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Traffic Equilibrium & Learning

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OUTLINE

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II. Overview of transport equilibrium

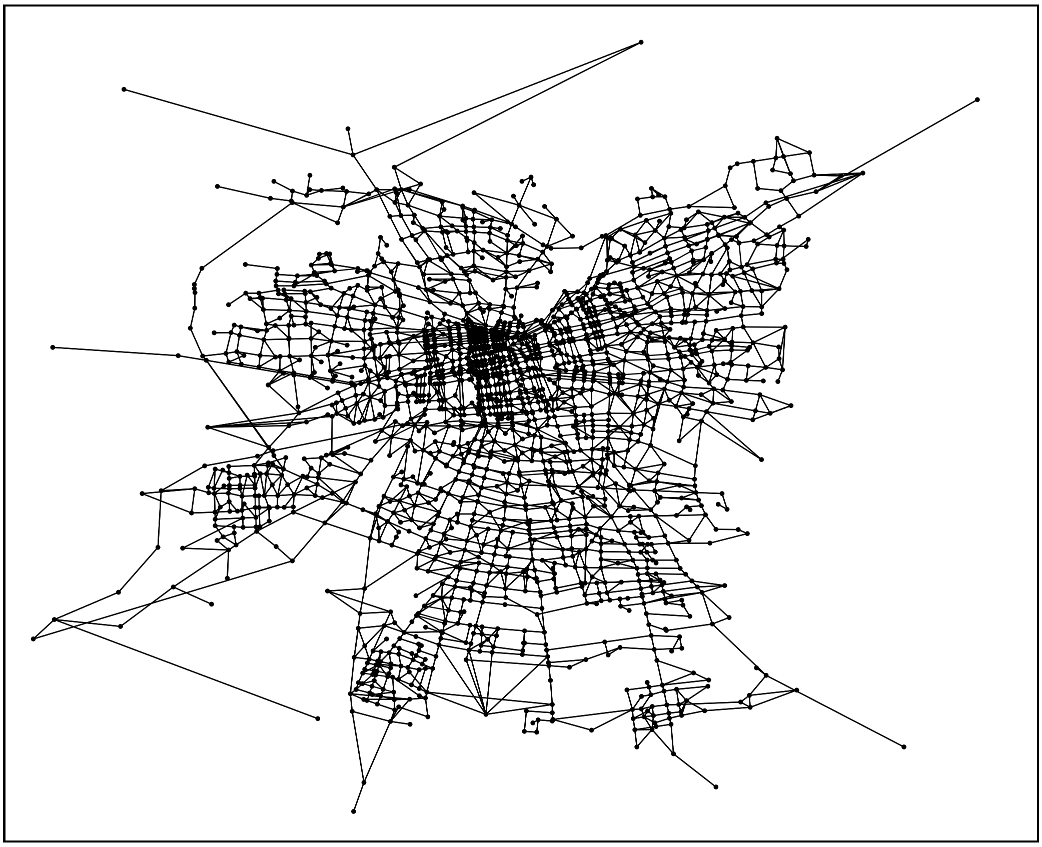
- Wardrop equilibrium
- Stochastic user equilibrium
- Markovian traffic equilibrium

III. Adaptive learning in transport

- Discrete-stochastic learning model
- Continuous adaptive dynamics
- Rest points and the underlying game
- Asymptotic convergence
- Potential and Lagrangian dynamics

I. Motivation

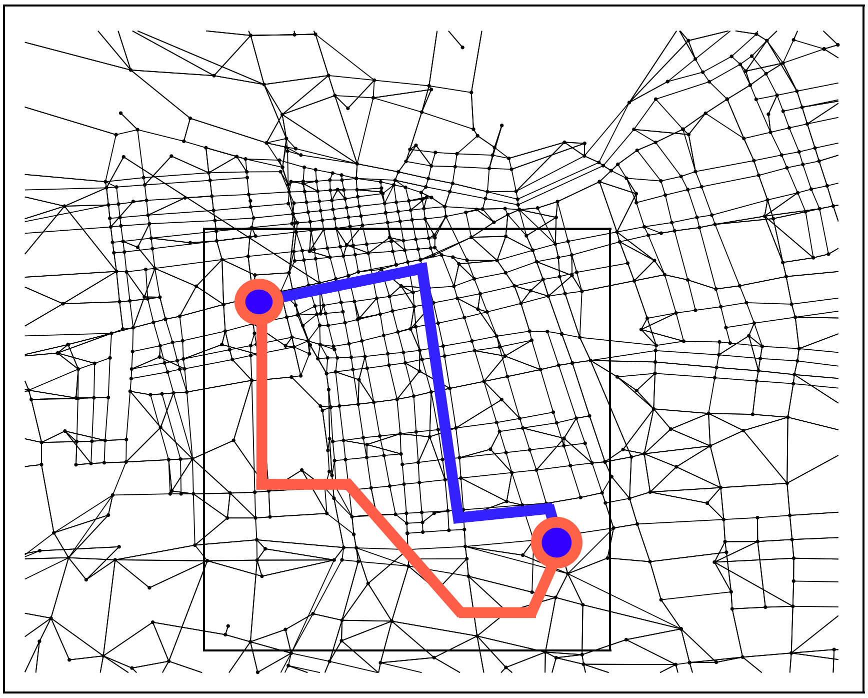
SANTIAGO NETWORK



2266 nodes / 7636 arcs / 409 destinations

SANTIAGO	
6.000.000	people
1.000.000	cars
50.000	taxis
6.000	buses
4	metro lines

Travel demand	
11.000.000	daily trips
8.000.000	motorized trips
1.750.000	car trips



