Statistical-mechanical forcing of ocean circulation

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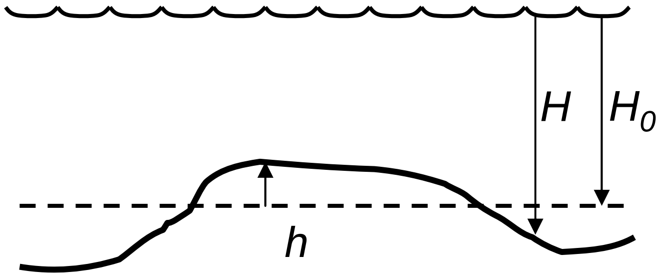
Canadian Centre for Climate Modelling and Analysis

Origins

Salmon, R., G. Holloway and M. C. Hendershott, 1976: The equilibrium statistical mechanics of simple quasi-geostrophic models. *J. Fluid Mech.*, 75, 691-703.

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0, \qquad q = \nabla \times \mathbf{u} + h$$

$$h = \frac{f(H_o - H)}{H_o}$$



Conserves E, moments of $Q_n = \int q^n \, dA$ (circulation, enstrophy…)

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- **Inviscid $\rightarrow$ cons of energy, enstrophy**
- **Spectrally truncated**
- **Equilibrium via maximization of entropy**

"…equilibrium…flow is positively correlated with bottom topography (anticyclonic flow over seamounts)"

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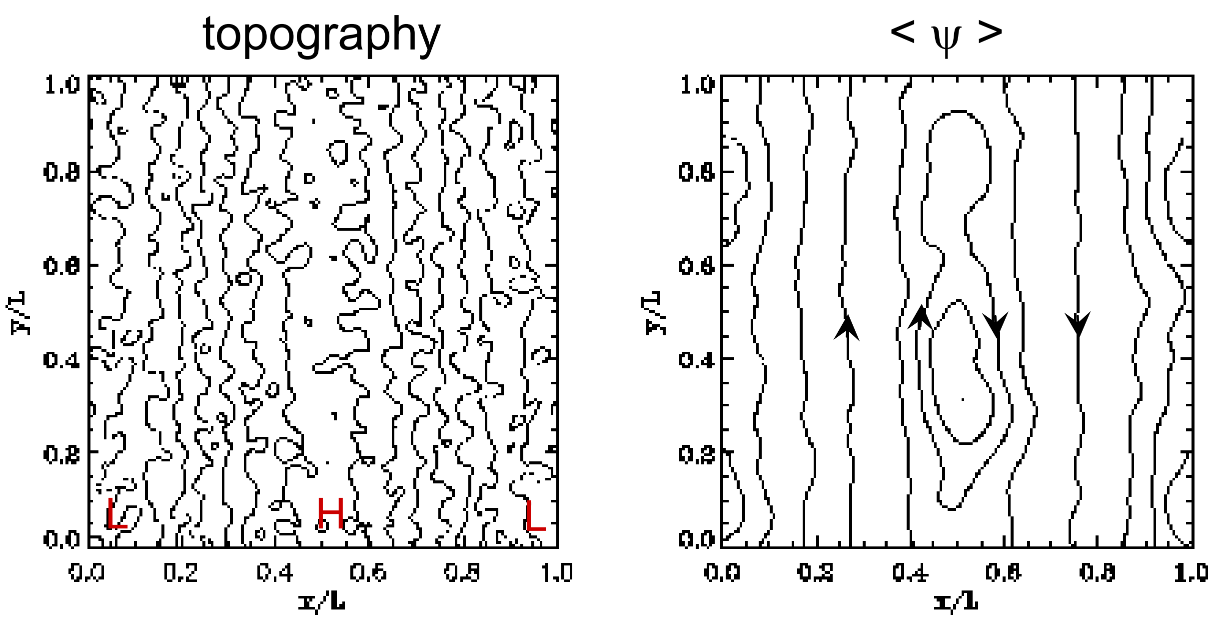
$$\boxed{\mu \langle \psi \rangle = \langle q \rangle}$$

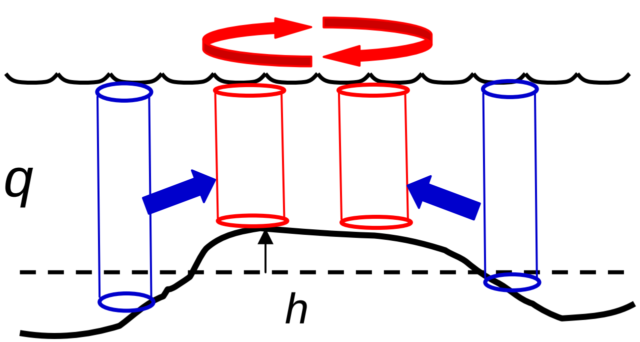
Example: circulation over bumpy ridge

- Consider 100 realizations from random initial conditions
- Ensemble mean flow develops on ~ eddy turnover time



