Exercises 1 — Finite Groups

Here G is a finite group and modules are finite dimensional unless stated.

1) Show that the natural map $M \otimes_k M^* \longrightarrow \mathrm{End}_k(M)$, $n \otimes f \longmapsto (m \mapsto f(m)n)$ is equivariant and an isomorphism when M is finite dimensional.

2) If M is endotrivial and $M = A \oplus B$ show that A is projective or B is projective. Deduce that M is of the form (indecomposable) $\oplus$ (projective).

3) Verify that (anything) $\otimes$ (projective) = (projective).

4) Verify that if M is projective then so is M^*, even when $\dim M$ is infinite.

5) If M is endotrivial, show that $X \mapsto X \otimes_k M$ defines an autoequivalence of the category kG-Mod.

6) Let $G = H \times F$, where F is a p'-group. Show that $T(G) = T(H) \times \mathrm{Hom}_{Grp}(F, k^\times)$.
Hint: if $[M] \in T(G)$ and $[\mathrm{Res}_H M] = [k]$ in $T(H)$ consider $\hat{H}^0(H, M)$ as a kG-module.

7) Here M may be infinite dimensional.
Define $M_{np} = \mho \Omega M$, where Ω and $\mho$ are calculated wrt to the projective cover or injective hull respectively. Show that M_{np} has no projective summands. Show that there is a natural map $M_{np} \to M$, it is injective, and the cokernel is projective. Thus $M \cong M_{np} \oplus (\mathrm{proj})$. Also $(M \oplus N)_{np} \cong M_{np} \oplus N_{np}$. Deduce that if M is stably a summand of a finite dimensional module it is (finite dimensional) $\oplus$ (proj).