The continuous limit of large random planar maps

Jean-François Le Gall

Université Paris-Sud Orsay and Institut universitaire de France

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Outline

Goal: To understand the continuous limit of large planar maps (planar maps are graphs drawn in the plane, or on the sphere) chosen uniformly at random in a certain class (p-angulations) viewed as metric spaces (for the graph distance)

- Expects universality of the limit
- Leads to an important continuous model (Brownian map)
- Gives insight into the properties of large planar maps.

Strong analogy with random paths and Brownian motion.

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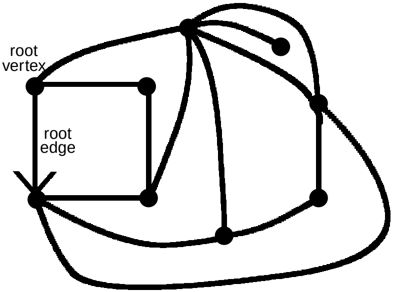
1. Introduction: Planar maps

Definition

A planar map is a proper embedding of a connected graph into the two-dimensional sphere (considered up to orientation-preserving homeomorphisms of the sphere).



A rooted quadrangulation

Faces = connected components of the complement of edges

p-angulation:

- each face has p adjacent edges

$p = 3$: triangulation
$p = 4$: quadrangulation

Rooted map: distinguished oriented edge

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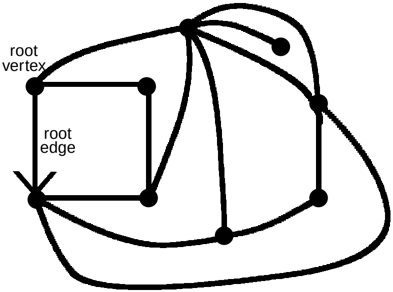

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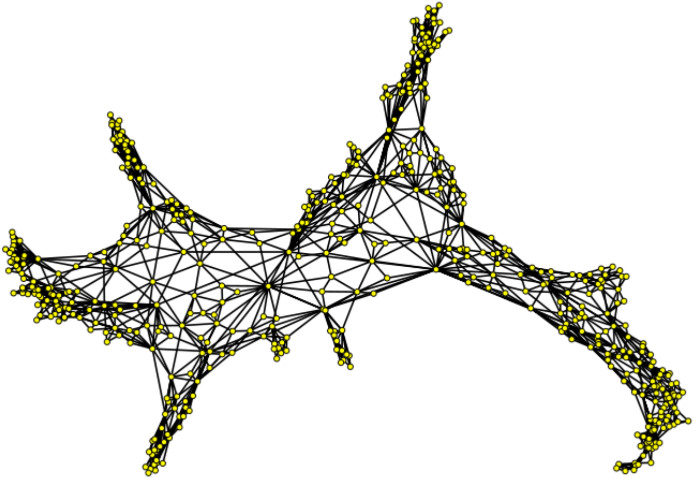


A large triangulation of the sphere (simulation by G. Schaeffer)
Can we get a continuous model out of this ?

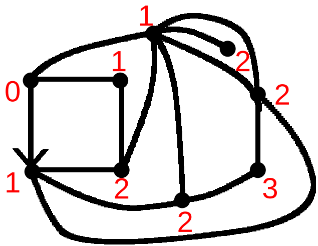
What is meant by the continuous limit ?

M planar map

- $V(M)$ = set of vertices of M
- d_{gr} graph distance on $V(M)$
- $(V(M), d_{\mathrm{gr}})$ is a (finite) metric space

$\mathbb{M}_n^p = \{$rooted $p-$angulations with n faces$\}$
(*modulo deformations of the sphere*)

$\mathbb{M}_n^p$ is a finite set



Goal

Let M_n be chosen uniformly at random in $\mathbb{M}_n^p$. For some $a > 0$,

$$(V(M_n), n^{-a} d_{\mathrm{gr}}) \underset{n\to\infty}{\longrightarrow} \text{“continuous limiting space”}$$

in the sense of the Gromov-Hausdorff distance.

Remarks.

a. Needs rescaling of the graph distance for a compact limit.

b. It is believed that the limit does not depend on p (universality).

