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On Metriplectic Dynamics

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Topics in Kinetic Theory

Victoria, BC

July 2, 2009

On Metriplectic Dynamics

(Form for Hamilt Dissipation)

Rayleigh

Incomplete System ← Cahn-Hilliard, Otto, Ricci Flows

Complete System → Metriplectic Dynamics

- Energy Conservation
- Entropy Production

Physics know: Dissipation term - Kurbatt, Viscosity etc

Kaufman, PJM, Grmela (1984)

(1986) Metriplectic Physica 18D 410 Kroshal

PJM (2009) J. Phys. Conf. Ser. 169 Kaufman

Overview

*) Ham System

*) Grad. Flows

*) Metriplectic System

* ~~Examples~~

1) Finite DoF

2) Generalized Landau Collision Vlasov

Ham. System

phase space manifold $\mathcal{Z} \ni z$

continuum system: NC variables

$\Rightarrow \quad \dot{q} = p \quad ; \quad \dot{p} = -\partial H/\partial q$

$\dot{z} = [z, H]$ coord. $\dot{z}^i = J^{ij} \dfrac{\partial H}{\partial z^j}$ $i, j \ldots M$

$[f,g]$ realization of Lie Algebra

*) bilinear

*) $[f,g] = -[g,f]$

*) Jacobi
$[[f,g],h] + \circlearrowleft = 0$

$S^{ijk} = J^{il} \dfrac{\partial}{\partial z^l} J^{jk} + \circlearrowleft \equiv 0$

$J_c = \begin{pmatrix} 0_N & I_N \\ -I_N & 0_N \end{pmatrix}$

$J(z)$ in general

$[f,g] = \dfrac{\partial f}{\partial z^i} J^{ij} \dfrac{\partial g}{\partial z^j}$

$dJ = 0$
$\Rightarrow$ force rotation

Casimir Invariants (1981)?

$[C, f] = 0 \ \forall f \iff J^{ij} \dfrac{\partial C}{\partial z^j} = 0$

Don't exist in can. Ham. systems

Darboux–Lie Thm
(coordinate entropies)

usual p.q planes spin

C = casimir

Lie-Poisson

$J^{ij} = c^{ij}_k z^k$

structure consts of Lie Algebra

$\dot{z} = [z, H + C] = [z, H]$

$J \longrightarrow \left[\begin{array}{cc|c} 0_N & I_N & 0 \\ -I_N & 0_N & \\ \hline & & 0 \end{array}\right]$

$F = H + C \qquad \dfrac{\delta F}{\delta z} = 0 \Rightarrow$ equil

Grad. System (Flows)

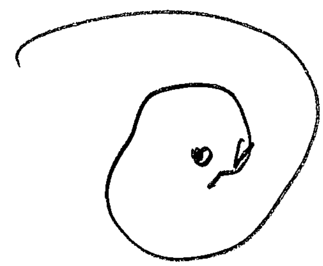
$$\dot z = -\nabla E$$

$$\dot E = \frac{\partial E}{\partial z^i}\left(-\frac{\partial E}{\partial z^i}\right) \le 0$$

$$\left.\frac{\partial E}{\partial z}\right|_{z_{eq}} = 0 \qquad \text{Lyapunov's Thm.}$$

z_{eq} asymptotically stable

Metric Flow



$$\dot z^i = g^{ij}(z)\frac{\partial E}{\partial z^j}$$

definite & symmetric

$$(f,g) = \frac{\partial f}{\partial z^i} g^{ij}\frac{\partial g}{\partial z^j}$$

degenerate?

∞-dim

$$\{A,B\} = \int \frac{\delta A}{\delta \psi^i} J^{ij}\frac{\delta B}{\delta \psi^j}\,dz \qquad \text{e.g.}$$

$$(A,B) = \int \frac{\delta A}{\delta \psi^i(z)} g^{ij}\frac{\delta B}{\delta \psi^j(z')}\,dz\,dz'$$

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Metriplectic System

$$\dot{z} = \{z, F\}_M = [z, F] + (z, F)$$

$$F = H + C = H + S$$

$$= E + TS$$

$\#$ $\quad \dot{H} = 0$

$$\dot{H} = [H, H + C] + (H, F)$$

$$= \underbrace{[H, H]}_{0} + \underbrace{[H, S]}_{0} + (H, H) + (H, S)$$

Build-In degeneracy

$$g^{ij} \frac{\partial H}{\partial z^j} = 0 \qquad \det g = 0$$

Projection

"H"-Theorem

$\#$ $\quad \dot{S} = \underbrace{[S, F]}_{0} + (S, F)$

$$\dot{S} = (S, S) \geq 0$$

Mostly It!

