# Asymmetrical Matching and Equilibrium in Hedonic Markets

## A Labor Market Example

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# Outline

# Review of the Literature

- Rosen 1974 and Mussa and Rosen (1978): Hedonic prices in pure competition (one dimensional case)
- Rochet and Choné (1998): Consider the case of monopoly pricing
- Carlier (2002): Solve a class of transportation problems using the h-Fenchel transform
- Ekeland (2003): Optimal matching theory
- Ekeland (2005): Hedonic markets in pure competition and multidimensional qualities
- Chiappori, McCann and Nesheim (2007): Hedonic price equilibria, stable matching and optimal transport equivalence

# Overview of the Problem

- We consider the equilibrium problem in a labor market with heterogeneous workers and employers
- The workers have to decide which job they want to execute and the quantity of goods they want to consume
- The employers have to decide how many workers to hire
- There is a consumption good and a quality good (the job)
- This problem is NOT a one-to-one matching but a $n$ to one matching

# Overview of the Problem

- Workers' problem:

$$\max_{z,\xi}\{(p(z)-\pi\xi)-u(x,z,\xi)\} \tag{1}$$

where:

- $x$ is the type of the worker
- $p(z)$ is the salary the worker receives
- $\pi$ is the unit price of the good

# Overview of the Problem

- Employers' problem:

$$\max_{z,n}\{v(y,z,n) - np(z)\} \tag{2}$$

where:

- $y$ is the type of the employer
- $n$ is the number of job $z$ that the employer will hire

# Overview of the Problem

- Mathematically speaking, we will eventually need to solve the following problem:

$$\inf_{p \in \mathcal{A}} \left\{ \int_Y \frac{[p^r(y)]^2}{2c(y)} d\nu(y) - \int_X p^w(x) d\mu(x) \right\} \tag{3}$$

- where:

$$p^w(x) := \max_{z,\xi} \{p(z) - w(x, z, \xi)\} \tag{4}$$

$$p^r(y) := \max_{z} \{r(y, z) - p(z)\} \tag{5}$$

# Main Results

Here are some of the main results we have obtained

1. The equilibrium exists
2. The equilibrium salary is not unique
3. The equilibrium is Pareto optimal

# Outline

1. Introduction
2. Model
   - Assumptions
   - General Setup
   - Definition of Equilibrium
3. Results and Sketch of Proofs
   - Existence
   - Non-Uniqueness
   - Social Planner
4. Conclusion and Further Research

# Assumptions

- Let $X \subset \mathbb{R}^{d_1}$, $Y \subset \mathbb{R}^{d_2}$, $Z \subset \mathbb{R}^{d_3}$, $N \subset \mathbb{R}^+$ and $W \subset \mathbb{R}^+$ be compact subsets.
    - $X$ set of worker's type,
    - $Y$ set of employer's type
    - $Z$ set of jobs
    - $N$ is the possible number of workers an employer can hire
    - $W$ is the possible quantity of good a worker can consume
- We are given non negative measures $\mu$ on $X$ and $\nu$ on $Y$ and typically $\mu(X) \neq \nu(Y)$.
- We assume that there is only one quantity good $\zeta$ which comes in infinite supply and sell at a price $\pi$ per unit. The workers choose the quantity $\xi \in W$ of good $\zeta$ she wants to consume.

# Outline

# Multidimensional Job

## Definition

A *job* $z$ is indivisible and units differ by their characteristics $(z_1, z_2, ..., z_{d_3}) \in Z$. The bundle $z$ is called a *multidimensional job* or a job.

# Optimization Problems

- For the workers of type $x$ the problem is given by:

$$\max_{z,\xi}\{(p(z) - \pi\xi) - u(x, z, \xi)\} \tag{6}$$

- For notational convenience we introduce the following function:

$$w(x, z, \xi) := u(x, z, \xi) + \pi\xi \tag{7}$$

and consider the worker's problem to be:

$$\min_{z,\xi}\{w(x, z, \xi) - p(z)\} \tag{8}$$

# Optimization Problems

- For the employer of type $y$ the problem is given by:

$$\max_{z,n}\{v(y,z,n)-np(z)\} \tag{9}$$

- We will consider a particular form for the function $v(y,z,n)$:

$$v(y,z,n)=nr(y,z)-\frac{n^2}{2}c(y) \tag{10}$$

# Optimization Problems

- For this particular form of the function $v(y, z, n)$ the optimization problem becomes:

$$\max_{z,n}\{nr(y,z) - \frac{n^2}{2}c(y) - np(z)\} \tag{11}$$

with the optimal $n$ given as a function of $y$:

$$\frac{p^r(y)}{c(y)} \tag{12}$$

and the value is given by:

$$\frac{[p^r(y)]^2}{2c(y)} \tag{13}$$

# Optimization Problems: Summary

- Workers:

$$\min_{z,\xi}\{w(x,z,\xi) - p(z)\} \tag{14}$$

- Employers:

$$\frac{[p^r(y)]^2}{2c(y)} \tag{15}$$

# Bid and Ask Salaries

## Definition

The *lowest ask salary* $a : Z \longrightarrow \mathbb{R}$ is given by:

$$a(z) = \min_{x,\xi}\{w(x,z,\xi)\} \tag{16}$$

and the *highest bid salary* $b : Z \longrightarrow \mathbb{R}$ is given by:

$$b(z) = \max_{y}\{r(y,z)\} \tag{17}$$

## Theorem

*If $a(z) > b(z)$ everywhere, then there is no possible equilibrium in the hedonic labor market.*

# Supply and Demand

## Definition

Given a salary system p, we define:

$$S(x) = \arg\min\{w(x,z,\xi) - p(z) \mid z \in Z, \xi \in W\} \tag{18}$$

$$D(y) = \arg\max\{r(y,z) - p(z) \mid z \in Z\} \tag{19}$$

We shall refer to $S(x)$ as the *supply* of type $x$ workers, and to $D(y)$ as the *demand* of type $y$ employers.