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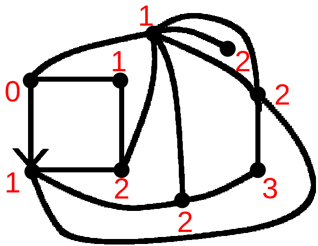

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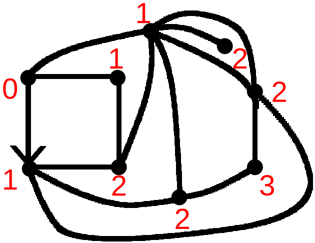

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The Gromov-Hausdorff distance

The Hausdorff distance. K_1, K_2 compact subsets of a metric space

$$d_{\mathrm{Haus}}(K_1, K_2) = \inf\{\varepsilon > 0 : K_1 \subset U_\varepsilon(K_2) \text{ and } K_2 \subset U_\varepsilon(K_1)\}$$

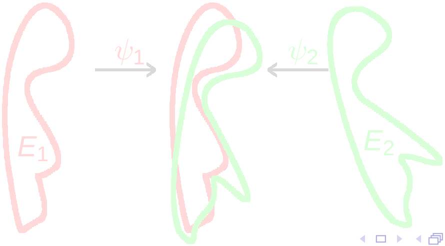
($U_\varepsilon(K_1)$ is the ε-enlargement of K_1)

Definition (Gromov-Hausdorff distance)

If (E_1, d_1) and (E_2, d_2) are two compact metric spaces,

$$d_{\mathrm{GH}}(E_1, E_2) = \inf\{d_{\mathrm{Haus}}(\psi_1(E_1), \psi_2(E_2))\}$$

the infimum is over all isometric embeddings $\psi_1 : E_1 \to E$ and $\psi_2 : E_2 \to E$ of E_1 and E_2 into the same metric space E.



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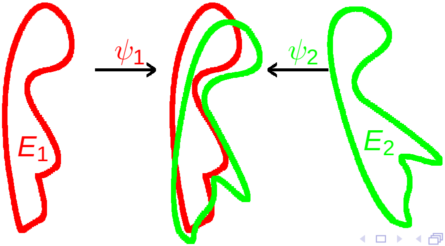

Gromov-Hausdorff convergence of rescaled maps

Fact

If $\mathbb{K} = \{$*isometry classes of compact metric spaces*$\}$, *then*

$(\mathbb{K}, d_{\mathrm{GH}})$ *is a separable complete metric space (Polish space)*

$\rightarrow$ It makes sense to study the convergence of

$$(V(M_n), n^{-a} d_{\mathrm{gr}})$$

as random variables with values in $\mathbb{K}$.

(Problem stated for triangulations by O. Schramm [ICM06])

Choice of a. The parameter a is chosen so that $\mathrm{diam}(V(M_n)) \approx n^a$.

$\Rightarrow a = \frac{1}{4}$ [cf Chassaing-Schaeffer PTRF 2004 for quadrangulations]

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- combinatorics [Tutte '60, four color theorem, etc.]
- theoretical physics
  - enumeration of maps related to matrix integrals ['t Hooft 74, Brézin, Itzykson, Parisi, Zuber 78, etc.]
  - large random planar maps as models of random geometry (quantum gravity, cf Ambjørn, Durhuus, Jonsson 95, Duplantier-Sheffield 08)
- probability theory: models for a Brownian surface
  - analogy with Brownian motion as continuous limit of discrete paths
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- metric geometry: examples of singular metric spaces
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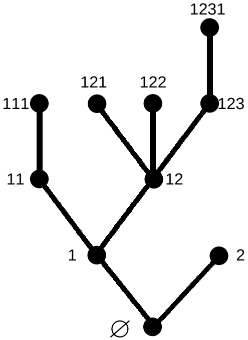
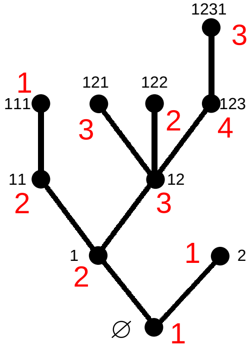
2. Bijections between maps and trees





A planar tree $\tau = \{\varnothing, 1, 2, 11, \ldots\}$

(rooted ordered tree)

the lexicographical order on vertices will play an important role in what follows

A well-labeled tree $(\tau, (\ell_v)_{v\in\tau})$

Properties of labels:

- $\ell_\varnothing = 1$
- $\ell_v \in \{1, 2, 3, \ldots\}, \forall v$
- $|\ell_v - \ell_{v'}| \leq 1$, if v, v' neighbors

Coding maps with trees, the case of quadrangulations

$$\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$$
$$\mathbb{M}_n^4 = \{\text{rooted quadrangulations with } n \text{ faces}\}$$

Theorem (Cori-Vauquelin, Schaeffer)

There is a bijection $\Phi : \mathbb{T}_n \longrightarrow \mathbb{M}_n^4$ such that, if $M = \Phi(\tau, (\ell_v)_{v\in\tau})$, then

$$V(M) = \tau \cup \{\partial\} \qquad (\partial \text{ is the root vertex of } M)$$
$$d_{\mathrm{gr}}(\partial, v) = \ell_v \quad , \forall v \in \tau$$

Key facts.

- Vertices of τ become vertices of M
- The label in the tree becomes the distance from the root in the map.

Coding of more general maps: Bouttier, Di Francesco, Guitter (2004).