Free Rigid Body

$$\dot{L} = \omega \times L$$

$$L = I\omega \qquad L_i = I_i \omega_i \quad \text{(No sum)}$$

ang. mom. — mom. inertia — ang. vel

$$\dot{\omega}_i = \sum_{j,k} \epsilon_{ijk}\, \omega_j\, (I_k \omega_k) \qquad i = 1, 2, 3$$

$$\dot{\omega}_1 = \omega_2 \omega_3 \left(\frac{I_3 - I_2}{I_1} \right)$$

$$H = \sum_i \frac{I_i \omega_i^2}{2}$$

PB

$$[f, g] = \left(\frac{\partial f}{\partial \omega} \times \frac{\partial g}{\partial \omega} \right) \cdot \omega$$

$$= \sum_{i,j,k} \epsilon_{ijk} \frac{\partial f}{\partial \omega_j} \frac{\partial g}{\partial \omega_k} \omega_i$$

LP w/ SO(3)

Casimir (recall) $\qquad \epsilon_{ijk} = C^{ij}_{k}$

$$L^2 = \sum_i I_i^2 \omega_i^2$$

check $\quad \dot{\omega}_i = [\omega_i, H] = +\left(\omega \times \frac{\partial H}{\partial \omega}\right)_i = +\epsilon_{ijk}\, \omega_j\, (I_k \omega_k)$ ✓

$$\dot{L^2} = 0 \Rightarrow \boxed{L^2 = \text{Entropy}}$$

$$(f,g) = -\sum_{i,j,k=1}^{3} z_k \left[\frac{\partial H}{\partial z^i}\frac{\partial H}{\partial z^j} - \delta_{ij}\frac{\partial H}{\partial z^k}\frac{\partial H}{\partial z^k} \right]$$

wrongHt

$$= g^{ij}\frac{\partial f}{\partial z^j}\frac{\partial g}{\partial z^j} \qquad \frac{\partial f}{\partial z^i}\frac{\partial g}{\partial z^j}$$

$$g^{ij} = \left[\nabla H \otimes \nabla H - \|\nabla H\|^2 I\right]_{ij}$$

FRB

$\boxed{\text{Birds}}$

stable ques

Vlasov-Landau 4

$$\frac{\partial f}{\partial t} + v\cdot\frac{\partial f}{\partial x} - E\cdot\frac{\partial f}{\partial v} = \left.\frac{\partial f}{\partial t}\right)_c$$

Ham. Part (PJM 1980)

$$\frac{\partial f}{\partial t} + [\mathcal{E}, f] = 0 \qquad \mathcal{E} = \frac{v^2}{2} - \phi \qquad \phi_x = -E$$

$$\{F, G\} = \int \frac{\delta F}{\delta f}\, \mathcal{J}\, \frac{\delta G}{\delta f}\, dx\, dv =$$

(1980)

$$\mathcal{J} = -[f, \cdot\,]$$

structure operator for sp. can. trans.

$$\{F, G\} = \int \left[\frac{\delta F}{\delta f}, \frac{\delta G}{\delta f}\right] f \; dx\, dv$$

$$H = \int \frac{v^2}{2} f + \int \frac{E^2}{2} = \int \frac{v^2}{2} f + \frac{1}{2}\int\!\!\int K\, f f$$

$$\frac{\delta H}{\delta f} = \mathcal{E} \qquad \frac{\partial f}{\partial t} = \{f, H\} = -[f, \mathcal{E}] \quad \checkmark$$

$$\frac{\partial f}{\partial t} = \{f, F\}_m = \{f, F\} + (f, F)$$

$$F = H + S$$

$$S[f] = \int \mathcal{C}(f)\, dx\, dv$$

candidate entropies

$$E = 0$$

$$\Rightarrow \left[f, \frac{v^2}{2}\right] = -\frac{\partial f}{\partial x}\cdot v$$

Generalized
Landau - Lenard - Balescu

$$(a,b) = \frac{\partial a}{\partial z^i} g^{ij} \frac{\partial a}{\partial z^j}$$

$$(A,B) = -\iint \left[\frac{\partial}{\partial v_i} \frac{\delta A}{\delta f(z)} - \frac{\partial}{\partial v_i'} \frac{\delta B}{\delta f(z')} \right] \times \left[\frac{\partial}{\partial v_j} \frac{\delta B}{\delta f(z)} - \frac{\partial}{\partial v_j'} \frac{\delta B}{\delta f(z')} \right] T_{ij}(z,z')\, dz\, dz'$$

$$\boxed{T_{ij} = \frac{w_{ij}}{2}(z,z')\, M(f(z))\, M(f(z'))}$$

$w_{ij}(z,z') = w_{ji}(z',z)$ etc

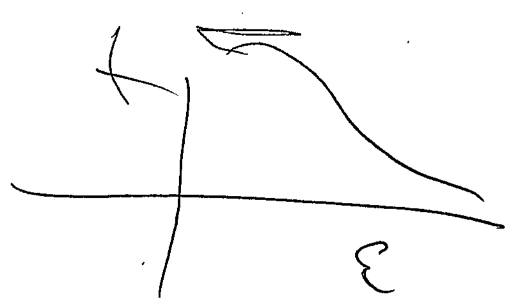
$$S[f] = \int s(f)\, dz$$

$$\boxed{\frac{d^2 s}{df^2} M = 1}$$ Matching

$$\frac{\delta}{\delta f}(H + S) = 0 \Rightarrow$$

$$\varepsilon_p + s'(f) = 0$$

$$f = (s')^{-1}(-\varepsilon)$$ monotone



choose

$f \Rightarrow s' \Rightarrow s$

$\Rightarrow s'' \Rightarrow M$

$\Rightarrow$ $\boxed{\text{Formal H-Theorem}}$

So What?

One lesson: Make up models
w/ H-Term