Morning peak	
500.000	car trips
29.000	OD pairs

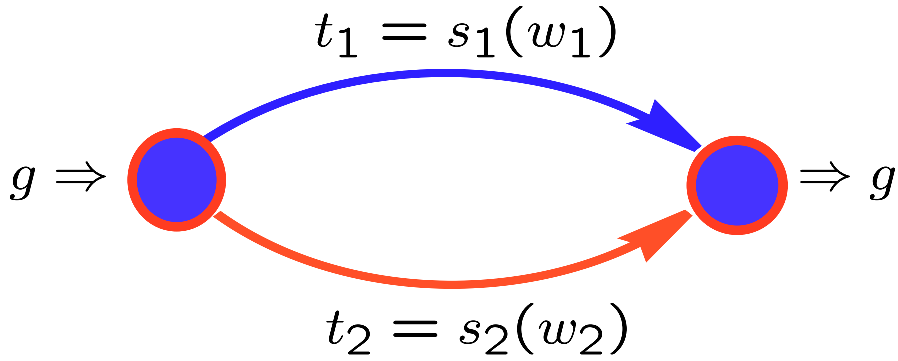
Which model is "most appropriate"?

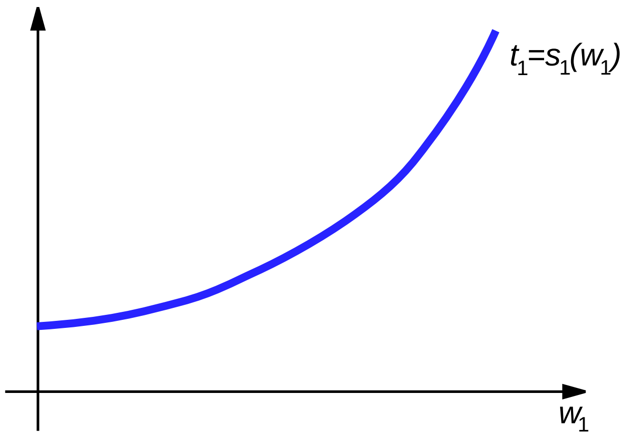
- Deterministic/stochastic equilibrium
- Repeated games with many players
- Stochastic processes: mean/std dev
- Fluid dynamics
- Simulation: car-following

To which extent are they connected?