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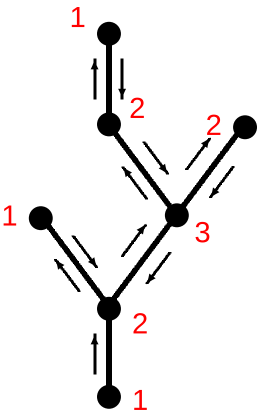
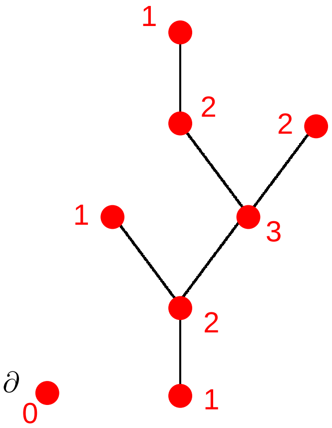


well-labeled tree



quadrangulation

Rules.

- add extra vertex ∂ labeled 0
- follow the contour of the tree, connect each vertex to the last visited vertex with smaller label

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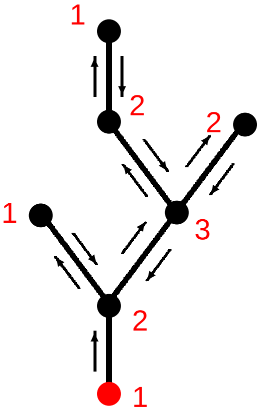

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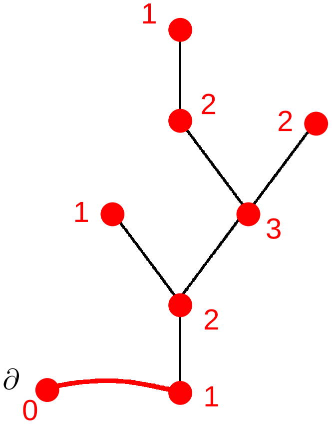

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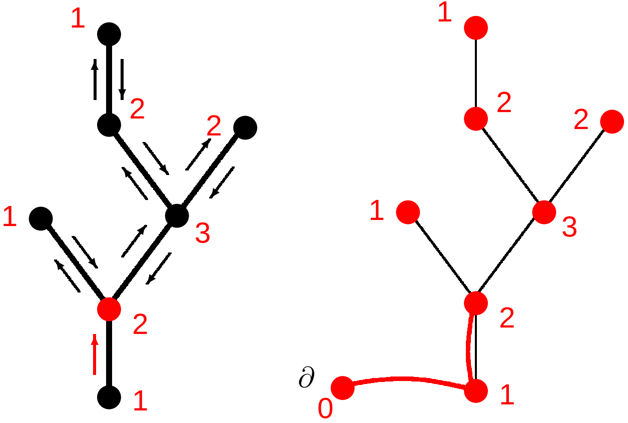

well-labeled tree quadrangulation

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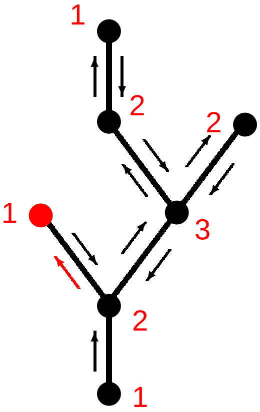

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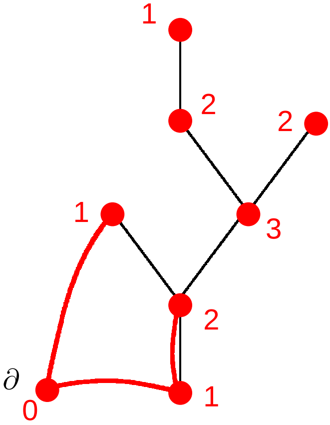

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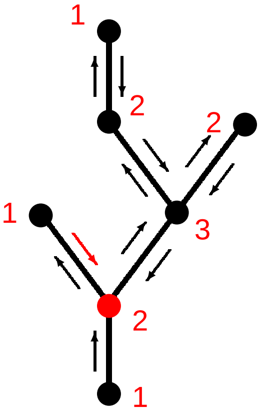

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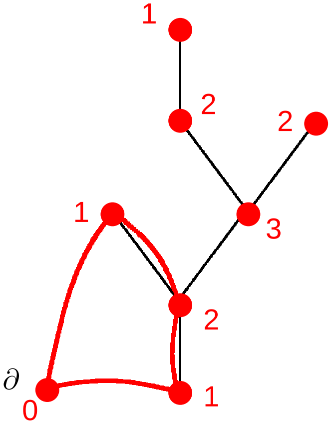