Flow in *pseudowestward* direction (shallower water on right in NH)

- Physics is advective rearrangement of PV, constrained by conservation of *E*, moments of *q*



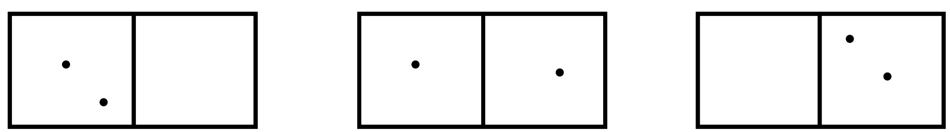
What is Entropy?

- **Entropy is the *INFORMATION DEFICIT* between detailed knowledge *(microstate)* and statistical knowledge *(macrostate)* of a system**

Simple example

- N particles in a partitioned box: $N = n_L + n_R$
- $N=2 \rightarrow 2^2=4$ possible configurations

Macrostate

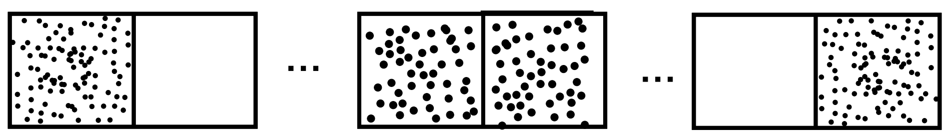


	n_L n_R 2 0	n_L n_R 1 1	n_L n_R 0 2
# Microstates $\frac{N!}{n_L!n_R!}$	1	2	1
Information deficit (bits)	**0**	**1**	**0**

Simple example

- ***N*** **particles in a partitioned box:** $N = n_L + n_R$
- N**=100** $\rightarrow$ 2^{100} **possible configurations**

Macrostate

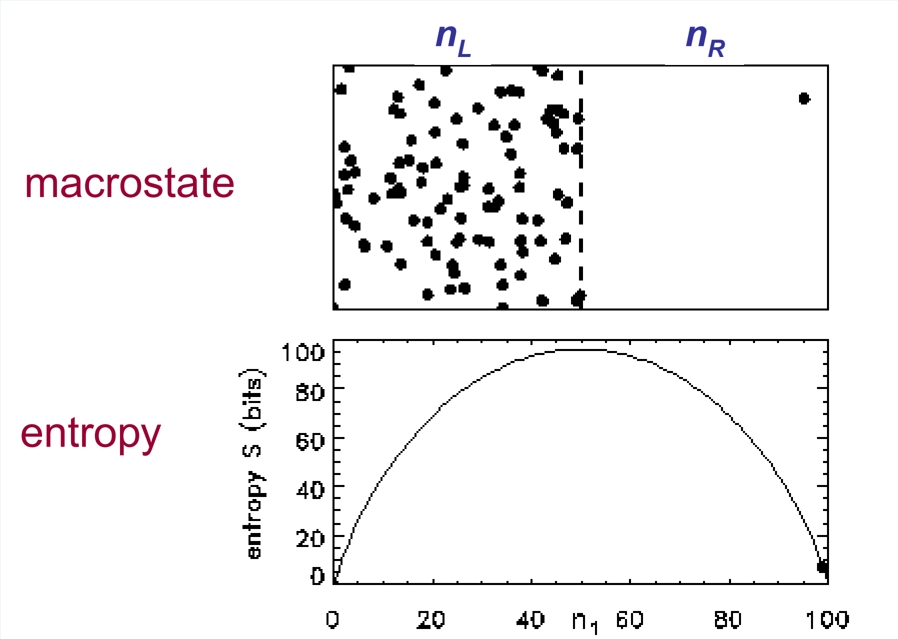


	n_L 100, n_R 0	… n_L 50, n_R 50 …	n_L 0, n_R 100
# Microstates $\dfrac{N!}{n_L!n_R!}$	1	1.0×10^{29}	1
Information deficit (bits)	**0**	**97**	**0**

Simple example

- **Now introduce *time dependence:***

probability *X* (unit time)$^{-1}$ a particle will undergo *transition* between two boxes



