) \frac{A_s^0}{A_s^\nu} ds + \int_0^{\cdot} \frac{F_s^\nu}{A_s^\nu} dA_s^0 - \int_0^{\cdot} \frac{F_s^\nu A_s^0}{|A_s^\nu|^2} dA_s^\nu = 0$$

where

$$F^\nu := \mathcal{E}^\nu Y^0$$

for $\nu \in \mathcal{U}$.

15. By using (1) and the fact that A^0 and A^ν are non-increasing, deduce from the previous result that

$$\begin{aligned} 1 \geq A^\nu &= \int_0^{\cdot} A_s^\nu \left(\phi_s^0 \nu_s - \delta(\nu_s)\right) ds + \int_0^{\cdot} \frac{A_s^\nu}{A_s^0} dA_s^0 \\ &\geq \int_0^{\cdot} A_s^\nu \left(\phi_s^0 \nu_s - \delta(\nu_s)\right) ds + \int_0^{\cdot} \frac{1}{A_s^0} dA_s^0 \end{aligned}$$

for all $\nu \in \mathcal{U}$.

16. Deduce from the above inequality and a formal argument that

$$\sup_{\zeta \in \mathbb{R}} (\phi^0 \zeta - \delta(\zeta)) < \infty \quad dt \times d\mathbb{P}\text{-a.e.}$$

17. Deduce that $\phi^0 \in [m, M]$ $dt \times d\mathbb{P}$-a.e.

18. Show that $\bar{p}(G) \geq p(G)$ and conclude.

Exercise 2 We let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\mathbb{F} := (\mathcal{F}_t)_{t \le T}$ be the filtration, satisfying the usual conditions, induced by a $\mathbb{P}$-Brownian motion W. We assume that $\mathcal{F}_T = \mathcal{F}$.
Let us consider the Black-and-Scholes one dimensional model with interest rate equal to 0, i.e. $r \equiv 0$, in which the dynamics of the risky asset is given by

$$X_{t,x}(s) = xe^{(\mu-\sigma^2/2)(s-t)+\sigma(W_s-W_t)} \ , \ \ t \le s \le T \ ,$$

where $\mu \in \mathbb{R}$ and $x, \sigma > 0$.

A financial strategy is described by a predictable square integrable process ϕ, which is associated to the amount of money invested in the risky asset. Given a financial strategy ϕ and an initial wealth y at time t, the induced wealth process is

$$Y^{\phi}_{t,y}(s) \quad = \quad y + \int_t^s (\phi_s / X_{t,x}(s)) dX_{t,x}(s) = y + \int_t^s \phi_s \mu ds + \int_t^s \phi_s \sigma dW_s \ ,$$

which does not depend on x. In the following, we say that a financial strategy is admissible if the portfolio process is $dt \times d\mathbb{P}$-a.e. non-negative. We denote by $\mathcal{A}_+(y)$ the set of admissible financial strategies for a given initial wealth $y \ge 0$.

The aim of this exercise is to study the quantile-hedging problem

$$v(t,x,p) = \inf \left\{ y \ge 0 \ : \ \exists\, \phi \in \mathcal{A}_+(y) \text{ s.t. } \mathbb{P}\left[Y^{\phi}_{t,y}(T) \ge g(X_{t,x}(T)) \right] \ge p \right\}$$

where g is a continuous and bounded function, and $p \in (0,1)$.

1. Let $\mathcal{L}^2_{\mathcal{P}}$ denote the set of predictable square integrable process. For $\alpha \in \mathcal{L}^2_{\mathcal{P}}$ and $p \in [0,1]$, set $P^{\alpha}_{t,p} := p + \int_t^{\cdot} \alpha_s dW_s$. Show that $v(t,x,p)$ coincides with

$$\inf \left\{ y \ge 0 \ : \ \exists\, (\phi, \alpha) \in \mathcal{A}_+(y) \times \mathcal{L}^2_{\mathcal{P}} \text{ s.t. } \mathbf{1}_{\{Y^{\phi}_{t,y}(T) \ge g(X_{t,x}(T))\}} \ge P^{\alpha}_{t,p}(T) \ge 0 \right\} \ .$$

From now on, we shall assume that $v \in C^{1,2}([0,T) \times (0,\infty) \times (0,1))$.