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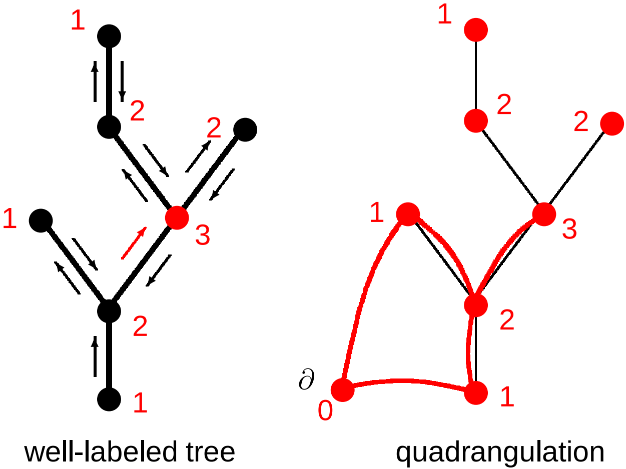

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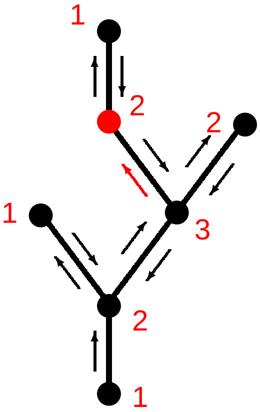

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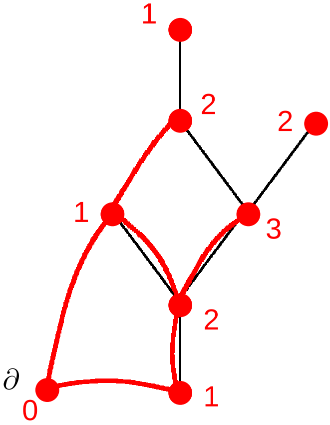

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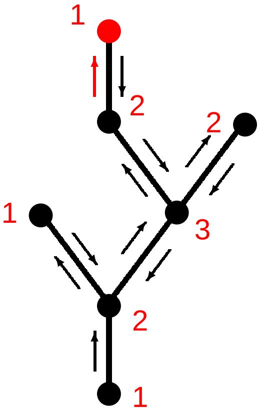

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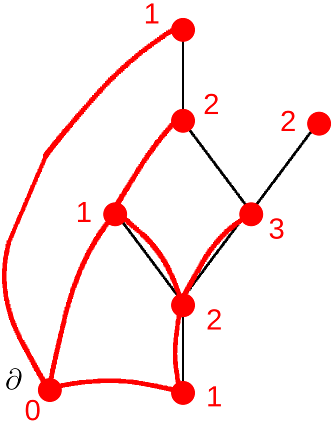

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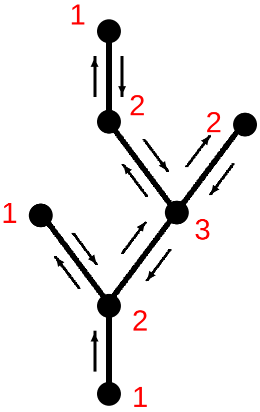

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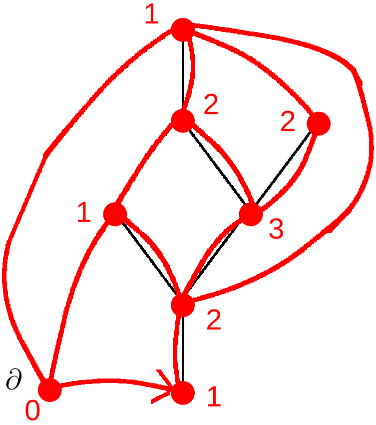

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General strategy

Understand continuous limits of trees ("easy")

in order to understand continuous limits of maps ("more difficult")

Key point. The bijections with trees allow us to handle distances from the root vertex, but **not** distances between two arbitrary vertices of the map (required if one wants to get Gromov-Hausdorff convergence)

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3. Asymptotics for trees

The case of planar trees

$$T_n^{\mathrm{planar}} = \{\text{planar trees with } n \text{ edges}\}$$

Theorem (reformulation of Aldous 1993)

For every n, let τ_n be a random tree uniformly distributed over T_n^{planar}. Then,

$$(\tau_n, \frac{1}{\sqrt{2n}} d_{\mathrm{gr}}) \longrightarrow (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}}) \qquad \text{as } n \to \infty$$

in distribution, in the Gromov-Hausdorff sense.
Here $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the CRT (Continuum Random Tree)

The notation $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ comes from the fact that the CRT is the tree coded by a Brownian excursion **e**

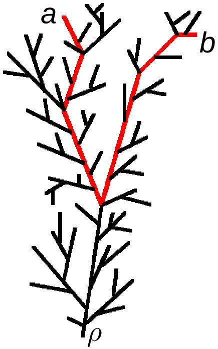
Definition of the CRT: notion of a real tree

Definition

A real tree is a (compact) metric space $\mathcal{T}$ such that:

- any two points $a, b \in \mathcal{T}$ are joined by a unique arc
- this arc is isometric to a line segment

It is a rooted real tree if there is a distinguished point ρ, called the root.