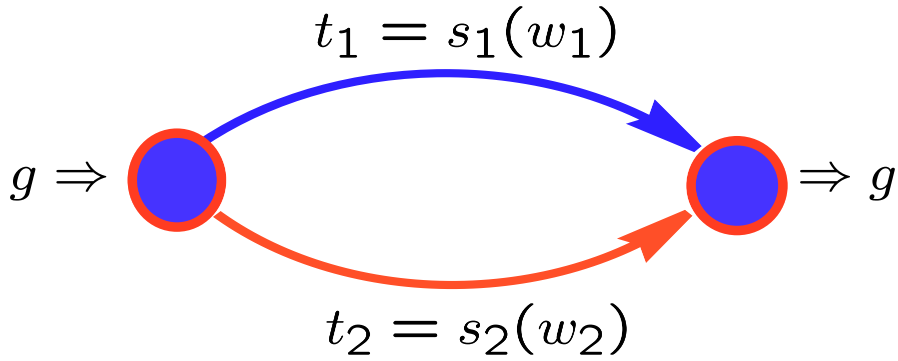
II. Overview of transport equilibrium

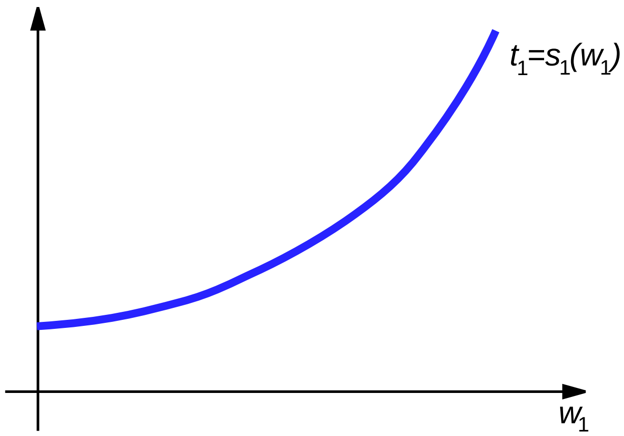
1. Wardrop Equilibrium





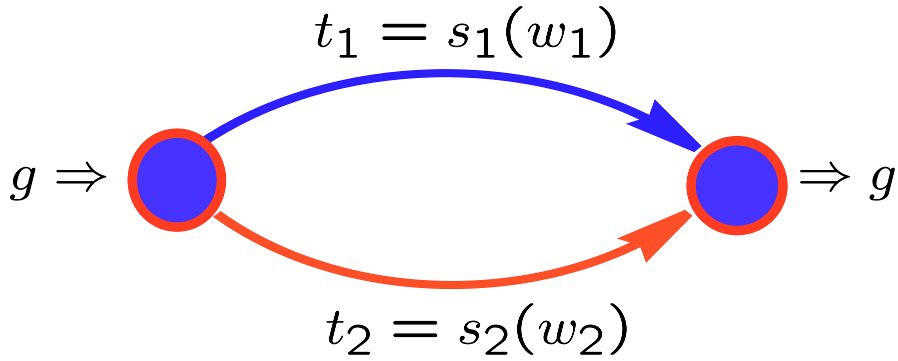
1. Wardrop Equilibrium

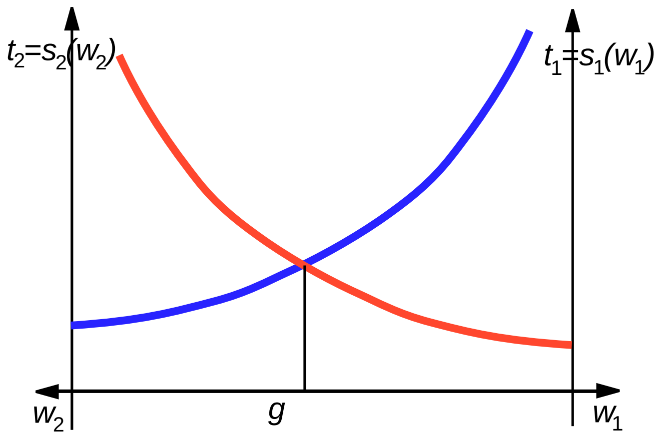




$$\begin{cases} g = w_1 + w_2 \\ w_1 > 0 \Rightarrow t_1 \leq t_2 \\ w_2 > 0 \Rightarrow t_2 \leq t_1 \end{cases}$$

1. Wardrop Equilibrium





$$\begin{cases} g = w_1 + w_2 \\ w_1 > 0 \Rightarrow t_1 \leq t_2 \\ w_2 > 0 \Rightarrow t_2 \leq t_1 \end{cases}$$

Network equilibrium (Wardrop'52)

$$\text{Given} \begin{cases} G = (N, A) & \text{network} \\ t_a = s_a(w_a) & \text{arc travel times} \\ g_i^d & \text{demands} \\ \mathcal{R}_i^d & \text{routes} \end{cases}$$

split the demands $g_i^d = \sum_{r \in \mathcal{R}_i^d} x_r$ into path flows $x_r \geq 0$ so that only shortest routes are used

$$\boxed{x_r > 0 \;\Rightarrow\; T_r = \tau_i^d}$$

where

$$T_r = \textstyle\sum_{a \in r} s_a(w_a) \quad \text{(route times)}$$

$$w_a = \textstyle\sum_{r \ni a} x_r \quad \text{(total arc flows)}$$

$$\tau_i^d = \min_{r \in \mathcal{R}_i^d} T_r \quad \text{(minimal time)}$$

Characterization (Beckman-McGuire-Winsten'56)

Network equilibrium $\Leftrightarrow$ NCP, VI, fixed point...
...or convex programming formulation

$$
(P) \quad \begin{cases}
\underset{w,v}{\text{Min}} \quad \displaystyle\sum_a \int_0^{w_a} s_a(x)\,dx \\
\text{s.t.} \\
w_a = \sum_d v_a^d & \text{(aggregate flow)} \\
g_i^d + \sum_{A_i^-} v_a^d = \sum_{A_i^+} v_a^d & \text{(flow conservation)} \\
v_a^d \geq 0 & \text{(non-negative flows)}
\end{cases}
$$

$$\Downarrow$$

There is a unique equilibrium w^*

Dual formulation (Fukushima'84)

Change of variables: $w_a \leftrightarrow t_a$

$$(D) \qquad \underset{t}{\text{Min}} \;\; \phi(t) = \underbrace{\sum_a \int_0^{t_a} s_a^{-1}(z)\, dz - \sum_{i,d} g_i^d \tau_i^d(t)}_{\text{strictly convex}}$$

$t \mapsto \tau_i^d(t)$ = minimum travel time
non-smooth and polyhedral concave function, solution of Bellman's equations

$$\tau_i^d = \min_{a \in A_i^+} [t_a + \tau_{j_a}^d]$$

Method of Successive Averages

- Compute $t_a^k = s_a(w_a^k)$
- Assign g_i^d to shortest routes
- Compute induced flows $\tilde{w}_a^k$
- Update $w^{k+1} = (1-\lambda_k)w^k + \lambda_k \tilde{w}^k$

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$$\boxed{w^{k+1} = (1-\lambda_k)w^k + \lambda_k \Phi(w^k)}$$

equilibrium $\Leftrightarrow$ fixed point

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equilibrium $\Leftrightarrow$ fixed point

Case Φ non-expansive: ($>$200 refs in MathSciNet)

Mann'53, Krasnoselskii'55, Edelstein'66,...

