Definition $M \in \mathrm{Stab}(kG)$ is endotrivial if there is a module N such that $M \otimes N \simeq k$ stably.

The stable isomorphism classes form a group $T(G)$.

For any finite subgroup F, $M{\downarrow_F}$ is endotrivial so
$M{\downarrow_F} = M' \oplus (\text{proj})$, $\dim M' < \infty$, M' endotrivial.

$$M \otimes M^* \xrightarrow{ev} k$$

restricts to

$$M{\downarrow_F} \otimes M^*{\downarrow_F} \xrightarrow{ev} k$$

$$\begin{tikzcd}
(M' \oplus (\text{proj})) \otimes (M'^* \oplus (\text{proj})) \arrow[r] & k \\
M' \otimes M'^* \arrow[u, "\simeq"] \arrow[r, "ev", "\simeq"'] & k \arrow[u, equal]
\end{tikzcd}$$

Proposition If M is endotrivial then its inverse is M^*.

M is endotrivial if and only if $M{\downarrow_F}$ is endotrivial for all finite (p)-subgroups F.

Note that $T(G) = 0$ if G has no p-torsion.

<u>Example</u> $G = C_p^1 * C_p^2$

Free product of two groups of order p.

G acts on a tree with stabilisers conjugate to C_p^1 or C_p^2 so it is of type Φ.

Canonical map $k\uparrow_{C_p^1}^G \to k$

$g \otimes x \mapsto gx$

Restrict this to C_p^1 $\quad k \oplus (\text{free}) \xrightarrow{} k$ $\quad$ Mackey formula (split)

C_p^2 $\quad (\text{free}) \to k$ $\quad$ "

Now consider $k\uparrow_{C_p^1}^G \oplus k\uparrow_{C_p^2}^G \longrightarrow k$.

On restriction to C_p^1 or C_p^2 this is a stable iso. Any torsion subgroup of G is conjugate to one of these two. Thus we have a stable isomorphism

$$k \simeq k\uparrow_{C_p^1}^G \oplus k\uparrow_{C_p^2}^G .$$

<u>Note:</u> The RHS is Gorenstein projective, since it is projective over a subgroup of finite index (ex)

Endotrivial modules need not be indecomposable.

Brown/Quillen complex

$\Delta(G)$ is a simplicial complex where the r-simplices are chains $P_0 < P_1 < \cdots < P_r$ of non-trivial:

- finite p-subgroups (Quillen)
- finite elementary abelian p-subgroups (Brown)
  - \+ many variants

G acts by conjugation.
The variants are all equivariantly homotopy equivalent.

Chain complex of kG-modules $C(\Delta(G)) \xrightarrow{\varepsilon} k$.

Can show that for any finite p-subgroup $1 \neq P < G$, $\Delta(G)^P$ is contractible, hence

$C(\Delta(G))\downarrow_P \longrightarrow k$ is chain homotopy equivalent to a complex of projectives

$\Omega^0_P\, C(\Delta(G))\downarrow_P \longrightarrow k$ is a stable iso

$(\Omega^0_G\, C(\Delta(G)))\downarrow_P \longrightarrow k$ stable iso

$\Omega^0_G\, C(\Delta(G)) \longrightarrow k$ stable iso over G.

<u>Theorem</u> If $C(\Delta(G))$ has homology in only finitely many degrees (e.g p-rank$(G) < \infty$) then $\Omega^0(C(\Delta(G))) \simeq k$.
So k decomposes as a sum, $k \simeq \bigoplus k_e$, one for each path component of $\Delta(G)/G$.

(i.e equivalence relation on non-trivial finite p-subgroups generated by $P \sim Q$ if $P \leq Q$ or P conjugate to Q)

stable $\rightsquigarrow$ $\widehat{End}_G(k) = \prod \widehat{End}_G(ke)$ — idempotents e.

In fact $\widehat{End}_G(k)$ is a commutative ring and the e are primitive. (ex).

We really have an end-e group $T_e(G)$, one for each idempotent.

$$T(G) = \prod_e T_e(G).$$