Fluctuations of traveling waves in an inhomogeneous medium

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Reaction-diffusion in a random environment

$$u_t = u_{xx} + f(x,u), \quad x \in \mathbb{R}, \ \ t > 0.$$

Solutions will exhibit a moving interface. . . .

(i) How does the solution evolve at large times? Mean behavior? Almost-sure behavior?

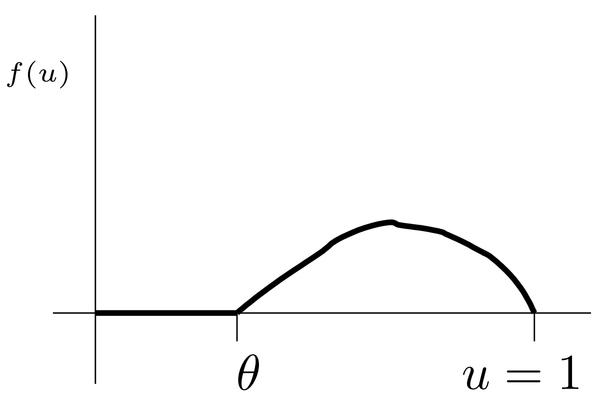
(ii) How does the interface fluctuate about its mean position?

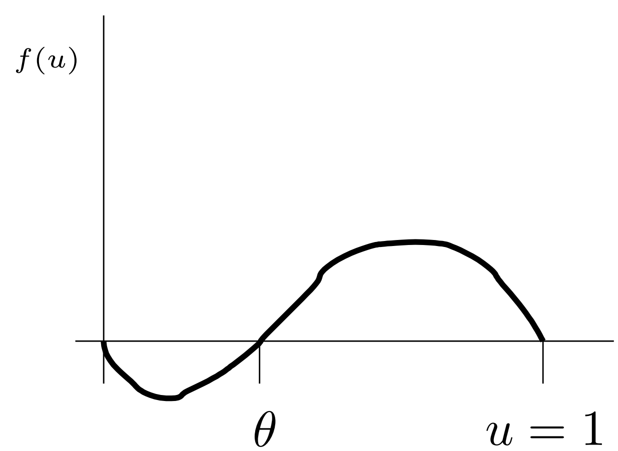
Reaction-diffusion in a homogeneous environment

$u(t,x)$ satisfies

$$u_t = u_{xx} + f(u), \quad x \in \mathbb{R}, \;\; t > 0$$
$$u(0,x) = u_0(x) \in [0,1]$$

$f(u)$ is nonlinear and $\int_0^1 f(u)\,du > 0$:

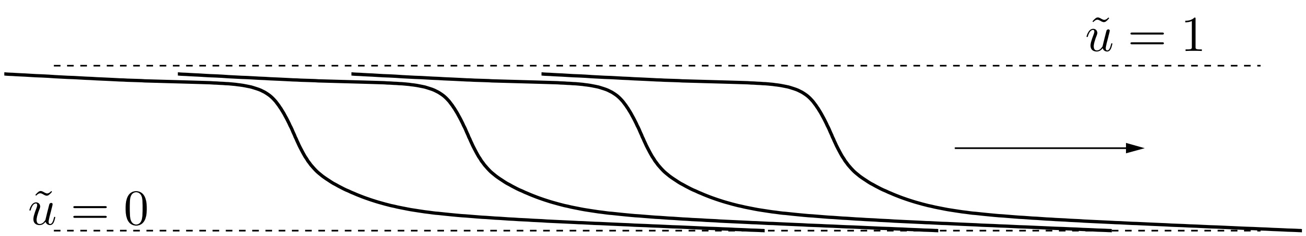




Traveling wave solutions exist.

$$\tilde{u}(t,x) = \tilde{u}(0, x - \tilde{c}t), \quad x \in \mathbb{R},\ t \in \mathbb{R}$$

$\tilde{c}$ is the unique **wave speed**, and $W(x) = \tilde{u}(0,x)$ is the **wave profile**.



Kolmogorov, Petrovsky, Piskunov (1937), Fisher (1937),

Kanel (1962), Aronson, Weinberger (1979).

Traveling wave solutions are attractors.

For the bistable and ignition-type nonlinearties, if $u(t,x)$ solves the initial value problem with appropriate "wave-like" initial data at $t = 0$, then

$$\sup_x |u(t,x) - \tilde{u}(t+\tau, x)| \le Ce^{-rt}, \quad \forall\, t \ge 0$$

for some shift $\tau \in \mathbb{R}$ and constants $C, r > 0$.