Problem: Our $\Phi(\cdot)$ is expansive!!!
Then, why does this method work?

MSA revisited (Baillon-C'06)

Reintroduce w back into Fukushima's dual
The equilibrium w^* solves

$$(\tilde{D}) \qquad \min_{w} \ \psi(w) := \phi(s_a(w_a) : a \in A)$$

and MSA can be re-written as a variable metric descent method

(MSA) $$\boxed{\frac{w^{k+1} - w^k}{\lambda_k} \in -D(w^k)^{-1} \partial\psi(w^k)}$$

$D(w) = \text{diag}(s'_a(w_a))$ is part of "$\nabla^2\psi(w)$"

Theorem
If $\sum \lambda_k = \infty$ and $\sum \lambda_k^2 < \infty$ then $w^k \to w^*$

2. Stochastic user equilibrium

Wardrop does not fit well with observations

Alternative: consider heterogeneous users

$$\left.\begin{aligned} \tilde{t}_a &= t_a + \epsilon_a \\ \tilde{T}_r &= \textstyle\sum_{a\in r} \tilde{t}_a \\ \tilde{\tau}_i^d &= \min_{r\in\mathcal{R}_i^d} \tilde{T}_r \end{aligned}\right\} \text{random variables}$$

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Stochastic assignment

$$x_r = g_i^d\,\mathbb{P}(\tilde{T}_r = \tilde{\tau}_i^d) \quad \text{(pbb } r \text{ is optimal route)}$$

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Stochastic assignment

$$x_r = g_i^d\,\mathbb{P}(\tilde{T}_r = \tilde{\tau}_i^d) \quad \text{(pbb } r \text{ is optimal route)}$$

plus equilibrium equations

$$t_a = s_a(w_a)$$

$$w_a = \textstyle\sum_{r\ni a} x_r$$

Logit model (Dial'71, Fisk'80)

i.i.d. Gumbel random errors (extremal distribution)

$$x_r = g_i^d \frac{\exp(-\beta T_r)}{\sum_{s \in \mathcal{R}_i^d} \exp(-\beta T_s)}$$

Drawbacks: i.i.d. & tractable only for small networks

Probit model (Daganzo'82)

Correlated Normal random errors

No explicit formula $\Rightarrow$ Montecarlo

Drawback: tractable only for very small networks

Discrete choice models

Random utilities: $u_1 + \varepsilon_1, \ldots, u_n + \varepsilon_n$

$$\pi_i = \mathbb{P}(u_i + \varepsilon_i \text{ is optimal}) = \frac{\partial \varphi}{\partial u_i}(u)$$

$$\varphi(u) = \mathbb{E}[\min\{u_1 + \varepsilon_1, \ldots, u_n + \varepsilon_n\}]$$

Example: Multinomial LOGIT

$$\begin{aligned} \pi_i &= \frac{\exp(-\beta u_i)}{\sum_j \exp(-\beta u_j)} \\ \varphi(u) &= -\tfrac{1}{\beta} \ln[\textstyle\sum_j \exp(-\beta u_j)] \end{aligned}$$

Variational formulation (Daganzo'82, Miyagi'85)

$$\operatorname*{Min}_{t} \quad \sum_{a} \int_{0}^{t_a} s_a^{-1}(x)\, dx - \sum_{i,d} g_i^d \, \varphi_i^d(T_r : r \in \mathcal{R}_i^d)$$

where $T_r = \sum_{a \in r} t_a$

Still intractable except for small networks

3. Markovian traffic equilibrium (Baillon-C'06)

$$\left.\begin{aligned} \tilde{t}_a &= t_a + \epsilon_a \\ \tilde{T}_r &= \textstyle\sum_{a \in r} \tilde{t}_a \\ \tilde{\tau}_i^d &= \min_{r \in \mathcal{R}_i^d} \tilde{T}_r \end{aligned}\right\} \text{random variables}$$

At every intermediate node i in his trip a user selects a *random optimal arc*

$$\operatorname*{Argmin}_{a \in A_i^+} \tilde{t}_a + \tilde{\tau}_{j_a}^d$$

$$\Downarrow$$

Markov chain for each destination d

Model derivation

Solving for the equilibrium laws in the Markov chains one gets that the expected in-flow

$$x_i^d = g_i^d + \sum_{a \in A_i^-} v_a^d$$

leaves node i according to

$$v_a^d = x_i^d \, \mathbb{P}(\tilde{t}_a + \tilde{\tau}_{j_a}^d \leq \tilde{t}_b + \tilde{\tau}_{j_b}^d \ \forall \, b \in A_i^+)$$

with

$$t_a = s_a(w_a)$$

$$w_a = \sum_d v_a^d$$

Variational formulation

$\tau_i^d = \mathbb{E}(\tilde{\tau}_i^d)$ turns out to be the unique solution of the stochastic Bellman's equations

$$\tau_i^d = \mathbb{E}(\min_{a \in A_i^+}[t_a + \tau_{j_a}^d + \epsilon_a^d])$$

Moreover, $t \mapsto \tau_i^d(t)$ is concave & smooth.