We also recall the first part of the geometric dynamic programming principle:

$$y > v(t,x,p) \Rightarrow \exists\, (\phi,\alpha) \in \mathcal{A}_+(y) \times \mathcal{L}^2_{\mathcal{P}} \text{ s.t. } Y^{\phi}_{t,y}(\theta) \ge v(\theta, X_{t,x}(\theta), P^{\alpha}_{t,p}(\theta))\ \forall\, \theta \in \mathcal{T}_{[t,T]}. \tag{2}$$

2. Set $\lambda := \mu/\sigma$. Assume that $(t_0, x_0, p_0) \in [0,T) \times (0,\infty) \times (0,1)$ is such that

$$\begin{aligned} -\partial_t v(t_0,x_0,p_0) - \tfrac{\sigma^2 x_0^2}{2} D_{xx} v(t_0,x_0,p_0) + \tfrac{(\lambda D_p v(t_0,x_0,p_0) - x_0 \sigma D_{xp} v(t_0,x_0,p_0))^2}{2 D_{pp} v(t_0,x_0,p_0)} &< 0 \\ D_{pp} v(t_0,x_0,p_0) &> 0 \ . \end{aligned} \tag{3}$$

Show that this implies that

$$-\partial_t v - \frac{\sigma^2 x^2}{2} D_{xx} v - \inf_{b\in\mathbb{R}} \left(-bD_p v\lambda + x\sigma b D_{xp} v + \frac{b^2}{2} D_{pp} v \right) < 0 \ , \ D_{pp} v > 0$$

on a neighborhood B_0 of (t_0, x_0, p_0).

3. Show that the above implies, for some $r, \eta > 0$ small enough,

$$\begin{array}{c} u\mu - \mathcal{L}^b_{X,P}\tilde{v} < -\eta \\ \text{for all } (u,b) \in \mathbb{R}^2 \text{ s.t. } |u\sigma - x\sigma D_x\tilde{v} - bD_p\tilde{v}| \le r \\ D_{pp}\tilde{v} > 0 \end{array} \tag{4}$$

on a neighborhood B_0 of (t_0, x_0, p_0), where

$$\tilde{v}(t,x,p) := v(t,x,p) - |t-t_0|^2 - |x-x_0|^4 - |p-p_0|^4$$

and

$$\mathcal{L}^b_{X,P} v := \partial_t\tilde{v} + x\mu D_x\tilde{v} + \frac{1}{2}\left(\sigma^2 x^2 D_{xx}\tilde{v} + 2x\sigma b D_{xp}\tilde{v} + b^2 D_{pp}\tilde{v}\right)$$

4. Set $(X^0, Y^0, P^0) := (X_{t_0,x_0}, Y^\phi_{t_0,y_0}, P^\alpha_{t_0,p_0})$ for some $y_0 > v(t_0,x_0,p_0)$ and $(\phi,\alpha) \in \mathcal{A}_+(y_0) \times \mathcal{L}^2_{\mathcal{P}}$ such that

$$Y^0(t\wedge\theta^0) \ge v(t\wedge\theta^0, X(t\wedge\theta^0), P^0(t\wedge\theta^0)) \ \text{ for } t \ge t_0, \tag{5}$$

where

$$\begin{aligned} \theta^0 &:= \inf\{s \ge t_0 \ : \ (s, X(s), P^0(s)) \notin B_0\} \wedge \theta \\ \theta &:= \inf\{s \ge t_0 \ : \ |Y^0(s) - \tilde{v}(s, X^0(s), P^0(s))| \ge \varepsilon\} \end{aligned}$$

for some $\varepsilon > 0$.

5. Deduce from (5) that

$$M_t := Y^0(t\wedge\theta^0) - \tilde{v}(t\wedge\theta^0, X^0(t\wedge\theta^0), P^0(t\wedge\theta^0)) - \zeta \ge -\zeta 1_{t<\theta^0}$$

for some $\zeta > 0$.