See Kanel (1962), Fife, McLeod (1977), Rothe (1981)

What if the environment varies (randomly) with x?

Suppose $f = f(x, u)$.

- What are the statistical properties of solutions to the Cauchy problem?
- Is there a wave speed?
- Is there a unique wave-like solution that attracts other solutions?

The random inhomogeneous environment

$$u_t = u_{xx} + g(x) f_0(u), \quad x \in \mathbb{R}, \quad t > 0$$
$$u(0, x) = u_0(x), \quad x \in \mathbb{R}$$

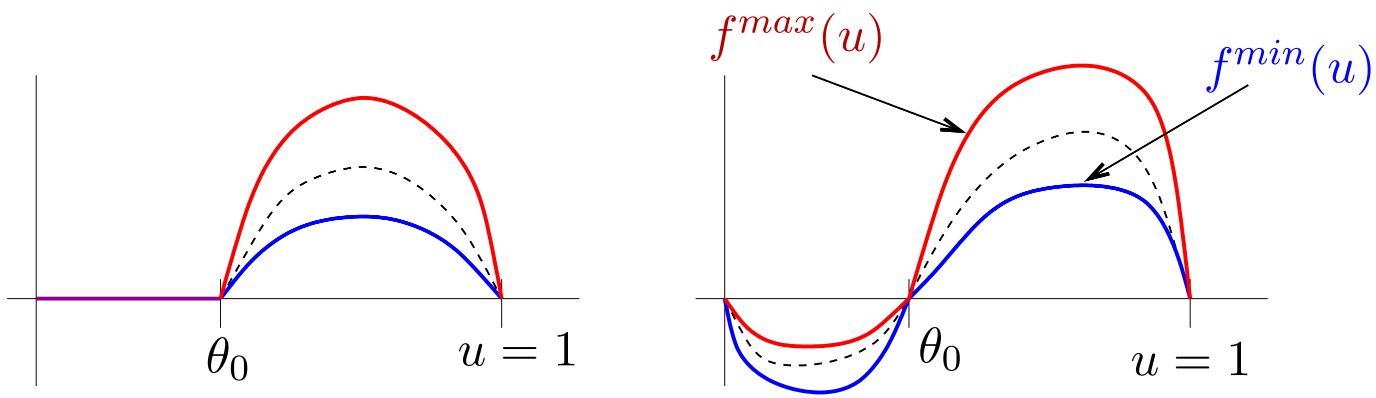
Initial condition $u_0(x)$ is "wave-like":

$$\lim_{x \to -\infty} u_0(x) = 1, \qquad u_0(x) \leq C e^{-\alpha x}$$

Suppose $g(x, \omega) : \mathbb{R} \times \Omega \to (0, \infty)$ is a stationary random field.

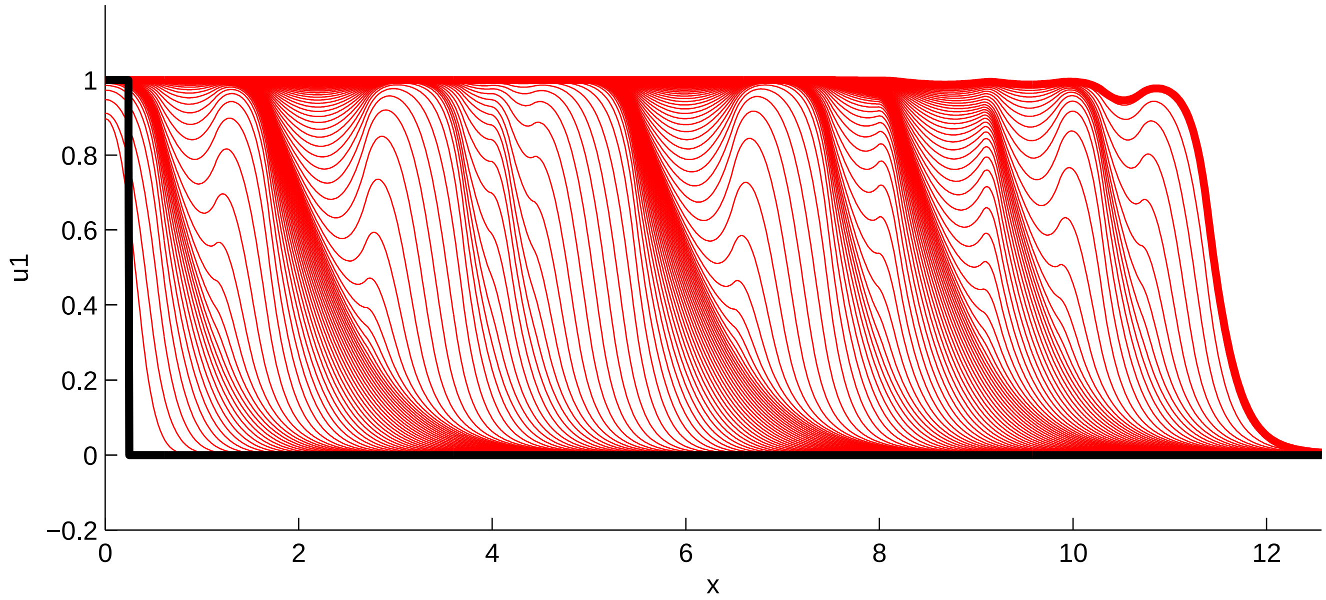
Let $\{\pi_x\}_{x \in \mathbb{R}}$ be a group of measure-preserving transformations which act ergodically on $(\Omega, \mathcal{F}, \mathbb{P})$ so that $g(x + h, \omega) = g(x, \pi_h \omega)$.

