Exercises 3

Groups are assumed to be of type Φ

1) Show that if Q_* is a complete resolution of k then $Q_* \otimes_k M$ is a complete resolution of M.

2) a) Verify that two complete resolutions of the same module must be chain homotopy equivalent

   b) Verify that two acyclic complexes of projectives that are chain homotopy equivalent have the same kernels, up to projective summands.

3) Suppose G has a subgroup H of finite index and p-torsion free. Show that for a module M TFAE

   i) M is Gorenstein projective

   ii) $M\downarrow_H$ is projective

   iii) $M\downarrow_K$ is projective for any $K \leq G$ finite index and p-torsion free.

   Hint: Because induction is also right adjoint to restriction for subgroups of <u>finite index</u>, we obtain a map $M \to M\downarrow\uparrow_H^G$, let M' be the cokernel. Check that $M\downarrow\uparrow_H^G$ is projective and the map splits over H, so $M'\downarrow_H$ is projective. Continue.

4) If G has a normal subgroup N that is p-torsion free and both G and G/N are of type Φ show that inflation gives a well-defined functor on stable categories $\mathrm{Stab}(kG/N) \to \mathrm{stab}(kG)$.

   Convince yourself that this will not work if N contains p-torsion, even for finite G.

7) a) Show that (Gorenstein projective) $\otimes$ (anything) = (Gorenstein projective)

   b) Show that Gorenstein projective and finite projective dimension $\Rightarrow$ projective

Ex 3 contd

5) Given G of type Φ and findim $G \le d$, show TFAE for a kG-module M:

a) M is Gorenstein projective (ie a kernel in a complete resolution)

b) There is an exact sequence $0 \to M \to P_m \to \cdots \to P_0 \to X \to 0$ for some $m \geq d$, P_i projective

c) $M \oplus (\text{proj})$ is of the form in (b)

d) $\forall\, m, n \geq 0$, $\forall N$, $\mathrm{Hom}_{\mathrm{Mod}\,\mathrm{proj}}(\Omega^n M, \Omega^n N) \to \mathrm{Hom}_{\mathrm{Mod}\,\mathrm{proj}}(\Omega^{n+m} M, \Omega^{n+m} N)$ is iso

e) $\Omega^\infty : \mathrm{Hom}_{\mathrm{Mod}\,\mathrm{proj}}(M, N) \longrightarrow \mathrm{Hom}_{\mathrm{Stab}}(M, N)$ is iso

f) $\mathrm{Ext}^i_{kG}(M, P) = 0 \quad \forall i \geq 1 \quad \forall P$ proj

g) $M \otimes_k N$ is projective whenever $\mathrm{projdim}\, N < \infty$.

6) We have $\varepsilon : \tilde{k} \to k$ stable iso, $\tilde{k}$ Gorenstein projective

$$\begin{array}{ccc} \tilde{k} \otimes_k \tilde{k} & \overset{m}{\dashrightarrow} & \tilde{k} \\ {\scriptstyle \varepsilon \otimes \varepsilon} \downarrow & & \downarrow {\scriptstyle \varepsilon} \\ k \otimes_k k & = & k \end{array}$$

Since $\tilde{k} \otimes_k \tilde{k}$ is Gorenstein projective there is a map m across the top that makes the diagram commute.

Since the maps are stable isos, m is unique mod proj.

Two possibilities for m are

$\varepsilon \otimes \mathrm{id} : \tilde{k} \otimes \tilde{k} \to k \otimes \tilde{k} \triangleq \tilde{k}$ and $\mathrm{id} \otimes \varepsilon$

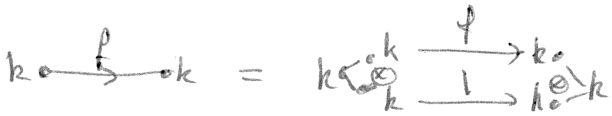
Let $\Delta : \tilde{k} \to \tilde{k} \otimes \tilde{k}$ be a map that is a stable inverse to m,

so $m \circ \Delta = \mathrm{id}$ $\quad \Delta \circ m = \mathrm{id}$ mod proj

Therefore, for any $f : \tilde{k} \to \tilde{k}$ we have

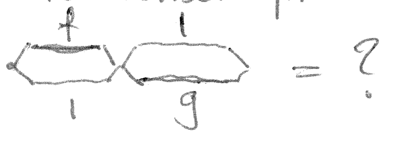
$$\begin{aligned} m \circ (f \otimes 1) \circ \Delta &= (1 \otimes \varepsilon) \circ (f \otimes 1) \circ \Delta \\ &= (f \otimes \varepsilon) \circ \Delta = f \circ (1 \otimes \varepsilon) \circ \Delta = f \circ m \circ \Delta = f. \end{aligned}$$

and similarly for $1 \otimes f$.

In diagrammatic form 

Use this formally to show that 

Deduce that $\mathrm{End}_{\mathrm{Stab}(kG)}(k)$ is commutative and composition corresponds to tensor product.

Hint: 

7) Let $Q_\cdot$ be a complete resolution of M. Define

$$\widehat{\mathrm{Ext}}^i_{kG}(M,N) = H^i(\mathrm{Hom}_{kG}(Q_\cdot, N)).$$

Check the following:

a) $\widehat{\mathrm{Ext}}^i_{kG}(M,N)$ does not depend on the complete resolution chosen.

b) $\widehat{\mathrm{Ext}}^i_{kG}(M,N) \cong \mathrm{Ext}^i_{kG}(M,N)$ for $i > \mathrm{fin\,dim}\, G$

c) $\widehat{\mathrm{Ext}}^0(M,N) \cong \mathrm{Hom}_{\mathrm{Modproj}}(\tilde{M},N) \cong \mathrm{Hom}_{\mathrm{Stab}}(M,N)$.