6. Set

$$\begin{aligned} A &:= \{|\psi| > r\} \ , \\ \psi &:= \phi - X^0\sigma D_x v(\cdot, X^0, P^0) - \alpha D_p v(\cdot, X^0, P^0) \ , \\ b &:= \phi\mu - \mathcal{L}^\alpha_{X,P} v(\cdot, X^0, P^0) \ . \end{aligned}$$

Define L by $L_{t_0} = 1$ and $dL_s = -L_s(b_s/\psi_s)1_{A(s)}dW_s$. Deduce from (4) that LM is a super-matingale bounded from below by $(-L_s\zeta 1_{s<\theta^0})_{s\geq t_0}$.

7. Deduce from the above questions that

$$0 = \mathbb{E}\left[-L_{\theta^0}\zeta 1_{\theta^0<\theta^0}\right] \leq \mathbb{E}\left[M_{\theta^0}L_{\theta^0}\right] \leq y_0 - v(t_0, x_0, p_0) - \zeta \ .$$

8. Show that this leads to a contradiction for y_0 sufficiently close to $v(t_0, x_0, p_0)$, and therefore that (3) can not hold.

We now consider the boundary condition at T. From now on, we admit that

$$\lim_{t\uparrow T,(x',p')\to(x,p)} v(t', x', p') =: \bar{v}(x, p)$$

is well-defined.

9. Let $\mathbb{Q}_t \sim \mathbb{P}$ be defined by $d\mathbb{Q}_t/d\mathbb{P} = H_t := \exp\left(-\frac{1}{2}\lambda^2(T-t) - \lambda(W_T - W_t)\right)$, and show that

$$\mathbb{E}^{\mathbb{Q}}\left[Y^{\phi}_{t,y}(T)\right] \leq y \text{ for all } \phi \in \mathcal{A}_+(y) \ .$$

10. Fix $(\phi, \alpha) \in \mathcal{A}_+(y) \times \mathcal{L}^2_{\mathcal{P}}$ such that

$$\mathbf{1}_{\{Y^{\phi}_{t,y}(T)\geq g(X_{t,x}(T))\}} \geq P^{\alpha}_{t,p}(T) \geq 0 \ .$$

Show that this implies that

$$\begin{aligned} H_tY^{\phi}_{t,y}(T) &\geq H_tP^{\alpha}_{t,p}(T)g(X_{t,x}(T)) \\ &\geq P^{\alpha}_{t,p}(T)g(x) - \left|H_tg(X_{t,x}(T)) - g(x)\right| \ , \end{aligned}$$

and deduce that

$$y \geq pg(x) - \mathbb{E}\left[\left|H_tg(X_{t,x}(T)) - g(x)\right|\right] \ .$$

11. Using the previous question and our assumptions on g, show that

$$\bar{v}(x, p) \geq pg(x) \ .$$

In order to provide an explicit solution to the above problem, we now introduce the Fenchel transform of v with respect to p:

$$V(t, x, q) := \sup_{p\in[0,1]} pq - v(t, x, p) \ .$$

13. Provide a upper-bound for

$$\lim_{t \uparrow T, (x', q') \to (x,q)} V(t', x', q') \ .$$

We admit that, for each $q > 0$ and $(t, x) \in [0, T) \times (0, \infty)$, the sup in the above definition is achieved by a point $\hat{p}(t, x, q) \in (0, 1)$ such that $D_{pp} v(t, x, \hat{p}(t, x, q)) > 0$. We also admit that V and $\hat{p}$ are $C^{1,2}([0, T) \times (0, \infty) \times (0, \infty))$.

9. Rewritte the partial derivatives of v appearing in (3) at the point $(t, x, \hat{p}(t, x, q))$ in terms of the derivatives of V at (t, x, q).

10. Deduce a PDE for V.

11. Using the Feynman-Kac representation theorem, deduce a upper-bound for V.

12. Deduce a lower-bound for v.

13. Show that this lower bound is actually a upper-bound and conclude.

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