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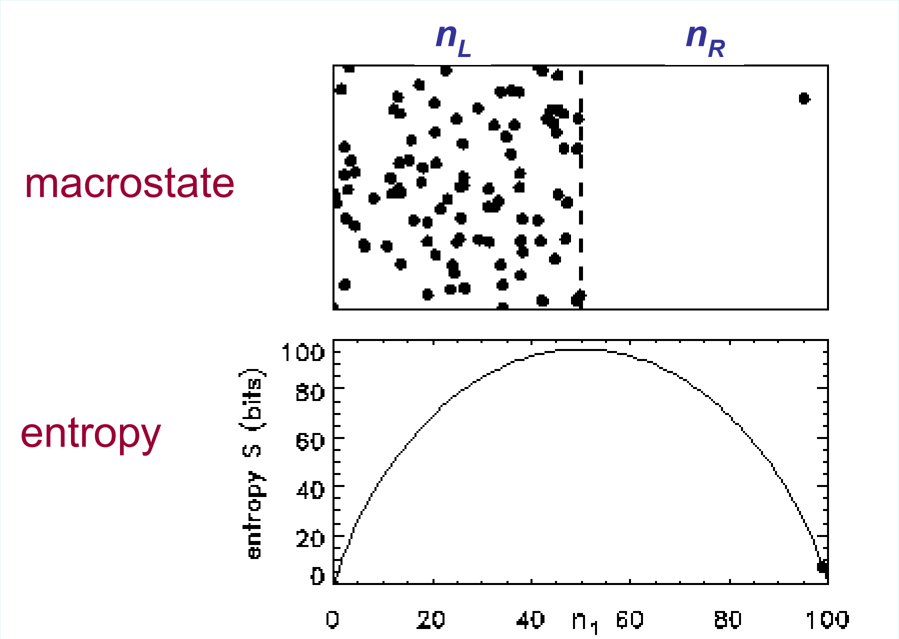
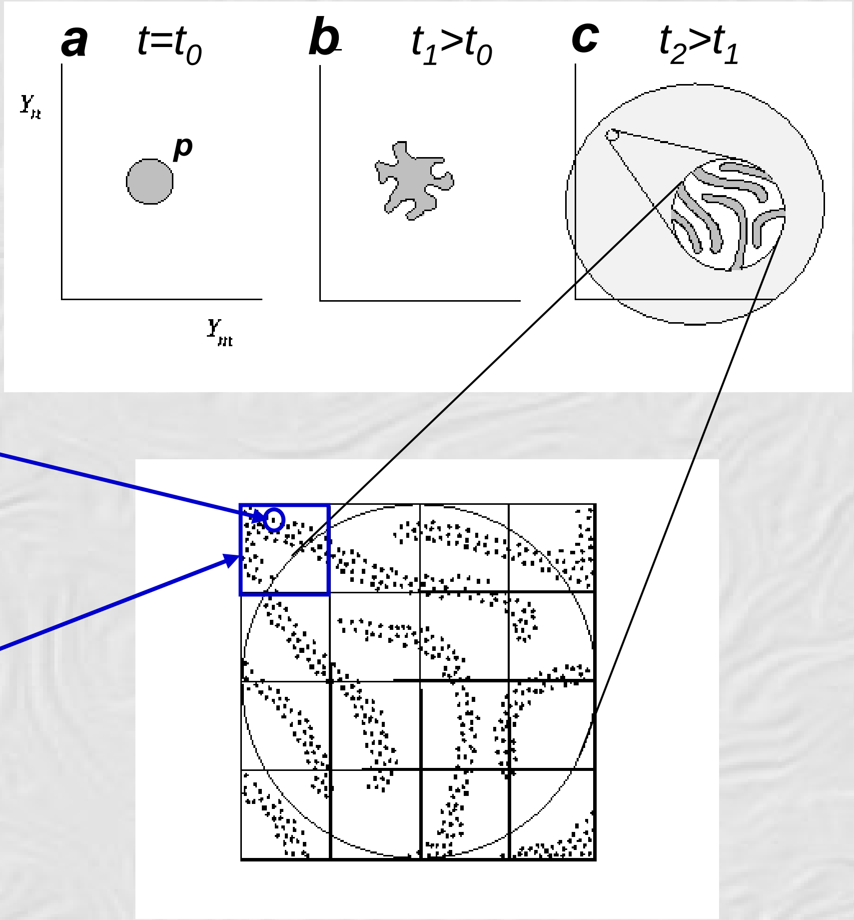


- **dS/dt $\geq$ 0 in ensemble mean $\rightarrow$ *arrow of time***
- **Maximizing *S* subject to constraints (n_L + n_R = *N here)* locates *equilibrium***