- $g(x) > 0$ is uniformly Lipschitz continuous in x.
- $f^{min}(u) \le g(x) f_0(u) \le f^{max}(u)$
- $\int_0^1 f^{min}(u)\, du > 0$



What does the solution look like?

The initial data is a step function (in black).
The plot shows $u(t, x)$ at regularly-spaced points in time, corresponding to one realization of $g(x, \omega)$.



The interface width does **not** spread out as $t \to \infty$.

Suppose $X^+(t)$ and $X^-(t)$ are defined by

$$\begin{aligned} X^+(t) &= \sup\{x \in \mathbb{R} \,|u(t,x) \geq \epsilon\} \\ X^-(t) &= \inf\{x \in \mathbb{R} \,|u(t,x) \leq 1-\epsilon\} \end{aligned}$$

then for some constant C,

$$|X^+(t) - X^-(t)| \leq C$$

holds for all t.

A Law of Large Numbers for the interface

Let $X(t,\omega)$ be the random interface position:

$$X(t,\omega) = \sup\{x \in \mathbb{R} \,|u(t,x,\omega) = \theta_0\}$$

Then $X(t,\omega)$ satisfies

$$\lim_{t\to\infty} \frac{X(t,\omega)}{t} = \tilde{c}, \quad \text{almost surely, and in } L^1(\Omega).$$

$\tilde{c} > 0$ is independent of the initial data.

N., Ryzhik (2008)

See Freidlin-Gärtner (1979) for a related result with K.P.P.-type nonlinearity.

For each ω, the average speed $X(t,\omega)/t$ is asymptotically independent of the initial data.

Moreover, for each ω, the entire solution profile also "forgets the initial data" very quickly.

Here is a **MOVIE** illustrating this phenomenon. . . .

A Generalized Traveling Wave:

There **exists** a global-in-time solution $\tilde{u}(t,x)$ of

$$\tilde{u}_t = \tilde{u}_{xx} + g_0(x) f(\tilde{u}), \quad x \in \mathbb{R}, \;\; t \in \mathbb{R}.$$

It is **unique** up to a time shift. Also, $\tilde{u}_t > 0$, for all $x \in \mathbb{R}$, $t \in \mathbb{R}$.

This solution is an **attractor**: if $u_0(x)$ is wave-like, then there is a time shift τ and constants $C, r > 0$ such that

$$\sup_{x \in \mathbb{R}} |u(x,t) - \tilde{u}(t+\tau, x)| \leq Ce^{-rt}$$

holds for all $t \geq 0$.

Mellet, Roquejoffre, Sire (2009),

N., Ryzhik (2008),

Mellet, N., Ryzhik, Roquejoffre (2009)

Statistical invariance of the generalized traveling wave:

We may normalize $\tilde{X}(0,\omega) = 0$, so that

$$\tilde{u}(T_k(\omega), x+k, \omega) = \tilde{u}(0, x, \pi_k \omega), \qquad \forall\, k \in \mathbb{R}$$

$T_k = T_k(\omega)$ is the hitting time to $x = k$: $\tilde{X}(T_k, \omega) = k$.
Increments $\Delta T_k = T_{k+1} - T_k$ are stationary with respect to k.

