**Definition.** A complete resolution of a $kG$-module $M$ is a commutative diagram

$$
\begin{array}{cccccccc}
\rightarrow & Q_{n+1} & \rightarrow & Q_n & \rightarrow & Q_{n-1} & \rightarrow & \cdots & \rightarrow Q_1 & \rightarrow Q_0 & \rightarrow Q_{-1} & \rightarrow Q_{-2} & \rightarrow & \\
\rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \rightarrow & \\
\rightarrow & P_n & \rightarrow & P_{n-1} & \rightarrow & \cdots & \rightarrow & P_1 & \rightarrow & P_0 & \rightarrow & M & \\
\end{array}
$$

where the $P_i$ and $Q_i$ are projective, $P_0$ is a projective resolution of $M$ and $Q_0$ is acyclic (i.e. exact). $n$ is called the coincidence index. We also require that $\text{Hom}(Q_0, P)$ be acyclic for any projective $P$.

For $n=0$ this is the same as the definition used for the Tate cohomology of finite groups.

**Theorem.** For groups $G$ of type I, any $kG$-module has a complete resolution, there is one with coincidence index $\leq \text{fndim} G$ and any two are chain homotopy equivalent.
Construction: Given $M$, take an injective resolution

$$M \downarrow$$

$$I_0 \rightarrow I_1 \rightarrow I_2 \rightarrow \cdots$$

Use the Horseshoe lemma,

$n \geq$ find $\lambda C$

Get

$$\Omega^n M \downarrow$$

$$\Omega^n I_0 \rightarrow \Omega^n I_1 \rightarrow \Omega^n I_2 \rightarrow \cdots$$

a projective resolution (in the wrong direction)

Add a normal projective resolution of $\Omega^n M$. 

Splice $\Rightarrow$ Cartan-Eilenberg resolution
For groups of type $k$ it is automatic that $\text{Hom}(k, P)$ is exact: since $P$ has finite injective dimension we can prove this by induction on $\text{rjdim}(P)$ (forgetting that $P$ is projective).

When $\text{rjdim}(P) = 0$, $P$ is injective and $\text{Hom}(k, P)$ is exact by definition of injective.

Otherwise $P \to I \to UP$, true for $UP$ by induction and the long exact cohomology sequence for $\text{Hom}(k, -)$ proves it for $P$.

Note: $Q_i \to Q_{i-1} \to \cdots$ exists because $\text{Hom}(k, P)$ is exact.

This allows us to fill in the extra morphisms in the definition of a complete resolution. It also allows us to show that any two complete resolutions of the same module are chain homotopy equivalent and that $\text{im} d_i$ is determined up to projective summand (ex).
If \[ \cdots \to Q_i \xrightarrow{d_i} Q_{i-1} \to \cdots \]
is a complete resolution of \( M \) we define \( \Omega^i M = \ker d_i \).
It is well defined up to projective summands
\( \Omega^i \Omega^j M = \Omega^{i+j} M \) etc.

The modules that can occur as kernels in an acyclic complex of projectives (with \( \text{Hom}(\cdot, M) \) exact) are called Gerstenstein projectives. They have many nice properties. (ex).

For any \( M \) we have a natural map \( \Omega^0 M \to M \).
It is a stable isomorphism, which we sometimes write \( M \simeq M \) and \( M \) is Gerstenstein projective.

of the role of CW complexes in homotopy theory.
Theorem TFAE

$$(kG\text{-Mod}, \Omega^0 \text{Hom}) \quad \xrightarrow{\text{complete resolution}} \quad (\text{Gorenstein Proj, Hom Med Proj})$$

$$(\Omega^0 \text{Hom}) \quad \xrightarrow{\text{complete resolution}} \quad (\text{acyclic complexes projectives, homotopy})$$

$D_0(kG\text{-Med})/D_0(kG\text{-Proj}) \xrightarrow{\text{complete resolution}} (\text{Hom}(-, R) \text{ exact})$$

These categories are triangulated in the usual way (analogously to the structures on the stable category for a finite group or on $D(kG)$).
In a triangulated category, \( X \xrightarrow{f} Y \xrightarrow{g} Z \) triangle
\[ f \text{ iso } \iff Z = 0 \]

Using this we can rephrase property \( \mathcal{F} \):

**Lemma**: \( f: X \to Y \) is a stable isomorphism if and only if \( \Phi_p : X \Phi_p \to Y \Phi_p \) is a stable isomorphism for every finite \((p)\)-subgroup.