Defining entropy for GFD

- To define ***microstates***, represent *p* by sampling of *N* realizations (*N* sufficiently large to resolve finest scales of *p*)

- To define ***macrostates***, divide phase space into *M* observably distinguishable cells, consider *observable* PDF $P_i = n_i / N$



Defining entropy for GFD

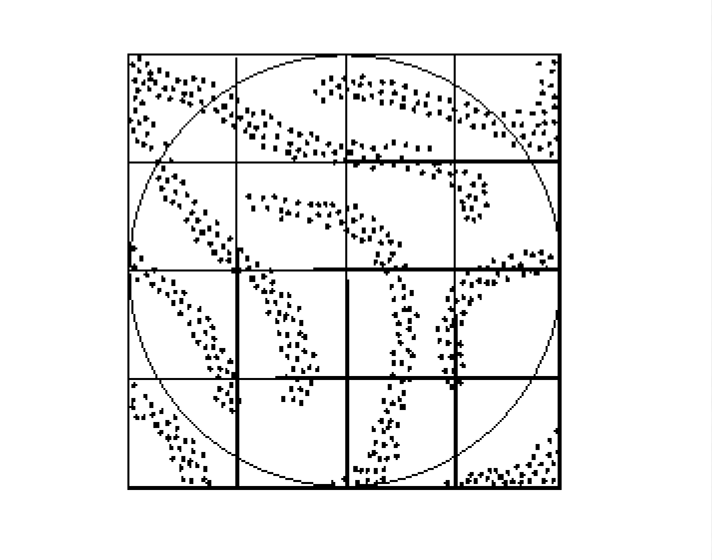
- *Multiplicity* (# microstates for given macrostate) is

$$w = \frac{N!}{\prod_i n_i!}$$

- Corresponding *information deficit* or *entropy* is

$$S = k \log w$$

(k = 1/log 2 if in bits)



Defining entropy for GFD

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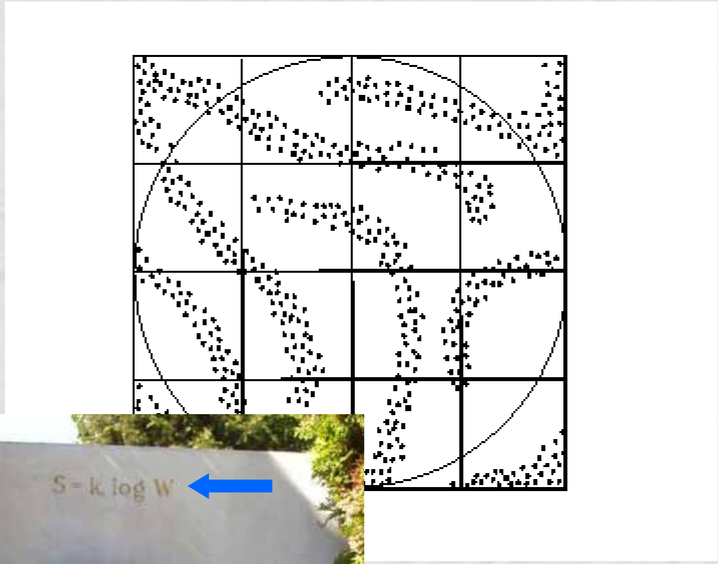

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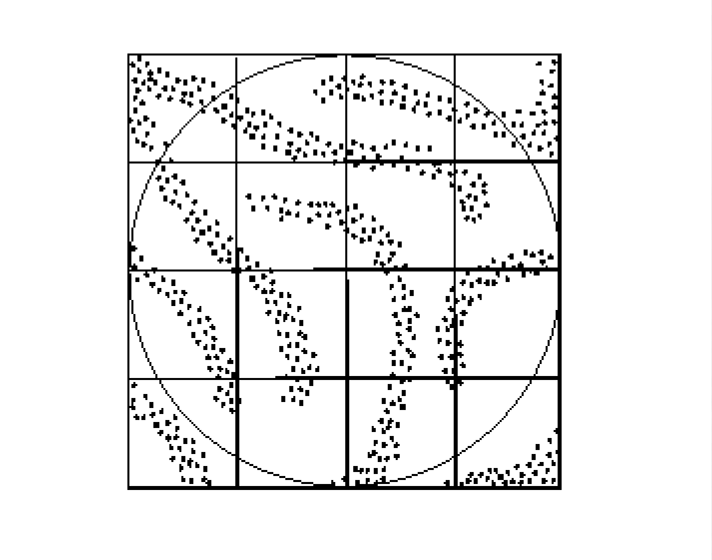
- Using $\log n_i \approx n_i \log n_i - n_i \,,\; P_i = n_i / N$