Remark. A real tree can have

- infinitely many branching points
- (uncountably) infinitely many leaves

Fact. The coding of discrete trees by contour functions (Dyck paths) can be extended to real trees.

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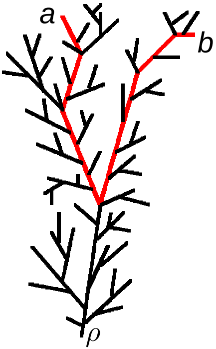

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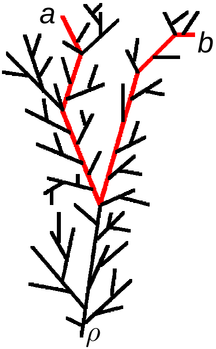

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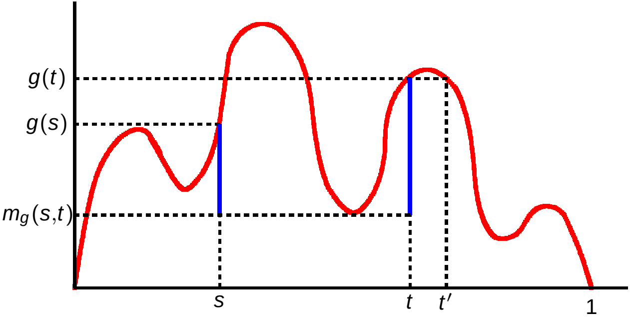
The real tree coded by a function g

$g : [0,1] \longrightarrow [0,\infty)$
continuous,
$g(0) = g(1) = 0$



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Proposition (Duquesne-LG)

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Remark. $\mathcal{T}_g$ inherits a "lexicographical order" from the coding.

The real tree coded by a function g

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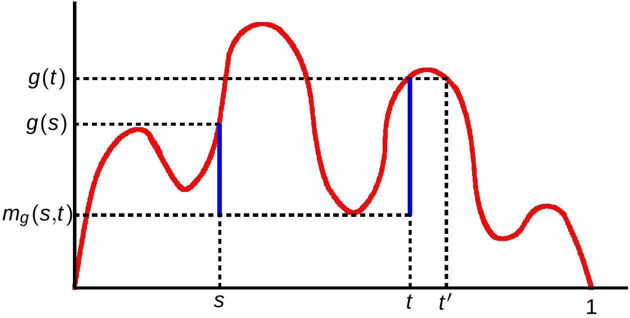

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Back to Aldous' theorem and the CRT

Aldous' theorem: τ_n uniformly distributed over T_n^{planar}

$$\Big(\tau_n, \frac{1}{\sqrt{2n}} d_{\text{gr}}\Big) \xrightarrow[n\to\infty]{\text{(d)}} (\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$$

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The limit $(\mathcal{T}_{\mathbf{e}}, d_{\mathbf{e}})$ is the (random) real tree coded by a Brownian excursion $\mathbf{e}$.

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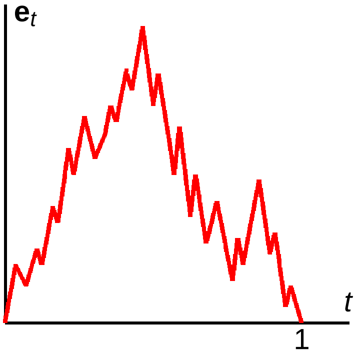

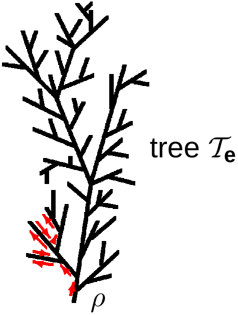

Assigning labels to a real tree

Need to assign (random) labels to the vertices of a real tree $(\mathcal{T}, d)$

$(Z_a)_{a\in\mathcal{T}}$: Brownian motion indexed by $(\mathcal{T}, d)$
= centered Gaussian process such that

- $Z_\rho = 0 \qquad (\rho$ root of $\mathcal{T})$
- $E[(Z_a - Z_b)^2] = d(a, b), \qquad a, b \in \mathcal{T}$

Labels evolve like Brownian motion along the branches of the tree:

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- Similar property for Z_b, but one uses
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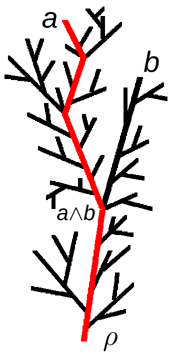

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The scaling limit of well-labeled trees

