Amalgamated Free Product \[ G = A * \mathbf{B} \]
Aut = stable automorphisms

Theorem: There is an exact sequence
\[ \hat{\text{Aut}}_c(k) \to \hat{\text{Aut}}_a(k) \times \hat{\text{Aut}}_b(k) \to \hat{\text{Aut}}_c(k) \to T(G) \xrightarrow{\text{res}_G^{\mathbf{A}} \times \text{res}_G^{\mathbf{B}}} T(A) \times T(B) \xrightarrow{\text{res}_G^c} T(C) \]

Naive approach: Given \( M \in h\mathbf{A}\)-Mod, \( N \in h\mathbf{B}\)-Mod, such that \( M \mid_c \cong N \mid_c \) (i.e., stably isomorphic) we can arrange representatives in the stable isomorphism classes such that \( M \mid_c \cong N \mid_c \) (genuine isomorphism) (ex).

Let \( \varphi: M \mid_c \to N \mid_c \) be such an isomorphism.

Define a \( hG \)-module \( C(M,N;\varphi) \) to be \( M \) as a vector space with \( G \)-action:

\[ a \cdot m = am \]
\[ b \cdot m = \varphi^{-1}(b \varphi(m)) \]

Call this \( \varphi^* N \), \( \varphi^* N \xrightarrow{\text{iso}} N \)

these agree on \( C \) so do define a \( hG \)-module.

Note: \( C(M,N;\varphi) \downarrow_A = M \), \( C(M,N;\varphi) \downarrow_B = \varphi^* N \cong N \)
More sophisticated approach:

Define \( D(M,N; \varphi) \) as follows:

\[
M \leftarrow \uparrow G \\
\varphi \downarrow \\
N \leftarrow \uparrow G
\]

\[
\begin{array}{cccc}
M \leftarrow \uparrow G & \rightarrow & M \uparrow G \\
\varphi \downarrow & & \downarrow \\
N \leftarrow \uparrow G & \rightarrow & N \uparrow G
\end{array}
\]

This only depends on stable data.

Fact: \( A \ast B \) acts on a graph; two orbits of vertices stabilisers \( A, B \) and one orbit of edges stabiliser \( C \)

Chain complex:

\[
\begin{array}{cccc}
M \leftarrow \uparrow G & \rightarrow & M \uparrow G & \rightarrow & C(M,N; \varphi)
\end{array}
\]

Tensor with \( C(M,N; \varphi) \):

\[
\begin{array}{cccc}
M \leftarrow \uparrow G & \rightarrow & M \uparrow G & \rightarrow & \varphi N \uparrow G & \rightarrow & C(M,N; \varphi)
\end{array}
\]

\[
\begin{array}{cccc}
g \downarrow & (g \downarrow, g \varphi(g)) \rightarrow & C(M,N; \varphi)
\end{array}
\]

So \( C(M,N; \varphi) \otimes D(M,N; \varphi) \).

To see that \( \delta \) is a group homomorphism note that by construction

\[
C(C(h, \hat{h}; \varphi_1) \otimes C(h, \hat{h}; \varphi_2)) = C(k \hat{h}, k \hat{h}; \varphi_1 \otimes \varphi_2)
\]
If $M$ and $N$ are endotrivial then so is $C(M,N;\varphi)$. This is because we only have to check the restrictions to finite subgroups and any finite subgroup is conjugate to a subgroup of $A$ or of $B$.

This proves exactness at $T(A)\times T(B)$.

The map $\hat{\text{Aut}}_c(k) \xrightarrow{\delta} T(c)$ is $\varphi \mapsto C(k,k;\varphi)$.

We need to check that

i) This is a group homomorphism

ii) It only depends on the stable class of $\varphi$ (later).

If $M \in k\text{-}\text{Mod}$ and $M \xrightarrow{\Theta_1} k$, $M \xrightarrow{\Theta_2} k$ then $M \otimes B_c \otimes B_c$ gives a map $\varphi = \Theta_2 \Theta_1^{-1} \in \hat{\text{Aut}}_c(k)$.

and $M \cong C(k,k;\varphi)$ (check).

This proves exactness at $T(c)$ and $\hat{\text{Aut}}_c(k)$. 
HNN extension \[ G = G = H \ast_{(f; A)} (A \leq H, \ f: A \rightarrow H). \]

\[ G = \langle H, t \mid t^{-1} = f(a) \rangle \]

**Theorem.** There is an exact sequence

\[ \hat{\text{Aut}}(H) \rightarrow \hat{\text{Aut}}_H(k) \rightarrow \text{Aut}_A(k) \overset{\delta}{\rightarrow} T(G) \overset{\text{res}_H^G}{\rightarrow} T(H) \overset{\text{res}_H^A - \text{res}_H^H}{\rightarrow} T(A) \]

**Claim:** Given \( M \in \text{Mod}_H \) and \( \phi: M \rightarrow f^*M \) \( f: H \rightarrow A \), i.e., \( \phi(a) = f(a) \phi(m) \)
we can arrange that \( \phi \) is a genuine isomorphism of modules.

**Define** \( E(M; \phi) \) to be \( M \) as a vector space, with \( G \) acting

\[ h \cdot m = hm, \]

\[ t \cdot m = \phi(m) \]

**Check:** \( (t a t^{-1}) m = \phi(a \phi^{-1}(m)) = f(a) \phi \phi^{-1}(m) = f(a) m \).

Any finite subgroup of \( G \) is conjugate to a subgroup of \( H \) and \( E(M; \phi) \downarrow_{H} \cong M \), so if \( M \) is endomivial so is \( E(M; \phi) \).

**Define** \( S(\phi) = E(k; \phi) \).
Consider $\text{gem } M \downarrow_{A}^{\uparrow G} \rightarrow M^{\uparrow G}$

\[\text{gem } M \downarrow_{(\text{gem })}^{\uparrow G} \rightarrow M^{\uparrow G}\]

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Let $F(M; \Theta)$ be the cone.
More generally, if $G$ is the fundamental group of a graph of groups we have

$$\hat{\text{Aut}}_G (k) \rightarrow \hat{T} \hat{\text{Aut}}_{G_v} (k) \rightarrow \hat{T} \hat{\text{Aut}}_{G_e} (k) \rightarrow T(G) \rightarrow \hat{\Pi} \Pi T(G_v) \rightarrow \hat{\Pi} \Pi T(G_e)$$