In this sense, the profile is **statistically invariant** with respect to reference point $x = k$.

An Invariance Principle

If the environment is sufficiently mixing, then

(i) There is $\kappa^2 \geq 0$ such that

$$\frac{X(t,\omega) - t\tilde{c}}{\sqrt{t}} \to N(0,\kappa^2), \qquad \text{as } t \to \infty.$$

(ii) If $\kappa^2 > 0$, the family of continuous process $\{Y_n(t)\}_{n=1}^{\infty}$ defined by

$$Y_n(t,\omega) = \frac{X(nt,\omega) - nt\tilde{c}}{\kappa\sqrt{n}}, \quad t \in [0,1],$$

converges weakly (as $n \to \infty$) to a standard Brownian motion on $[0,1]$, in the sense of weak convergence of measures on $C([0,1])$ with the topology of uniform convergence.

N. (2009)

The mixing condition

Define the family of σ-algebras

$$\mathcal{F}_k^- = \sigma\left(g(x,\omega) \middle| \, x \le k\right)$$
$$\mathcal{F}_k^+ = \sigma\left(g(x,\omega) \middle| \, x \ge k\right)$$

$$\mathcal{F}_k^- \subset \mathcal{F}_{k+1}^- \subset \mathcal{F}, \qquad \text{and} \qquad \mathcal{F} \supset \mathcal{F}_k^+ \supset \mathcal{F}_{k+1}^+$$

We say the environment is ϕ-mixing if for all $j \ge k$ and any $\xi \in L^2(\Omega, \mathcal{F}_k^-, \mathbb{P})$ and $\eta \in L^2(\Omega, \mathcal{F}_j^+, \mathbb{P})$,

$$\left|\mathbb{E}\left[\xi\eta\right] - \mathbb{E}[\xi]\mathbb{E}[\eta]\right| \le \sqrt{\phi(j-k)}\left(\mathbb{E}[\xi^2]\mathbb{E}[\eta^2]\right)^{1/2}$$

for $\phi(n) : \mathbb{Z}^+ \to [0,\infty)$ is nonincreasing. If $\sum_{n\ge1}\sqrt{\phi(n)} < \infty$, then the invariance principle holds.

How do we obtain a CLT for $X(t,\omega)$?

$$\frac{X(t,\omega) - t\tilde{c}}{\sqrt{t}} \to N(0,\sigma^2)$$

Some challenges:

- $X(t,\omega)$ depends on the environment in a nonlinear way, through solution of the PDE.
- What does a mixing assumption on the environment imply about the process $X(t,\omega)$?

Consider the hitting times

$$T_k(\omega) = \inf\{t \geq 0 | \tilde{X}(t, \omega) = k\}.$$

Then

$$\frac{T_n - \tilde{\tau} n}{\sqrt{n}} = \frac{1}{\sqrt{n}} \sum_{k=0}^{n-1} \left(\Delta T_k - \mathbb{E}[\Delta T_k] \right),$$

where $\Delta T_k = T_{k+1} - T_k$, and $\mathbb{E}[\Delta T_k] = \tilde{\tau} = \tilde{c}^{-1}$.

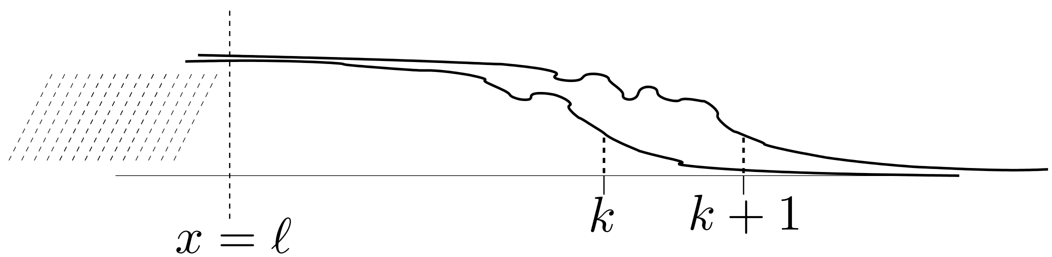
For the traveling wave, these increments are $\{\Delta T_k\}_k$ stationary!

To derive a CLT for T_n, we need to estimate the **dependence** among terms ΔT_k in the sum.