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$$\tau_i^d = \mathbb{E}(\min_{a \in A_i^+}[t_a + \tau_{j_a}^d + \epsilon_a^d])$$

Moreover, $t \mapsto \tau_i^d(t)$ is concave & smooth.

...and then MTE can be characterized by

$$(D) \qquad \underset{t}{\text{Min}} \quad \sum_a \int_0^{t_a} s_a^{-1}(x)\, dx - \sum_{i,d} g_i^d \tau_i^d(t)$$

exactly as in the deterministic case!

Stochastic MSA

- Compute $t_a^k = t_a(w_a^k)$
- Solve stochastic Bellman's equations
- Compute Markov chain flows v_a^d
- Aggregate $\tilde{w}_a^k = \sum_d v_a^d$
- Update $w^{k+1} = (1-\lambda_k)w^k + \lambda_k \tilde{w}^k$

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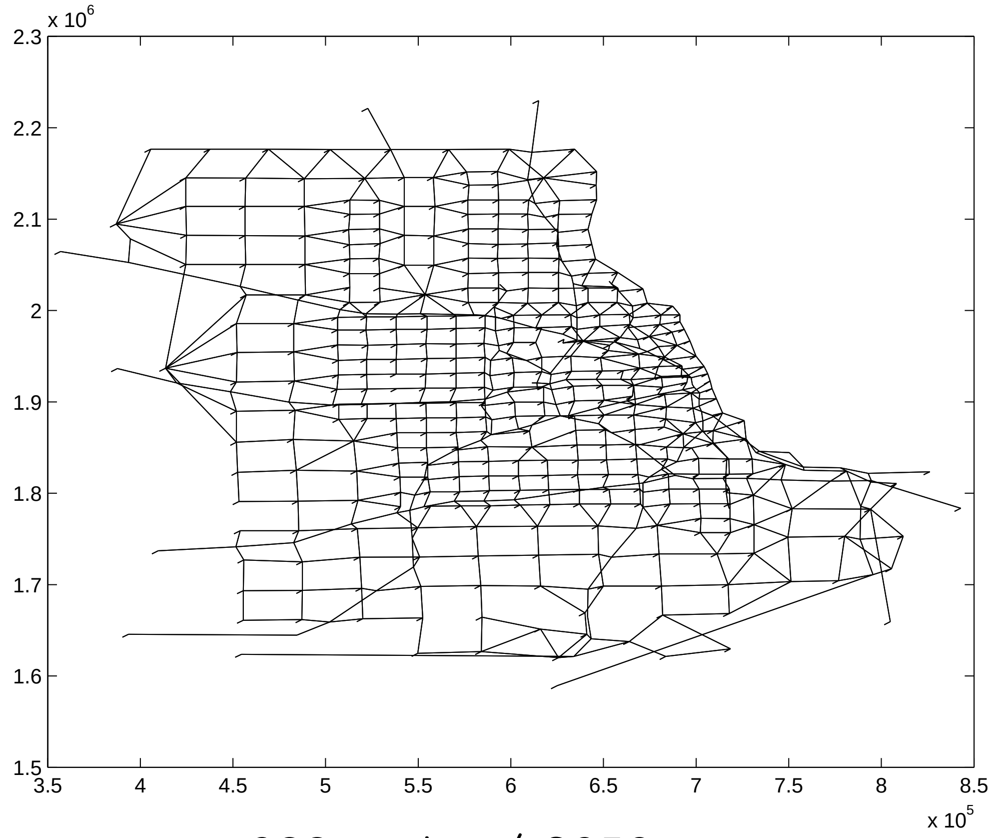
May be rewritten as

