Endotrivial Modules For Infinite Groups

Why study endotrivial modules for infinite groups:

- You can't say much about all modules. Look for some small subclass where you can do more.
- For finite groups, endotrivial modules or their generalisation, endopermutation modules, occur as sources of simple modules for $p$-solvable groups and in the description of the source algebra of a nilpotent block. Their classification for finite $p$-groups was a major achievement.
- Connected to a lot of work from '70s and '80s on cohomology of infinite groups.
- It forces us to look carefully at stable categories and suggests how those might be described.

Joint work with Nadia Mazza
Notation

Always: \( k \) is a field of finite characteristic \( p \) for this course. But in fact everything works for \( k \) \((p\text{-local})\), finite global dimension, noetherian.

For now: \( G \) is a finite group, \( kG \)-modules are finite-dimensional unless stated otherwise.

Definition

A \( kG \)-module \( M \) is **endotrivial** if there is another module \( N \) such that \( M \otimes k N \cong k \oplus (\text{proj}) \).

Note that \((\text{proj}) \oplus \text{(anything)} \cong (\text{proj})\), so if we write \( M \oplus M' \) when \( M \oplus (\text{proj}) = M' \oplus (\text{proj}) \) then the equivalence classes \([M]\) form an abelian group under \( \otimes_k \). Call it \( T(G) \). (\( N \) is the inverse of \( M \).)

Each equivalence class contains an indecomposable module \( M \) such that every other element is of the form \( M \oplus (\text{proj}) \).

Syzgies

Given \( M \), find a surjection from a projective module \( P \to M \) and let \( \Omega M \) be the kernel. \( \Omega M \to P \to M \).

\( \Omega M \) is well defined up to projective summand (Schanuel's theorem) so \([\Omega M]\) is well defined. We can also go in the other direction using injective modules, \( M \to I \to \Omega M \).

For finite groups \( \text{proj} \Rightarrow \text{inj} \) so we write \( \Omega^i \) instead.

Iterating gives \([\Omega^i M]\) \text{ for } i \in \mathbb{Z} ; \text{ these are the kernels in a projective/injective resolution of } M \).

Check that \( \Omega (M \otimes N) \cong (\Omega M) \otimes N \).
Now suppose that \( M \) is endotrivial, so \( M \otimes N \cong k \).
Then \( \Omega M \otimes \Omega' N \cong \Omega(M \otimes \Omega' N) \cong M \otimes \Omega' N \cong k \).
So \( \Omega M \) is endotrivial.

Clearly \( k \) is endotrivial, so \( \Omega' k \) is endotrivial, and \( \Omega k \otimes \Omega' k \cong k \); we obtain a homomorphism \( \mathbb{Z} \to \tau(G) \).
\( \tau(G) \) also contains all 1-dimensional representations of \( G \).

There are natural restrictin maps \( \tau(G) \to \tau(H) \) for \( H \leq G \).

Recall that \( \text{Hom}_k(M, N) \) is considered to be a \( kG \)-module via \( (gf)(m) = gf(g'm) \) \((f \in \text{Hom}_k(M, N) \) etc.).

In particular we have the dual \( M^* = \text{Hom}_k(M, k) \).

**Lemma.** If \( M \otimes N = k \otimes \text{proj} \) then \([N] = [M^*]\). The natural evaluation map \( \text{ev}: M \otimes M^* \to k \) is split over \( kG \) and the kernel is projective.

**Proof**

\[
\begin{array}{ccc}
M \otimes N & \cong & k \otimes \text{proj} \\
\downarrow \phi & & \cong \downarrow \phi \\
M \otimes M^* & \xrightarrow{\text{ev}} & k
\end{array}
\]

where \( \phi: N \to M^* \) by \( \phi(m)(n) = \pi(m \otimes n) \) \((\phi \) will be a stable iso).

Thus \( k \mid M \otimes M^* \) splitting \( \text{ev} \) is a summand of \( N \mid N \otimes M \otimes M^* \cong M^* \otimes \text{proj} \).

But \( M = M^* \otimes \text{proj} \), \( M^* \) indecomposable.

Thus \( N = M^* \otimes \text{proj} \) \((N \not \text{proj if } p \mid \text{odd})\).

Now \([N] = [M^*]\) and \( M \otimes M^* \cong k \otimes \text{proj} \), so the kernel must be projective.
If we relax the condition that $M$ be finite dimensional we do not get any more endotrivial modules.

**Lemma.** If $M, N$ are possibly infinite dimensional $kG$ modules such that $M \otimes N = k \oplus (\text{proj})$ then $M = M' \oplus (\text{proj})$, $M'$ finite dimensional and endotrivial.

**Proof:** $M \otimes N = k \oplus (\text{proj})$; write a generator of $k$ as $\sum m_i \otimes n_i$ and let $M' = \langle m_i \rangle_{kG} \leq M$. Then $\dim_k M' < \infty$ and $k \leq M' \otimes N \leq M \otimes N \to k$, so $k \mid M' \otimes N$. Thus $M' \otimes N \otimes M \cong M' \oplus (\text{proj})$.

Somehow we deduce that $M = (\text{End}_{kG} M') \oplus (\text{proj})$.

*Note:* An advanced version of Krull-Schmidt forms a sum of countably generated modules with local endomorphism rings.

See exercises.