- Remark: We need to distinguish between the demand for jobs and the quantity of a particular job demanded.

# Supply and Demand

## Definition

A *supply distribution* associated with p is a positive measure $\alpha_{X \times Z \times W}$ on $X \times Z \times W$ such that:

- $\alpha_{X \times Z \times W}$ is carried by the graph of $S(x)$
- its marginal $\alpha_X = \mu$

Similarly, a *demand distribution* associated with $p$ is a positive measure $\beta_{Y \times Z}$ on $Y \times Z$ such that:

- $\beta_{Y \times Z}$ is carried by the graph of D(y)
- its marginal $\beta_Y$ is equal to $\nu$

# Admissible Salary System

## Definition

A job $z \in Z$ will be called *marketable* is $a(z) \leq b(z)$. The set of marketable qualities will be denoted by $Z_1$.

## Definition

A salary system $p : Z \longrightarrow \mathbb{R}$ will be called *admissible* if:

$$\forall z \in Z_1, \qquad a(z) \leq p(z) \leq b(z) \tag{20}$$

# Outline

# Equilibrium

## Definition

An *equilibrium* is a triplet $(p, \alpha_{X\times Z\times W}, \beta_{Y\times Z})$ where p is an admissible salary system and $\alpha_{X\times Z\times W}$ and $\beta_{Y\times Z}$ are supply and demand distributions associated with p, such that:

$$\forall \varphi \in \mathcal{K}(Z), \qquad \int_{X\times W} \varphi(z) d\alpha(x, z, \xi) = \int_Y n(y)\varphi(z) d\beta(y, z)$$

where $\mathcal{K}(Z)$ is the space of continuous functions with compact support on $Z$.

From now on, we will denote the market clearing condition in the following way:

$$\alpha_Z[A] = \eta_Z[A] \qquad \forall A \in Z$$

Let us write down explicitly all the conditions on $(p, \alpha, \beta)$ implied by the definition of equilibrium.

- $p : Z \longrightarrow \mathbb{R}$ is continuous, and $p(z) \in [a(z), b(z)]$ whenever $a(z) \leq b(z)$
- the marginal $\alpha_X$ is equal to $\mu$
- the conditional probability $P_x^\alpha$ is carried by $S(x)$
- the marginal $\beta_Y$ is equal to $\nu$
- the conditional probability $P_y^\beta$ is carried by $D(y)$
- the marginals $\alpha_Z$ and $\eta_Z$ coincide on $Z$:

$$\alpha_Z[A] = \eta_Z[A] \quad \forall A \subset Z$$

# Outline

# Existence Theorem

**Theorem (Existence)**

*Under the standing assumptions, there is an equilibrium.*

# Sketch of the Proof of Existence Theorem

1. Start by proving that the following problem has a solution:

$$(P) : \inf_{p \in \mathcal{A}} I(p) = \inf_{p \in \mathcal{A}} \left\{ \int_Y \frac{[p^r(y)]^2}{2c(y)} d\nu(y) - \int_X p^w(x) d\mu(x) \right\}$$

   This is accomplish using the same techniques as in Ekeland 2005,.i.e., take a minimizing sequence and work a little bit.

2. Prove that the map $I(p)$ is convex and the set $\mathcal{A}$ is non-empty, closed and convex.
3. Compute the subdifferential of the map $I(p)$, compute the normal cone of $\mathcal{A}$ and show that the optimality condition $0 \in \partial I(p) + N_{\mathcal{A}}(p)$ give a characterization of the equilibrium.

# Outline

# Non-Uniqueness

## Proposition (Non uniqueness of equilibrium salary)

*Let p be a solution of problem (P). Then, $p_b^{ww}(z)$ and $p_a^{rr}(z)$ are also solutions. More generally, if q is an admissible price schedule such that:*

$$p^{rr}(z) \leq q(z) \leq p^{ww}(z) \qquad \forall z \in Z \tag{22}$$

*then q is a solution of problem (P).*

"Proof": Use the properties of $r-$convex and $w-$concave functions and their conjugate.

# Outline

# Social Planner's Problem

With every pair of demand and supply distribution we associate the following number which correspond to the social planner's problem:

$$J(\alpha_{X\times Z\times W},\beta_{Y\times Z})$$
$$\triangleq \int_{Y\times Z} n(y)r(y,z)-\frac{n(y)^2}{2}c(y)d\beta-\int_{X\times Z\times W} w(x,z,\xi)d\alpha$$

# Pareto Optimality of Equilibrium Allocations

**Theorem (Pareto optimality of equilibrium allocations)**

*Let $(p, \alpha_{X\times Z\times W}, \beta_{Y\times Z})$ be an equilibrium. Take any pair of supply and demand distributions $\alpha'_{X\times Z\times W}$ and $\beta'_{Y\times Z}$ such that $\alpha'_Z = \eta'_Z$. Then*

$$J(\alpha'_{X\times Z\times W}, \beta'_{Y\times Z}) \leq J(\alpha_{X\times Z\times W}, \beta_{Y\times Z})$$
$$= \int_Y \frac{[p^r(y)]^2}{2c(y)} d\nu - \int_X p^w(x) d\mu$$

"Proof": Use conditional expectation and the fact that the utility functions are separable with respect to the salary $p(z)$.

# Further Research

- Change the assumption about the function $v(y, z, n)$
- Introduce a budget constraint in our problem
- Allow the employers to stay out of the labor market
- Find a good numerical example
- Calibration with real data
- Developing a numerical method to solve multidimensional cases
- Consider the case where the employers can choose different qualities