(SMSA) $$\boxed{\frac{w^{k+1} - w^k}{\lambda_k} = -D(w^k)^{-1}\nabla\psi(w^k)}$$

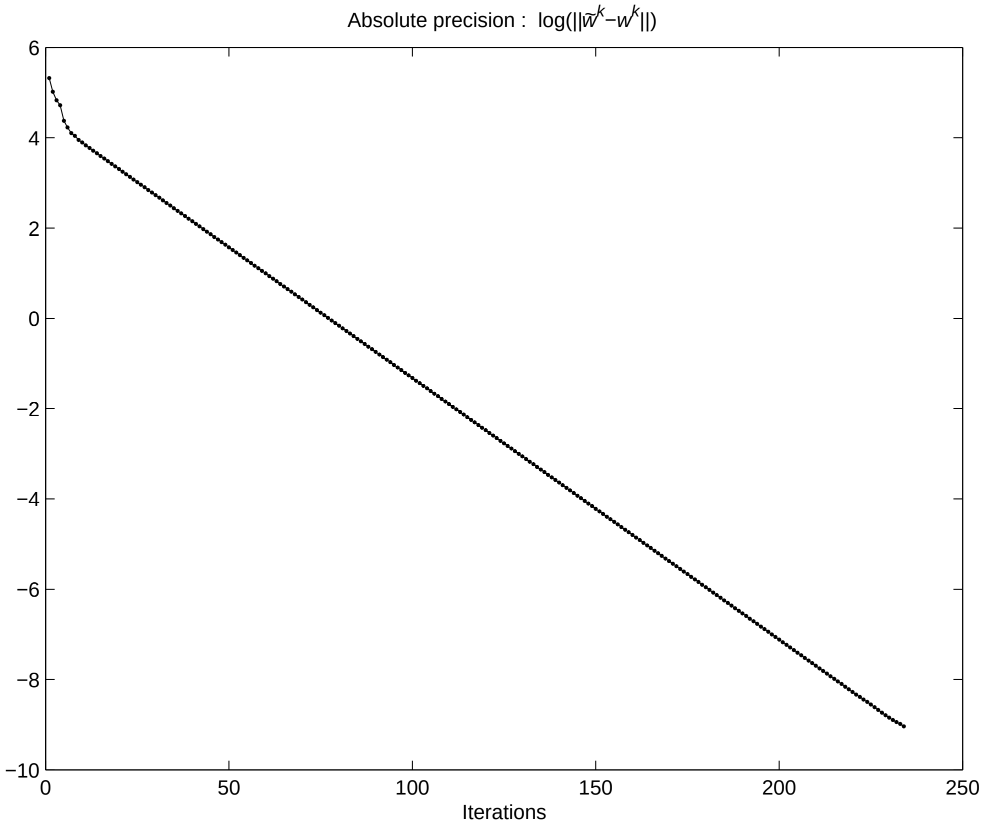
and converges to stochastic equilibrium w^*

Example: Chicago network



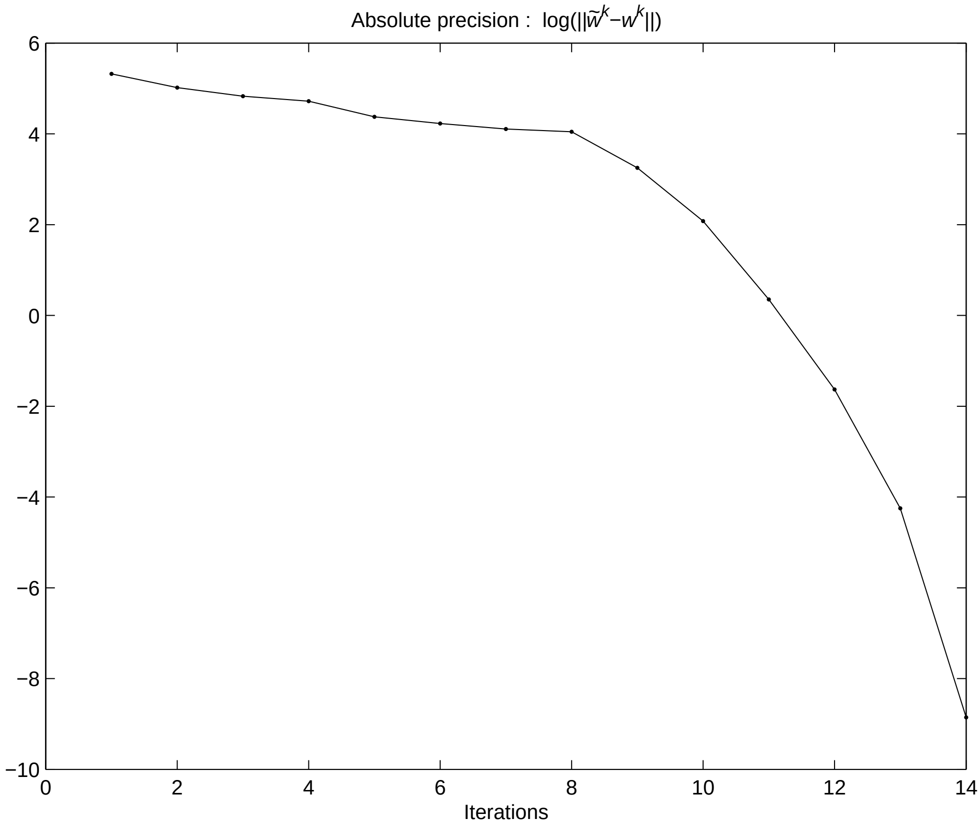
933 nodes / 2950 arcs

SMSA iterations



Execution time (1.6Mhz Pentium): 29[min]

