Exercises 4

1) Let $e \in \widehat{\mathrm{End}}_{kG}(k)$ be an idempotent.
Show that if $P < G$ is a finite p-subgroup then $\mathrm{res}^G_P(e) = 0$ or 1 and that if P' is in the same component of $\Delta(C)/G$ as P (i.e. you can get from one to the other by a chain of inclusions and conjugations of non-trivial p-subgroups) then $\mathrm{res}^G_P(e) = \mathrm{res}^G_{P'}(e)$.
Deduce that the idempotents corresponding to the components of $\Delta(C)/G$ are primitive.

2) Calculate $\widehat{\mathrm{End}}_{C_p \times \mathbb{Z}}(k)$ and $\widehat{\mathrm{Aut}}_{C_p \times \mathbb{Z}}(k)$. (Easy if you know the right spectral sequences, needs some work if not.)

3) Calculate $T(SL_2(\mathbb{Z}))$ at different primes $(SL_2(\mathbb{Z}) \cong C_6 *_{C_2} C_4)$.

4) Calculate $T(C_{p^2} *_{C_p} C_{p^2})$.

5) There is an obvious surjection $C_4 *_{C_2} C_4 \to Q_8$. Calculate the inflation map $T(Q_8) \to T(C_4 *_{C_2} C_4)$.

6) Calculate $T(C_p * \mathbb{Z})$ and $T(C_p \times \mathbb{Z})$ (it is an HNN extension).
What happened to the 1-dim representations of $\mathbb{Z}$?

7) Calculate $T(C_p \times \mathbb{Z} \times \mathbb{Z})$. Is it finitely generated?

8) Calculate $T(\mathbb{Z}/p^\infty)$. ($\mathbb{Z}/p^\infty$ means the p-torsion in $\mathbb{Q}/\mathbb{Z}$; it acts on a graph with finite stabilisers).

9) For $G = A *_C B$, M a kA-module, N a kB-module such that $M\downarrow_C \simeq N\downarrow_C$ stably, show that $M \simeq M'$, $N \simeq N'$ such that $M'\downarrow_C \cong N'\downarrow_C$, a genuine isomorphism.
Hint: Make M, N Gorenstein projective. Let F be a very big free module for G and set $M' = M \oplus F\downarrow_A$, $N' = N \oplus F\downarrow_B$. Use the Eilenberg trick.

10) Show that any stable automorphism $\phi : M \to M$ can be realised as a genuine automorphism $\phi' : M' \to M'$.
Hint: take M Gorenstein projective. Find a projective

Exercises continued

Module P big enough to allow maps $P \xrightarrow{e} M$ and $M \xrightarrow{f} P$.
Consider $M \oplus P^{\mathbb{N}}$ and matrix

$$\varphi' = \begin{bmatrix} \varphi & e & & & \\ f & 0 & 1 & & \\ & 1 & 0 & 1 & \\ & & 1 & 0 & \ddots \\ & & & & \ddots \end{bmatrix}.$$

(This is not strong enough for the construction $E(M;\Theta)$ in general, but it suffices for Q6,7 above. You could try and formulate and prove the more general version).

11) Check that $\varphi \mapsto C(k,k;\varphi)$ is a group homomorphism (or use D).
Hint: use Exercises3, Q6.

12) Show that if M is endotrivial then the natural map $M \otimes_k M^* \longrightarrow \mathrm{End}_k(M)$ is a stable isomorphism, hence $\mathrm{End}_k(M) \simeq k$ stably.