Similarly for $N$; $k \oplus (\text{proj}) = M \otimes N = M \otimes N \oplus (\text{proj})$. 
\( T(C_2) = \begin{cases} 0 & q = 2 \\
\mathbb{Z}/2 & q \neq 2 \end{cases} \)

\( T(Q_8) = \begin{cases} \mathbb{Z}/4 & k \text{ contains no cube root of } 1 \\
\mathbb{Z}/4 \oplus \mathbb{Z}/2 & k \text{ contains a cube root of } 1 \end{cases} \)

\( T(Q_{2^n}) = \begin{cases} \mathbb{Z}/4 \oplus \mathbb{Z}/2 & 2^n > 16 \\
\mathbb{Z} & 2^n \geq 8 \end{cases} \)

\( T(D_{2^n}) = \mathbb{Z} \oplus \mathbb{Z} \)

Theorem. If all maximal elementary abelian \( p \)-subgroups of \( G \) have rank \( \geq 3 \) then \( T(G) \cong \mathbb{Z} \), generated by \( Z_k \).

Theorem. Suppose that \( P \) has at least one maximal elementary abelian \( p \)-subgroup of rank 2 and \( P \) is not semidihedral. Then \( T(P) \) is free abelian on \( r \) generators, where \( r \) is defined by \( c = \# \text{ conjugacy classes of maximal elementary abelian } p \)-subgroups of rank 2 and \( r = c + 1 \) if \( \text{rank}(P) = 2 \), \( r = c + 1 \) if \( \text{rank}(P) > 2 \).

The other cases are dealt with separately. Extravasional and almost extravasional are particularly tricky.

Carlson - Thevenaz, Bour - Alpern Dade.

For general finite groups there is no classification:

\[ \text{Pic}(T(G)) = \ker \text{Res}_P^G : T(G) \to T(P), \quad P \text{ Sylow} \]

\( \text{Pic}(T(G)) \) is finite - see Balmer, Grodal

In all cases \( T(G) \) is finitely generated (originally Puig)
Groups of type $\Phi$

Definition

A group $G$ is of type $\Phi$ (over $k$) if for any $kG$-module $M$, $M \otimes_k \mathbb{F}$ is of finite projective dimension (i.e. some projective resolution steps) for all finite subgroups $\mathbb{F}$ implies that $M$ is of finite projective dimension.

Note that, since we are taking $k$ to be a field of characteristic $p$, we only need to check for non-elementary abelian $p$-subgroup and $M \otimes_k \mathbb{F}$ projective.

The finitistic dimension of $G$ is $\text{fdim } G = \sup \{ \text{projdim } M : \text{projdim } M < \infty \}$.

For groups of type $\Phi$, $\text{fdim } G < \infty$. Otherwise for each $i \in \mathbb{N}$ let $M_i$ have $i < \text{projdim } M < \infty$ and consider $M = \bigoplus M_i$.

Then $M \otimes_k \mathbb{F}$ is projective for any finite $F \leq G$, so $\text{projdim } M < \infty$.

Proposition

Let $G$ act admissibly on a contractible CW-complex of finite dimension with finite stabilizers. Then $G$ is of type $\Phi$.

Proof

Let $X$ be the complex, chain complex $C_*(X)$

$$0 \to C_0(X) \to \cdots \to C_3(X) \to C_2(X) \to C_1(X) \to C_0(X)$$

Each $C_i(X)$ is a sum of permutation modules $k \mathbb{F}_i$, $F$-finite.

Tensor with $M$:

$$G \to C_n(X) \otimes_k M \to C_{n-1}(X) \otimes_k M \to C_{n-2}(X) \otimes_k M$$

$k \mathbb{F} \otimes_k M \cong M \otimes_k \mathbb{F}$, so if each $M \otimes_k \mathbb{F}$ is projective, this is a projective resolution of $M$. 

$G$ is of finite virtual cohomological dimension (over $k$) if $G$ has a subgroup $H$ of finite index such that $\text{projdim}_{kH}X < \infty$. Such $G$ are of type $\Phi$.

E.g. $SL_n(\mathbb{Z})$, lattices in connected Lie groups $(\mathbb{Z}/p)^n$, $\mathbb{Z}/p^n$ are of type $\Phi$ (they act on a tree) but are not of finite vcd.

$\mathbb{Z}^n$ is not of type $\Phi$.

Define a category $\text{Mod}_{kG}(\mathcal{M})$ to have the same objects as $kG$-Mod, but $\text{Hom}_{\text{Mod}_{kG}(\mathcal{M})}(M, N) = \text{Hom}_{kG}(\mathcal{M}, N)/(\text{factors through a projective})$.

$\Omega : \text{Hom}_{\text{Mod}_{kG}(\mathcal{M})}(M, N) \to \text{Hom}_{\text{Mod}_{kG}(\mathcal{M})}(\Omega M, \Omega N)$

Define $\text{Hom}_{\text{Stab}(\mathcal{M})}(M, N) = \Omega^w \text{Hom}(M, N) = \lim_{\Omega} \text{Hom}_{\text{Mod}_{kG}(\mathcal{M})}(\Omega^w M, \Omega^w N)$.

This is difficult to calculate with.

From now on all groups are of type $\Phi$ and there is no restriction on the $kG$-modules we consider.

It follows easily from the definition any injective $kG$-module has projective dimension $\leq \text{findim} G$.

Proposition Any projective module has injective dimension $\leq \text{findim} G$.

Proof See exercises.