SMSA-Newton iterations



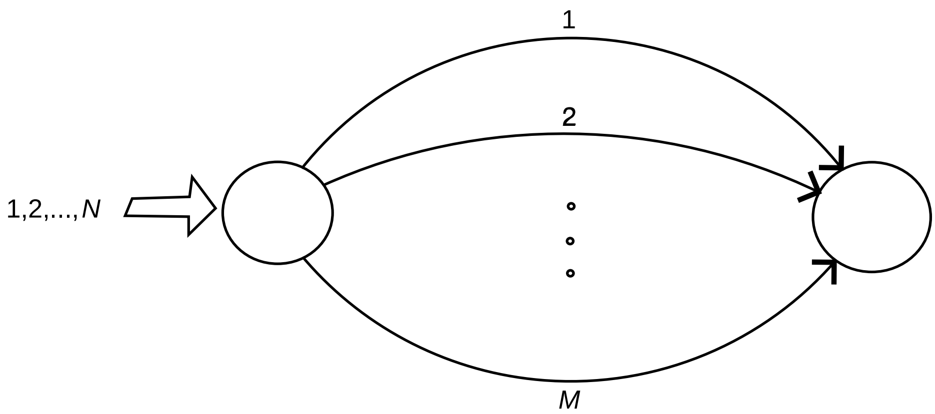
Execution time (1.6Mhz Pentium): 11[min]

$\boxed{\text{III. Adaptive learning in transport}}$

(Melo-Sorin-C'08)

Discrete-stochastic learning model

Dynamical models that sustain equilibrium?



$$
\begin{aligned}
i &= 1, \ldots, N \text{ drivers} \\
r &= 1, \ldots, M \text{ routes} \\
c_u^r &= \text{cost of route } r \text{ with } u \text{ drivers}
\end{aligned}
$$

State variable & decision rule

$$x^{ir} \quad = \quad \text{perception of driver } i \text{ on route } r$$

$$\pi^{ir} \quad = \quad \text{pbb for driver } i \text{ to choose route } r$$

$$\pi^{ir} = \frac{\exp(-\beta x^{ir})}{\sum_{\ell} \exp(-\beta x^{i\ell})}$$

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Learning process

x_{n-1}^{ir}	$\leadsto$	π_n^{ir}	$\leadsto$	r_n^i	$\leadsto$	u_n^r	$\leadsto$	c_n^r	$\leadsto$	x_n^{ir}
state		pbb's		routes		loads		costs		update

$$x_n^{ir} = \begin{cases} (1-\lambda_n)x_{n-1}^{ir} + \lambda_n c_n^r & \text{if } r = r_n^i \\ x_{n-1}^{ir} & \text{if } r \neq r_n^i \end{cases}$$

$$\Rightarrow \quad \boxed{x_n - x_{n-1} = \lambda_n[\tilde{c}_n - x_{n-1}]}$$

Expected continuous-time dynamics

The stochastic process

$$\frac{x_n - x_{n-1}}{\lambda_n} = \tilde{c}_n - x_{n-1}$$

converges almost surely to an internally chain transitive set (ICT) of the related dynamics

$$\frac{dx}{dt} = \mathbb{E}(\tilde{c}|x) - x$$

Ljung'77, Benaim-Hirsch'96, Benaim-Hofbauer-Sorin'05

Adaptive dynamics

The equation $\frac{dx}{dt} = \mathbb{E}(\tilde{c}|x) - x$ can be written as

$$\boxed{\frac{dx^{ir}}{dt} = \pi^{ir}(x)[C^{ir}(x) - x^{ir}]}$$

$$C^{ir}(x) = \mathbb{E}(c^r_\bullet | i \text{ chooses } r)$$

Adaptive dynamics

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$$C^{ir}(x) = \mathbb{E}(c^r_\bullet | i \text{ chooses } r)$$

Explicitly

$$\begin{aligned} C^{ir}(x) &= F^{ir}(\Pi(x)) \\ \Pi(x) &= (\pi^{ir}(x)) \end{aligned}$$

with

$F^{ir}(\pi) = \mathbb{E}[c^r_{U^r} | X^{ir} = 1]$

$U^r = \sum_j X^{jr} =$ load of route r

X^{ir} Bernoulli r.v. $\mathbb{P}(X^{ir}\!=\!1) = \pi^{ir}$

Example: 2 drivers $\times$ 2 routes

$$\begin{array}{rcl} \frac{dx}{dt}^{1a} & = & \pi^a(x^1)[C^a(x^2) - x^{1a}] \qquad \text{(driver 1)} \\ \frac{dx}{dt}^{1b} & = & \pi^b(x^1)[C^b(x^2) - x^{1b}] \end{array}$$

$$\begin{array}{rcl} \frac{dx}{dt}^{2a} & = & \pi^a(x^2)[C^a(x^1) - x^{2a}] \qquad \text{(driver 2)} \\ \frac{dx}{dt}^{2b} & = & \pi^b(x^2)[C^b(x^1) - x^{2b}] \end{array}$$

$$\begin{array}{rcl} \pi^a(x) & = & \exp(-\beta x^a)/[\exp(-\beta x^a) + \exp(-\beta x^b)] \\ \pi^b(x) & = & \exp(-\beta x^b)/[\exp(-\beta x^a) + \exp(-\beta x^b)] \end{array}$$

$$\begin{array}{rcl} C^a(x) & = & c_1^a \pi^b(x) + c_2^a \pi^a(x) \\ C^b(x) & = & c_1^b \pi^a(x) + c_2^b \pi^b(x) \end{array}$$