Recall $\mathbb{T}_n = \{\text{well-labeled trees with } n \text{ edges}\}$

$(\theta_n, (\ell_v^n)_{v\in\theta_n})$ uniformly distributed over $\mathbb{T}_n$

Rescaling:

- Distances on θ_n are rescaled by $\frac{1}{\sqrt{n}}$ (Aldous' theorem)
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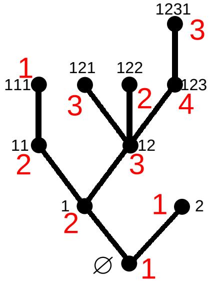

Fact

The scaling limit of $(\theta_n, (\ell_v^n)_{v\in\theta_n})$ is $(\mathcal{T}_{\mathbf{e}}, (\overline{Z}_a)_{a\in\mathcal{T}_{\mathbf{e}}})$, where

- *$\mathcal{T}_{\mathbf{e}}$ is the CRT*
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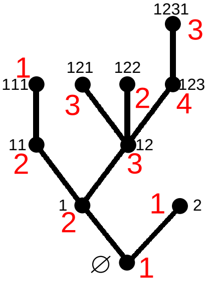

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Application to the radius of a planar map

Recall

- Schaeffer's bijection : quadrangulations $\leftrightarrow$ well-labeled trees
- labels on the tree correspond to distances from the root in the map

Theorem (Chassaing-Schaeffer 2004)

Let R_n be the maximal distance from the root in a quadrangulation with n faces chosen at random. Then,

$$n^{-1/4}R_n \xrightarrow[n\to\infty]{(d)} \left(\frac{8}{9}\right)^{1/4}(\max Z - \min Z)$$

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Extensions to much more general planar maps (including triangulations, etc.) by

- Marckert-Miermont (2006), Miermont, Miermont-Weill (2007), ...

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4. The scaling limit of planar maps

$\mathbb{M}_n^{2p} = \{$rooted $2p-$angulations with n faces$\}$ (bipartite case)
M_n uniform over $\mathbb{M}_n^{2p}$, $V(M_n)$ vertex set of M_n, $d_{\rm gr}$ graph distance

Theorem (The scaling limit of $2p$-angulations)

At least along a sequence $n_k \uparrow \infty$, we have

$$(V(M_n), c_p \frac{1}{n^{1/4}} d_{\rm gr}) \xrightarrow[n\to\infty]{\rm (d)} (\mathbf{m}_\infty, D)$$

in the sense of the Gromov-Hausdorff distance.
Furthermore, $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$ where

- *$\mathcal{T}_{\mathbf{e}}$ is the CRT (re-rooted at vertex ρ_* minimizing Z)*
- *$(Z_a)_{a\in\mathcal{T}_{\mathbf{e}}}$ is Brownian motion indexed by $\mathcal{T}_{\mathbf{e}}$, and $\overline{Z}_a = Z_a - \min Z$*
- *$\approx$ equivalence relation on $\mathcal{T}_{\mathbf{e}}$: $a \approx b \Leftrightarrow \overline{Z}_a = \overline{Z}_b = \min_{c\in[a,b]} \overline{Z}_c$ ($[a,b]$ lexicographical interval between a and b in the tree)*
- *D distance on $\mathbf{m}_\infty$ such that $D(\rho_*, a) = \overline{Z}_a$*
  D induces the quotient topology on $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$

Interpretation of the equivalence relation $\approx$

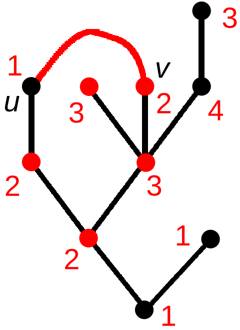
Recall Schaeffer's bijection:
$\exists$ edge between u and v if

- $\ell_u = \ell_v - 1$
- $\ell_w \geq \ell_v \ , \ \forall w \in]u, v]$

Explains why in the continuous limit

$$a \approx b \quad \Rightarrow \quad \overline{Z}_a = \overline{Z}_b = \min_{c \in [a,b]} \overline{Z}_c$$

$$\Rightarrow \quad a \text{ and } b \text{ are identified}$$



Key point: Prove the converse (no other pair of points are identified)

Remark: Equivalence classes for $\approx$ contain 1, 2 or 3 points.

Consequence and open problems

Corollary

The topological type of any Gromov-Hausdorff sequential limit of $(V(M_n), n^{-1/4}d_{\mathrm{gr}})$ is determined:

$$\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx \quad \textit{with the quotient topology.}$$

Open problems

- Identify the distance D on $\mathbf{m}_\infty$
  (would imply that there is no need for taking a subsequence)
- Show that D does not depend on p
  (universality property, expect same limit for triangulations, etc.)

