# Endotrivial Modules For Infinite Groups

Why study endotrivial modules for infinite groups.

- You can't say much about all modules. Look for some small subclass where you can do more.
- For finite groups, endotrivial modules or their generalisation, endopermutation modules, occur as sources of simple modules for p-solvable groups and in the description of the source algebra of a nilpotent block. Their classification for finite p-groups was a major achievement.
- Connected to a lot of work from '70s and '80s on cohomology of infinite groups
- It forces us to look carefully at stable categories and suggests how those might be described.

Joint work with Nadia Mazza

## Notation

Always: $k$ is a field of finite characteristic $p$ for this course. But in fact everything works for $k$ ($p$-local), finite global dimension, noetherian.

For now: $G$ is a finite group, $kG$-modules are finite-dimensional unless stated otherwise.

Definition

A $kG$-module $M$ is <u>endotrivial</u> if there is another module $N$ such that $M \otimes_k N \cong k \oplus (\text{proj})$.

Note that $(\text{proj}) \otimes (\text{anything}) = (\text{proj})$, so if we write $M \sim M'$ when $M \oplus (\text{proj}) = M' \oplus (\text{proj})$ then the equivalence classes $[M]$ form an abelian group under $\otimes_k$. Call it $T(G)$. ($N$ is the inverse of $M$).

Each equivalence class contains an indecomposable module $M$ such that every other element is of the form $M \oplus (\text{proj})$ (ex)

<u>Syzygies</u> Given $M$, find a surjection from a projective module $P \twoheadrightarrow M$ and let $\Omega M$ be the kernel. $\Omega M \to P \to M$.

$\Omega M$ is well defined up to projective summand (Schanuel's lemma) so $[\Omega M]$ is well defined. We can also go in the other direction using injective modules. $M \to I \to \mho M$.

For finite groups proj $\Leftrightarrow$ inj so we write $\Omega^{-1}$ instead.

Iterating gives $[\Omega^r M]$ $r \in \mathbb{Z}$; these are the kernels in a projective/injective resolution of $M$.

Check that $\Omega(M \otimes N) \cong (\Omega M) \otimes N$.

Now suppose that $M$ is endotrivial, so $M\otimes N\simeq k$.
Then $\Omega M\otimes\Omega^{-1}N \simeq \Omega(M\otimes\Omega^{-1}N) \cong M\otimes\Omega\Omega^{-1}N \simeq M\otimes N\simeq k$.
So $\Omega M$ is endotrivial.

Clearly $k$ is endotrivial, so $\Omega^r k$ is endotrivial, and $\Omega^r k\otimes\Omega^s k\simeq\Omega^{r+s}k$;
we obtain a homomorphism $\mathbb{Z}\to T(G)$.
$T(G)$ also contains all 1-dimensional representations of $G$.

There are natural restriction maps $T(G)\to T(H)$ for $H\leq G$.

Recall that $\mathrm{Hom}_k(M,N)$ is considered to be a $kG$-module via $(gf)(m) = gf(g^{-1}m)$ $\quad$ ($f\in\mathrm{Hom}_k(M,N)$ etc.).
In particular we have the dual $M^* = \mathrm{Hom}_k(M,k)$.

<u>Lemma</u> If $M\otimes N = k\oplus(\mathrm{proj})$ then $[N]=[M^*]$. The natural evaluation map $ev: M\otimes_k M^*\to k$ is split over $kG$ and the kernel is projective.

<u>Proof</u>

$$\begin{tikzcd}
M\otimes N \arrow[r, "\cong"] \arrow[d, "1\otimes\varphi"'] & k\oplus(\mathrm{proj}) \arrow[r, "\pi"] \arrow[l, dashed, bend right, "\text{split}"'] & k \arrow[d, equal] \\
M\otimes M^* \arrow[rr, "ev"] & & k
\end{tikzcd}$$

commutes

where $\varphi: N\to M^*$ by $(\varphi(m))(n) = \pi(m\otimes n)$ $\quad$ ($\varphi$ will be a stable iso).