Rest points: definition

$$\boxed{\frac{dx^{ir}}{dt} = \pi^{ir}(x)[C^{ir}(x) - x^{ir}]}$$

$$\mathcal{E} = \{\text{rest points}\} = \{x : x^{ir} = C^{ir}(x)\}$$

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$$x = C(x) \;\Leftrightarrow\; x = F(\Pi(x))$$

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$$x = C(x) \;\Leftrightarrow\; x = F(\Pi(x))$$

$$\Leftrightarrow \begin{cases} x = F(\pi) \\ \pi = \Pi(x) \end{cases}$$

Rest points: definition

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$$\mathcal{E} = \{\text{rest points}\} = \{x : x^{ir} = C^{ir}(x)\}$$

$$x = C(x) \iff x = F(\Pi(x))$$

$$\iff \begin{cases} x = F(\pi) \\ \pi = \Pi(x) \end{cases}$$

It follows that

Proposition $x \rightleftarrows \pi$ is a bijection on $\mathcal{E}$

$$\Pi(\mathcal{E}) = \{\text{rest probabilities}\}$$

Rest points: characterization

Theorem

x is a rest point $\Leftrightarrow$ the point $\pi = \Pi(x)$ is a Nash equilibrium of an underlying N-person game defined by

$$\text{Strategies} \quad \to \quad \pi^i \in \Delta(R)$$

$$\text{Payoffs} \quad \to \quad G^i(\pi) = -\langle \pi^i, F^i(\pi) \rangle - \frac{1}{\beta} \sum_r \pi^{ir} \ln \pi^{ir}$$

Proof: $\pi \in \Pi(\mathcal{E}) \Leftrightarrow \pi = \Pi(F(\pi))$

$$\Pi^i(x) = \text{Argmax} \; - \langle \pi^i, x^i \rangle - \frac{1}{\beta} \sum_r \pi^{ir} \ln \pi^{ir}$$

Rest points: existence/uniqueness

Theorem:

- there exist rest points
- exactly one of them is symmetric: $\hat{x}^{ir} = \hat{x}^{jr}$
- if $\beta\delta < 2$ then $\hat{x}$ is the only rest point

$\boxed{\delta = \max_{r,u}[c^r_u - c^r_{u-1}]}$ congestion jump

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The rest points bifurcate as $\beta\delta$ increases
The study of bifurcations is open

Rest points: attractors

Theorem:

- if $\beta\delta < 2$ then $\widehat{x}$ is a local attractor
- if $\beta\delta < \frac{2}{N-1}$ then
  - $\widehat{x}$ is a global attractor
  - the learning process converges $x_n \stackrel{a.s.}{\longrightarrow} \widehat{x}$.

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Simulations: convergence for all values of $\beta\delta$

No theoretical explanation available

Potential function

The dynamics may be reconsidered using an alternative Lagrangian representation, based on the following

Theorem: The map F admits a potential

$$F(\pi) = \nabla H(\pi)$$

where

$$H(\pi) = \sum_r \mathbb{E}(c_1^r + c_2^r + \cdots + c_{U^r}^r).$$

Dynamics in Lagrangian form

Denoting

$$\begin{aligned} H_\beta(\pi) &= H(\pi) + \tfrac{1}{\beta}\textstyle\sum_{ir} \pi^{ir} \ln(\pi^{ir}) \\ \mathcal{L}(\pi;\lambda) &= H_\beta(\pi) - \textstyle\sum_i \lambda^i [\sum_r \pi^{ir} - 1] \end{aligned}$$

the adaptive dynamics can be written

$$\boxed{\dot{x} = -\tfrac{1}{\beta}\nabla_x L(x;\lambda(x))}$$

where

$$\begin{aligned} L(x;\lambda) &= \mathcal{L}(\pi(x,\lambda);\lambda) \\ \pi^{ir}(x,\lambda) &= \exp(-\beta(x^{ir} - \lambda^i)) \\ \lambda^i(x) &= -\tfrac{1}{\beta}\ln(\textstyle\sum_r \exp(-\beta x^{ir})) \end{aligned}$$

Rest points as extremals

For $\pi = \Pi(x)$ the following are equivalent

(a) $x \in \mathcal{E}$

(b) $\nabla_x L(x, \lambda(x)) = 0$

(c) π is a Nash equilibrium

(d) $\nabla_\pi \mathcal{L}(\pi, \lambda) = 0$ for some $\lambda \in \mathbb{R}^M$

(e) π is a critical point of $H_\beta(\cdot)$ on $\Delta(R)^N$

Moreover, if $\beta\delta < 1$ then H_β is strongly convex and $\widehat{\pi} = \Pi(\widehat{x})$ is its unique minimum.