STILL MUCH CAN BE PROVED ABOUT THE LIMIT !

The limiting space $(\mathbf{m}_\infty, D)$ is called the Brownian map [Marckert, Mokkadem 2006, with a different approach]

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Two theorems about the Brownian map

Theorem (Hausdorff dimension)

$$\dim(\mathbf{m}_\infty, D) = 4 \qquad a.s.$$

(Already "known" in the physics literature.)

Theorem (topological type, LG-Paulin 2007)

Almost surely, $(\mathbf{m}_\infty, D)$ is homeomorphic to the 2-sphere $\mathbb{S}^2$.

Consequence: for n large, no separating cycle of size $o(n^{1/4})$ in M_n, such that both sides have diameter $\geq \varepsilon n^{1/4}$

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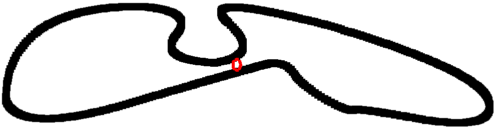

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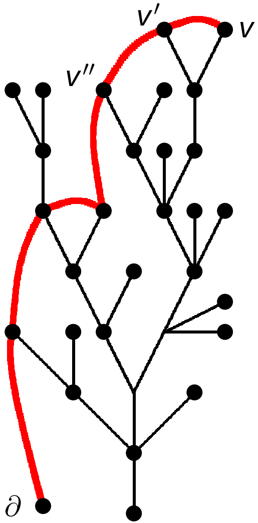
5. Geodesics in the Brownian map

Geodesics in quadrangulations

Use Schaeffer's bijection between quadrangulations and well-labeled trees.

To construct a geodesic from v to ∂:

- Look for the last visited vertex (before v) with label $\ell_v - 1$. Call it v'.
- Proceed in the same way from v' to get a vertex v''.
- And so on.
- Eventually one reaches the root ∂.



Simple geodesics in the Brownian map

Brownian map: $\mathbf{m}_\infty = \mathcal{T}_{\mathbf{e}}/\approx$, root ρ_*
$\prec$ lexicographical order on $\mathcal{T}_{\mathbf{e}}$
Recall $D(\rho_*, a) = \overline{Z}_a$ (labels on $\mathcal{T}_{\mathbf{e}}$)

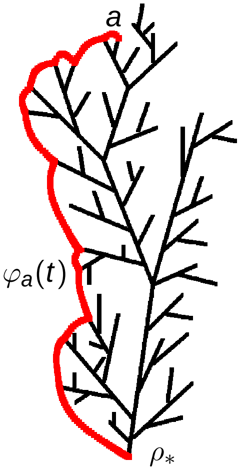
Fix $a \in \mathcal{T}_{\mathbf{e}}$ and for $t \in [0, \overline{Z}_a]$, set

$$\varphi_a(t) = \sup\{b \prec a : \overline{Z}_b = t\}$$

(same formula as in the discrete case !)

Then $(\varphi_a(t))_{0\le t\le \overline{Z}_a}$ is a geodesic from ρ_* to a

(called a simple geodesic)



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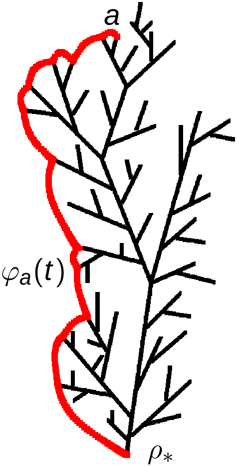

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How many simple geodesics from a given point ?

- If a is a leaf of $\mathcal{T}_{\mathbf{e}}$, there is a unique simple geodesic from ρ_* to a
- Otherwise, there are
  - 2 distinct simple geodesics if a is a simple point
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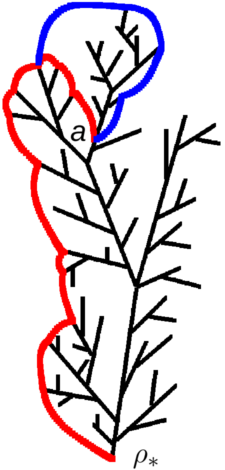
(3 is the maximal multiplicity in $\mathcal{T}_{\mathbf{e}}$)



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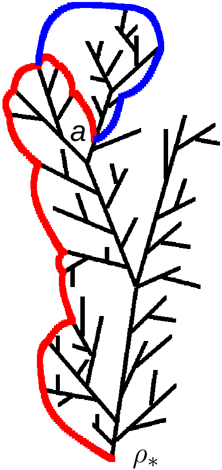

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The main result about geodesics