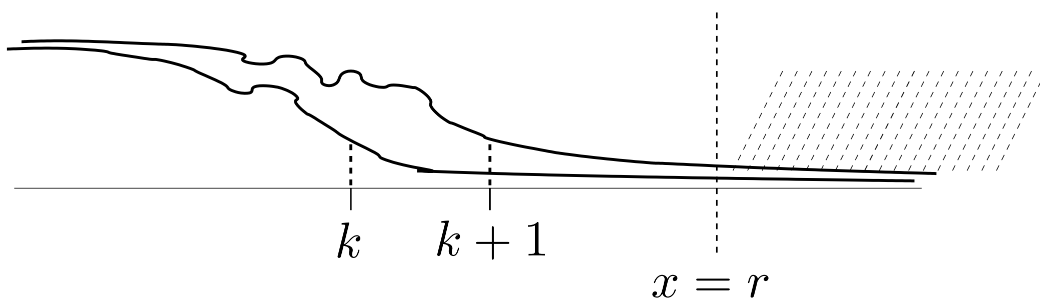
Stability of the wave under perturbations of the environment enables us to show that

$$\Delta T_k = T_{k+1} - T_k$$

does not depend strongly on the **distant past**:



or **distant future**:



ΔT_k depends primarily on the local environment near $x = k$.

Define the **modified environment**

$$\hat{g}(x,\omega) = \begin{cases} g(x,\omega), & x \geq 0 \\ g(0,\omega), & x \leq 0 \end{cases}$$

Theorem: Let $z(t,x,\omega)$ solve the **modified initial value problem**

$$z_t = z_{xx} + \hat{g}(x,\omega) f_0(z), \quad x \in \mathbb{R}, \quad t \geq 0$$

with deterministic initial condition $z(0,x,\omega) = z_0(x)$ that is wave-like. There are constants $C, C_\tau, r > 0$ and a random variable $\tau(\omega)$ such that for almost surely with respect to $\mathbb{P}$, both $|\tau(\omega)| \leq C_\tau$ and

$$\sup_{x\in\mathbb{R}} |z(t,x,\omega) - \tilde{u}(t+\tau(\omega),x,\omega)| \leq Ce^{-rt}$$

hold for all $t > 0$.

For $N > 0$ define the **modified environment**

$$\hat{g}_N(x,\omega) = \begin{cases} g(x,\omega), & x \leq N \\ g(N,\omega), & x \geq N \end{cases}$$

Let $z^N(t,x,\omega)$ solve the modified problem

$$z_t = z_{xx} + \hat{g}^N(x,\omega) f_0(z), \quad x \in \mathbb{R}, \quad t \geq 0$$

with deterministic initial condition $z(0,x,\omega) = z_0(x)$. There are non-random constants $C, C_\tau, r > 0$, $p_0 \in (0,1)$, and a random variable $\tau_N(\omega)$ such that, almost surely, both $|\tau_N(\omega)| \leq C_\tau$ and

$$\sup_{x\in\mathbb{R}} |z^N(t,x,\omega) - u(t + \tau_N(\omega), x, \omega)| \leq Ce^{-rt}$$

holds for all $t \in [0, p_0 N]$, for all $N > 0$.

Corollary: The increments

$$\Delta T_k \quad \text{and} \quad \Delta T_j$$

depend primarily on local environements near $x = k$ and $x = j$. The mixing condition on the environment then implies that they are approximately independent if $|k - j|$ is large.

Using a martingale approximation argument, one can then derive the CLT for

$$\frac{T_n - \tilde{\tau} n}{\sqrt{n}}$$

Summary:

- Homogenization in a stationary, random environment: $X(t)/t \to \tilde{c}$.
- Solutions to the Cauchy problem converge exponentially fast to the generalized traveling wave.
- Traveling wave respects statistical invariance of the medium (akin to the t.w. property in homogeneous environment)
- Stability of the wave implies approximate local dependence of the wave on the environment.
- Interface behaves like a Brownian motion with positive drift.

Some CLTs for other stochastic, nonlinear PDEs:

- Burgers' equation with random flux/I.C. – Wehr, Xin, (1997, 2000)

$$u_t + \frac{1}{2}\left(a(x,\omega)u^2\right)_x = 0$$

- Stochastic Hamilton-Jacobi equations – Rezakhanlou (2000)

$$u_t^\epsilon + H(Du^\epsilon, \frac{x}{\epsilon}, \omega) = 0$$

- Semilinear heat equation with random source –
  Varadhan, Zygouras (2008)

$$u_t = u_{xx} - u^2 + \lambda(t,\omega)\delta_0(x)$$

This is the end!

Thank you for your attention.