Thus $k \mid M\otimes M^*$ splitting $ev$ $\quad$ ($\mid$ is a summand of)

$N \mid N\otimes M\otimes M^* \simeq M^*\oplus(\mathrm{proj})$

But $M = M'\oplus(\mathrm{proj})$, $M'$ indecomposable.
Thus $N = M'^*\oplus(\mathrm{proj})$ $\quad$ ($N$ not proj if $p \mid |G|$).
Now $[N]=[M^*]$ and $M\otimes M^*\simeq k\oplus(\mathrm{proj})$, so the kernel must be projective.

If we relax the condition that $M$ be finite dimensional we do not get any more endotrivial modules.

Lemma If $M, N$ are possibly infinite dimensional $kG$-modules such that $M \otimes N = k \oplus (\text{proj})$ then $M = \bar{M} \oplus (\text{proj})$, $\bar{M}$ finite dimensional and endotrivial.

Proof $M \otimes N = k \oplus (\text{proj})$; write a generator of $k$ as $\sum m_i \otimes n_i$ and let $M' = \langle m_i \rangle_{kG} \leq M$. Then $\dim_k M' < \infty$ and $k \leq M' \otimes N \leq M \otimes N \to k$, so $k \mid M' \otimes N$. Thus $M \mid M' \otimes N \otimes M \cong M' \oplus (\text{proj})$.

Somehow we deduce that $M = (\text{fin dim } \bar{M}) \oplus (\text{proj})$.

e.g. An advanced version of Krull-Schmidt (RHS a sum of countably generated modules with local endomorphism rings)

or see exercises

Similarly for $N$; $k \oplus (\text{proj}) = M \otimes N = \bar{M} \otimes \bar{N} \oplus (\text{proj})$.

## Finite $p$-groups

$$T(C_q) = \begin{cases} 0 & q=2 \\ \mathbb{Z}/2 & q \neq 2 \end{cases}$$

$$T(Q_8) = \begin{cases} \mathbb{Z}/4 & k \text{ contains no cube root of } 1 \\ \mathbb{Z}/4 \oplus \mathbb{Z}/2 & k \text{ contains cube root of } 1 \end{cases}$$

$$i = \begin{pmatrix} 1 & & \\ 1 & 1 & \\ & 1 & 1 \end{pmatrix} \qquad j = \begin{pmatrix} 1 & & \\ \omega & 1 & \\ & \omega^2 & 1 \end{pmatrix}$$

$$\underset{2^n \geq 16}{T(Q_{2^n})} = \mathbb{Z}/4 \oplus \mathbb{Z}/2 \quad \text{regardless}$$

$$\underset{2^n \geq 8}{T(D_{2^n})} = \mathbb{Z} \oplus \mathbb{Z} \qquad K \to k[D_{2^n}/\underset{\text{not central}}{C_2}] \to k$$

<u>Theorem</u> If all maximal elementary abelian $p$-subgroups of have rank $\geq 3$ then $T(\ ) \cong \mathbb{Z}$, generated by $\Omega k$.

<u>Theorem</u> Suppose that $P$ has at least one maximal elementary abelian $p$-subgroup of rank 2 and $P$ is not semidihedral. Then $T(P)$ is free abelian on $r$ generators, where $r$ is defined by $c = \#$ conjugacy classes of maximal elementary abelian $p$-subgroups of rank 2 and $r = c$ if $\mathrm{rank}(P) = 2$, $r = c+1$ if $\mathrm{rank}(P) > 2$.

The other cases are dealt with separately. Extraspecial and almost extraspecial are particularly tricky.
Carlson - Thévenaz, Bouc - Alperin Dade.