Define the skeleton of $\mathcal{T}_{\mathbf{e}}$ by $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}}) = \mathcal{T}_{\mathbf{e}} \backslash \{$leaves of $\mathcal{T}_{\mathbf{e}}\}$ and set

$$\mathrm{Skel} = \pi(\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})) \qquad (\pi : \mathcal{T}_{\mathbf{e}} \to \mathcal{T}_{\mathbf{e}}/\approx = \mathbf{m}_\infty \text{ canonical projection})$$

Then

- the restriction of π to $\mathrm{Sk}(\mathcal{T}_{\mathbf{e}})$ is a homeomorphism onto Skel
- $\dim(\mathrm{Skel}) = 2$ (recall $\dim(\mathbf{m}_\infty) = 4$)

Theorem (Geodesics from the root)

Let $x \in \mathbf{m}_\infty$. Then,

- *if $x \notin \mathrm{Skel}$, there is a unique geodesic from ρ_* to x*
- *if $x \in \mathrm{Skel}$, the number of distinct geodesics from ρ_* to x is the multiplicity $m(x)$ of x in Skel (note: $m(x) \leq 3$).*

Remarks

- Skel is the cut-locus of $\mathbf{m}_\infty$ relative to ρ: cf classical Riemannian geometry [Poincaré, Myers, ...], where the cut-locus is a tree.
- same results if ρ_* replaced by a point chosen "at random" in $\mathbf{m}_\infty$.
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Confluence property of geodesics

Fact: Two simple geodesics coincide near the root.
(easy from the definition)

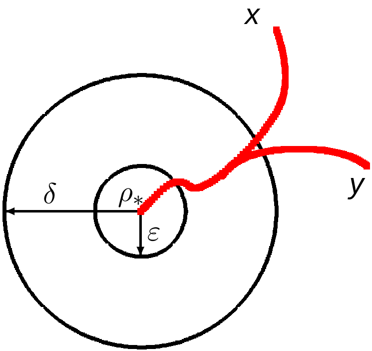
Corollary

Given $\delta > 0$, there exists $\varepsilon > 0$ s.t.

- *if $D(\rho_*, x) \geq \delta$, $D(\rho_*, y) \geq \delta$*
- *if γ is any geodesic from ρ_* to x*
- *if γ' is any geodesic from ρ_* to y*

then

$$\gamma(t) = \gamma'(t) \quad \textit{for all } t \leq \varepsilon$$



"Only one way" of leaving ρ_* along a geodesic.
(also true if ρ_* is replaced by a typical point of $\mathbf{m}_\infty$)

Uniqueness of geodesics in discrete maps

M_n uniform distributed over $\mathbb{M}_n^{2p} = \{2p-\text{angulations with } n \text{ faces}\}$
$V(M_n)$ set of vertices of M_n, ∂ root vertex of M_n, d_{gr} graph distance

For $v \in V(M_n)$, $\mathrm{Geo}(\partial \to v) = \{\text{geodesics from } \partial \text{ to } v\}$

If γ, γ' are two discrete paths (with the same length)

$$d(\gamma,\gamma') = \max_i d_{\mathrm{gr}}(\gamma(i),\gamma'(i))$$

Corollary

Let $\delta > 0$*. Then,*

$$\frac{1}{n}\#\{v \in V(M_n) : \exists \gamma,\gamma' \in \mathrm{Geo}(\partial \to v),\ d(\gamma,\gamma') \geq \delta n^{1/4}\} \underset{n\to\infty}{\longrightarrow} 0$$

Macroscopic uniqueness of geodesics, also true for
"approximate geodesics"= paths with length $d_{\mathrm{gr}}(\partial, v) + o(n^{1/4})$

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Macroscopic uniqueness of geodesics, also true for "approximate geodesics"= paths with length $d_{\rm gr}(\partial, v) + o(n^{1/4})$

Exceptional points in discrete maps

M_n uniformly distributed $2p$-angulation with n faces
For $v \in V(M_n)$, and $\delta > 0$, set

$$\mathrm{Mult}_\delta(v) = \max\{k : \exists \gamma_1, \ldots, \gamma_k \in \mathrm{Geo}(\partial, v),\ d(\gamma_i, \gamma_j) \geq \delta n^{1/4} \text{ if } i \neq j\}$$

(number of "macroscopically different" geodesics from ∂ to v)

Corollary

1. *For every $\delta > 0$,*

$$P[\exists v \in V(M_n) : \mathrm{Mult}_\delta(v) \geq 4] \underset{n\to\infty}{\longrightarrow} 0$$

2. *But*

$$\lim_{\delta\to 0} \Big(\liminf_{n\to\infty} P[\exists v \in V(M_n) : \mathrm{Mult}_\delta(v) = 3] \Big) = 1$$

There can be at most 3 macroscopically different geodesics from ∂ to an arbitrary vertex of M_n.

Remark. ∂ can be replaced by a vertex chosen at random in M_n.

Exceptional points in discrete maps

M_n uniformly distributed $2p$-angulation with n faces
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