**Definition** \( M \in \text{Stab}(kG) \) is endotrivial if there is a module \( N \) such that \( M \otimes N \cong k \) stably.

The stable isomorphism classes form a group \( T(G) \).

For any finite subgroup \( F \), \( M^{F} \) is endotrivial so

\[
M^{F} = M^{' \otimes (\text{proj})} , \quad \dim M^{'} < \infty , \quad M^{'} \text{ endotrivial}.
\]

\[
M \otimes M^{*} \xrightarrow{\text{ev}} k
\]

resmcts \( 1 \circ \)

\[
M^{F} \otimes M^{F} \xrightarrow{\text{ev}} k
\]

\[
(M^{' \otimes (\text{proj})}) \otimes (M^{' \otimes (\text{proj})}) \twoheadrightarrow k
\]

\[
\cong \quad \uparrow \quad \parallel
\]

\[
M^{'} \otimes M^{'} \xrightarrow{\text{ev}} k
\]

**Proposition** If \( M \) is endotrivial then its inverse is \( M^{k} \).

\( M \) is endotrivial if and only if \( M^{F} \) is endotrivial for all finite \((n)\)-subgroups \( F \).

**Note** that \( T(G) = 0 \) if \( G \) has no \( p \)-torsion.
Example \[ G = C_p^1 \ast C_p^2 \]

Free product of two groups of order \( p \).

\( G \) acts on a tree with stabilizers conjugate to \( C_p^1 \) or \( C_p^2 \) so it is of type \( \Phi \).

Canonical map \[ k^+_{C_p^1} \to k \]

\[ g \cdot x \mapsto gx \]

Restrict this to \( C_p^1 \): \[ k @ (free) \to k \] Mackey formula

\[ C_p^2 \] (free) \[ \to k \] "

Now consider \[ k^+_{C_p^1} \oplus k^+_{C_p^2} \to k \].

On restriction to \( C_p^1 \) or \( C_p^2 \) this is a stable iso.

Any torsion subgroup of \( G \) is conjugate to one of these two. Thus we have a stable isomorphism

\[ k \cong k^+_{C_p^1} \oplus k^+_{C_p^2} \].

Note: The RHS is Gorenstein projective, since it is projective over a subgroup of finite index (ex).

Endotrivial modules need not be indecomposable.
Brown/Quillen complex

\( \Delta(G) \) is a simplicial complex where the \( p \)-simplices are chains \( p_0 < p_1 < \cdots < p_r \) of non-trivial:

- finite \( p \)-subgroups (Quillen)
- finite elementary abelian \( p \)-subgroups (Brown)
- many variants

\( G \) acts by conjugation.
The variants are all equivariantly homotopy equivalent.

Chain complex of \( kG \)-modules \( C(\Delta(G)) \rightarrow k \).

Can show that for any finite \( p \)-subgroup \( H \leq G \),
\( \Delta(G)_p \) is contractible, hence

\[
\begin{align*}
\Omega^c_0 C(\Delta(G))_p & \rightarrow k \\
\Omega^c_0 C(\Delta(G))_p & \rightarrow k \\
\Omega^c_0 C(\Delta(G))_p & \rightarrow k \\
\Omega^c_0 C(\Delta(G))_p & \rightarrow k
\end{align*}
\]

is a chain homotopy equivalence to a complex of projectives
is a stable iso
stable iso
stable iso over \( G \).

**Theorem**

If \( C(\Delta(G)) \) has homology in only finitely many degrees (e.g. \( p-\text{rank}(\cdot) < \infty \)) then \( \Omega^c_0 C(\Delta(G)) \cong k \).

So \( k \) decomposes as a sum, \( k \cong \oplus k e \), one for each path component of \( \Delta(G)/G \).

(i.e. equivalence relation on non-trivial finite \( p \)-subgroups generated by \( P \cap Q \) if \( P \leq G \) or \( P \conjugate Q \to Q \))
In fact $\hat{\text{End}}(k)$ is a commutative ring and the $e$ are primitive. (cc).

We really have an end-e group $T_e(G)$, one for each idempotent.

$$T(G) = \prod_e T_e(G).$$