For general finite groups there is no classification

$$\mathrm{Pic}(T(G)) = \ker \mathrm{Res}^G_P : T(G) \to T(P), \quad P \text{ Sylow}$$

$\mathrm{Pic}(T(G))$ is finite — see Balmer, Grodal

In all cases $T(G)$ is finitely generated (originally Puig)

# Groups of type $\Phi$

Definition

A group $G$ is of type $\Phi$ (over $k$) if for any $kG$-module $M$, $M\downarrow_F$ is of finite projective dimension (i.e. some projective resolution stops) for all finite subgroups $F$ implies that $M$ is of finite projective dimension.

Note that, since we are taking $k$ to be a field of characteristic $p$, we only need to check for $F$ an elementary abelian $p$-subgroup and $M\downarrow_F$ projective.

The finitistic dimension of $G$ is $\operatorname{findim} G = \sup\{\operatorname{projdim} M \;;\; \operatorname{projdim} M < \infty\}$.

For groups of type $\Phi$, $\operatorname{findim} G < \infty$. Otherwise for each $i \in \mathbb{N}$ let $M_i$ have $i \leq \operatorname{projdim} M < \infty$ and consider $M = \bigoplus M_i$.

Then $M\downarrow_F$ is projective for any finite $F < G$, so $\operatorname{projdim} M < \infty$ ※.

Proposition Let $G$ act admissibly on a contractible CW-complex of finite dimension with finite stabilisers. Then $G$ is of type $\Phi$.

Proof Let $X$ be the complex, chain complex $C(X)$

$$0 \to C_n(X) \to \cdots \to C_2(X) \to C_1(X) \to C_0(X) \to k$$

Each $C_i(X)$ is a sum of permutation modules $k\uparrow_F^G$, $F$ finite.

Tensor with $M$:

$$0 \to C_n(X)\otimes M \to \cdots \to C_1(X)\otimes M \to C_0(X)\otimes M \to M$$

$k\uparrow_F^G \otimes M \cong M\downarrow_F\uparrow^G$, so if each $M\downarrow_E$ is projective, this is a projective resolution of $M$.

$G$ is of finite virtual cohomological dimension (over $k$) if $G$ has a subgroup $H$ of finite index such that $\operatorname{projdim}_{kH} k < \infty$. Such $G$ are of type $\Phi$.

e.g. $Sl_n(\mathbb{Z})$, lattices in connected Lie groups ...

$(\mathbb{Z}/p)^{\mathbb{N}}$, $\mathbb{Z}/p^\infty$ are of type $\Phi$ (they act on a tree) but are not of finite vcd.

$\mathbb{Z}^{\mathbb{N}}$ is not of type $\Phi$.

Define a category $\mathrm{ModProj}(kG)$ to have the same objects as $kG$-Mod, but $\operatorname{Hom}_{\mathrm{ModProj}}(M, N) = \operatorname{Hom}_{kG}(M,N)/(\text{factors through a projective})$.

$$\Omega : \operatorname{Hom}_{\mathrm{ModProj}}(M,N) \to \operatorname{Hom}_{\mathrm{ModProj}}(\Omega M, \Omega N)$$

$$\begin{array}{ccccc} \Omega M & \to & P_M & \to & M \\ \downarrow{\scriptstyle \Omega f} & & \downarrow & & \downarrow{\scriptstyle f} \\ \Omega N & \to & P_N & \to & N \end{array}$$

Define $\operatorname{Hom}_{\mathrm{Stab}}(M,N) = \Omega^\omega \operatorname{Hom}(M,N) = \varinjlim_{\Omega} \operatorname{Hom}_{\mathrm{ModProj}}(\Omega^r M, \Omega^r N)$.

This is difficult to calculate with.

From now on all groups are of type $\Phi$ and there is no restriction on the $kG$-modules we consider.

It follows easily from the definition any injective $kG$-module has projective dimension $\leq \operatorname{findim} G$.

<u>Proposition</u> Any projective module has injective dimension $\leq \operatorname{findim} G$

<u>Proof</u> See exercises.