Amalgamated Free Product $G = A *_C B$

$\widehat{\mathrm{Aut}}$ = stable automorphisms

Theorem There is an exact sequence

$$\widehat{\mathrm{Aut}}_G(k) \longrightarrow \widehat{\mathrm{Aut}}_A(k) \times \widehat{\mathrm{Aut}}_B(k) \longrightarrow \widehat{\mathrm{Aut}}_C(k) \xrightarrow{\delta} T(G) \xrightarrow{(\mathrm{res}^G_A, \mathrm{res}^G_B)} T(A) \times T(B) \xrightarrow{\mathrm{res}^A_C - \mathrm{res}^B_C} T(C)$$

Naive approach: Given $M \in kA\text{-Mod}$, $N \in kB\text{-Mod}$, such that $M\downarrow_C \simeq N\downarrow_C$ (i.e. stably isomorphic) we can arrange representatives in the stable isomorphism classes such that

$M\downarrow_C \cong N\downarrow_C$ (genuine isomorphism) (ex)

Let $\varphi : M\downarrow_C \longrightarrow N\downarrow_C$ be such an isomorphism.

Define a kG-module $C(M,N;\varphi)$ to be M as a vector space with G action $*$

$$a * m = am$$

$$b * m = \varphi^{-1}(b\,\varphi(m)) \qquad \text{call this } \varphi^* N; \quad \varphi^* N \xrightarrow[\text{iso}]{\varphi} N$$

these agree on C so do define a kG-module.

Note $C(M,N;\varphi)\downarrow_A = M$, $C(M,N;\varphi)\downarrow_B = \varphi^* N \cong N$

More sophisticated approach:

Define $D(M,N;\varphi)$ as follows:

$$\begin{array}{ccc} M{\downarrow_C}{\uparrow^G} & \longrightarrow & M{\uparrow^G} \\ {\scriptstyle \varphi\uparrow^G}\downarrow & & \\ N{\downarrow_C}{\uparrow^G} & \longrightarrow & N{\uparrow^G} \end{array}$$

gives

$$M{\downarrow_C}{\uparrow^G} \longrightarrow M{\uparrow^G} \oplus N{\uparrow^G}$$

$$g\otimes m \longmapsto (g\otimes m,\ g\otimes\varphi(m))$$

Let $D(M,N;\varphi)$ be the cone. This only depends on stable data.

Fact: $A *_C B$ acts on a graph; two orbits of vertices stabilisers A, B and one orbit of edges stabiliser C

Chain complex

$$k{\uparrow_C^G} \longrightarrow k{\uparrow_A^G} \oplus k{\uparrow_B^G} \longrightarrow k \qquad \text{exact}$$

Tensor with $C(M,N;\varphi)$

$$M{\downarrow_C}{\uparrow^G} \longrightarrow M{\uparrow^G} \oplus \varphi^{*}N{\uparrow^G} \longrightarrow C(M,N;\varphi)$$

$$g\otimes m \longmapsto (g\otimes m,\ g\otimes\varphi(m))$$

So $C(M,N;\varphi) \simeq D(M,N;\varphi)$.

To see that δ is a group homomorphism note that by construction

$$C(\tilde{k},\tilde{k};\varphi_1)\otimes C(\tilde{k},\tilde{k};\varphi_2) = C(\tilde{k}\otimes\tilde{k},\tilde{k}\otimes\tilde{k};\varphi_1\otimes\varphi_2)$$

If M and N are endotrivial then so is $C(M,N;\varphi)$.
This is because we only have to check the restrictions to finite subgroups and any finite subgroup is conjugate to a subgroup of A or of B.

This proves exactness at $T(A)\times T(B)$

The map $\widehat{\mathrm{Aut}}_{C}(k) \xrightarrow{\delta} T(G)$ is $\varphi \longmapsto C(k,k;\varphi)$

We need to check that i) this is a group homomorphism

ii) It only depends on the stable class of φ

(later).

If $M \in kG\text{-Mod}$ and $M{\downarrow_A} \xrightarrow[\simeq]{\theta_1} k$, $M{\downarrow_B} \xrightarrow[\simeq]{\theta_2} k$ then
$M{\downarrow_A}{\downarrow_C} = M{\downarrow_B}{\downarrow_C}$ gives a map $\varphi = \theta_2\theta_1^{-1} \in \mathrm{Aut}_C(k)$.
and $M \cong C(k,k;\varphi)$ (check)

This proves exactness at $T(G)$ and $\widehat{\mathrm{Aut}}_C(k)$.

HNN extension $\quad G = \quad G = H *_{(f,A)}$

$A \leq H, \quad f: A \hookrightarrow H.$

$G = \langle H, t \mid tat^{-1} = f(a) \rangle$

Theorem There is an exact sequence

$$\hat{Aut}_G(k) \longrightarrow \hat{Aut}_H(k) \longrightarrow Aut_A(k) \xrightarrow{\delta} T(G) \xrightarrow{res^G_H} T(H) \xrightarrow{res^H_A - f^* res^H_{f(A)}} T(A)$$

Claim: given $M \in kH\text{-Mod}$ and $M\downarrow_A \xrightarrow[\Theta]{\sim} f^* M \downarrow_{f(A)}$ i.e. $\Theta(am) = f(a)\Theta(m)$ we can arrange that Θ is a genuine isomorphism of modules.

Define $E(M;\Theta)$ to be M as a vector space, with G action

$$h * m = hm$$
$$t * m = \Theta(m)$$

Check: $(tat^{-1}) * m = \Theta(a\,\Theta^{-1}(m)) = f(a)\,\Theta\Theta^{-1}(m) = f(a) * m.$

Any finite subgroup of G is conjugate to a subgroup of H and $E(M;\Theta)\downarrow_H \cong M$, so if M is endotrivial so is $E(M;\Theta)$.

Define $\delta(\Theta) = E(k;\Theta)$.

<u>Or</u> Consider

$$\begin{array}{ccc} g\otimes m & M\downarrow_A\uparrow^G & \longrightarrow M\uparrow^G \\ \downarrow & \downarrow & \\ gt^{-1}\otimes\theta(m) & M\downarrow_{f(A)}\uparrow^G & \longrightarrow M\uparrow^G \end{array}$$

gives

$$\begin{aligned} M\downarrow_A\uparrow^G &\longrightarrow M\uparrow^G \\ g\otimes m &\longmapsto g\otimes m - gt^{-1}\otimes\theta(m) \end{aligned}$$

Let $F(M;\theta)$ be the cone.

More generally, if G is the fundamental group of a graph of groups we have

$$\widehat{\mathrm{Aut}}_G(k) \longrightarrow \prod \widehat{\mathrm{Aut}}_{G_v}(k) \longrightarrow \prod \widehat{\mathrm{Aut}}_{G_e}(k) \longrightarrow T(G) \longrightarrow \prod_{\text{vertices}} T(G_v) \longrightarrow \prod_{\text{edges}} T(G_e)$$