$$S = - k \, \Sigma_i \, P_i \log P_i$$

or

$$\boxed{S = - k \int P \log P \, dY}$$



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"…equilibrium…flow is positively correlated with bottom topography (anticyclonic flow over seamounts)"

$$\boxed{\mu\langle\psi\rangle = \langle q\rangle}$$

Bretherton, F. P. and D. Haidvogel, 1976: Two-dimensional turbulence above topography. *J. Fluid Mech.*, 78, 129-154.

- Small-scale dissipation dissipates enstrophy, conserves energy
- No explicit truncation
- Equilibrium via minimization of enstrophy

"…initially turbulent flow tends to a steady state with streamlines parallel to contours of constant depth, anticyclonic around a bump"

$$\boxed{\mu\psi = q}$$

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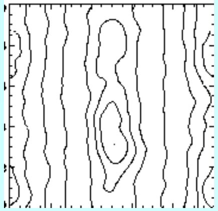
Carnevale, G. F. and J. S. Frederiksen, 1987: Nonlinear stability and statistical mechanics of flow over topography. *J. Fluid Mech.,* 175, 157-181*.*

"…in the limit of infinite resolution the canonical mean state is statistically sharp, that is, without eddy energy on any scale, and is identical to the nonlinearly stable minimum enstrophy state"

SHH76



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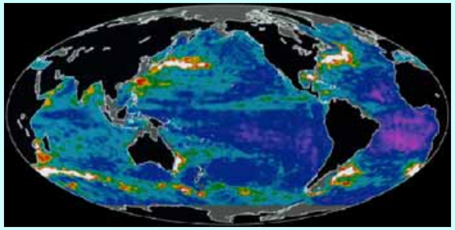
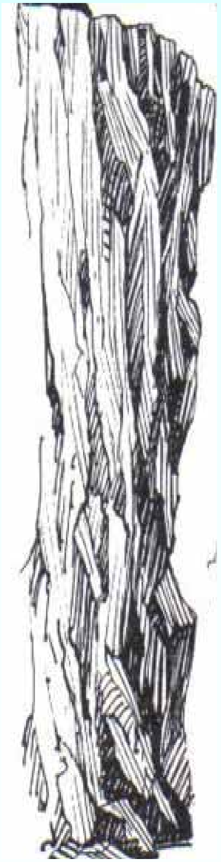
World Ocean





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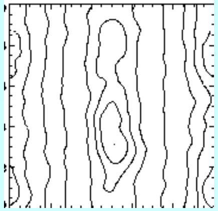

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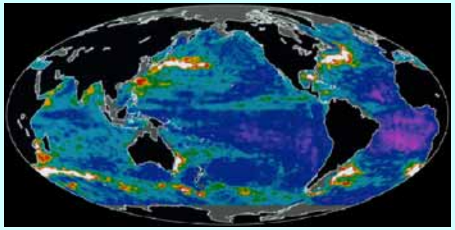
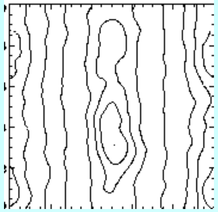


Equilibrium→ Disequilibrium

SHH76



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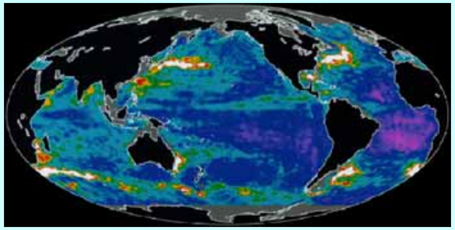

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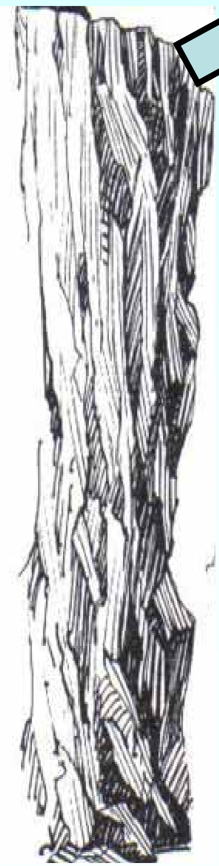


Holloway (*JFM* 1978)

Carnevale, Frisch & Salmon (*J Phys* 1981)

Moment closure implies d*S*/dt≥0

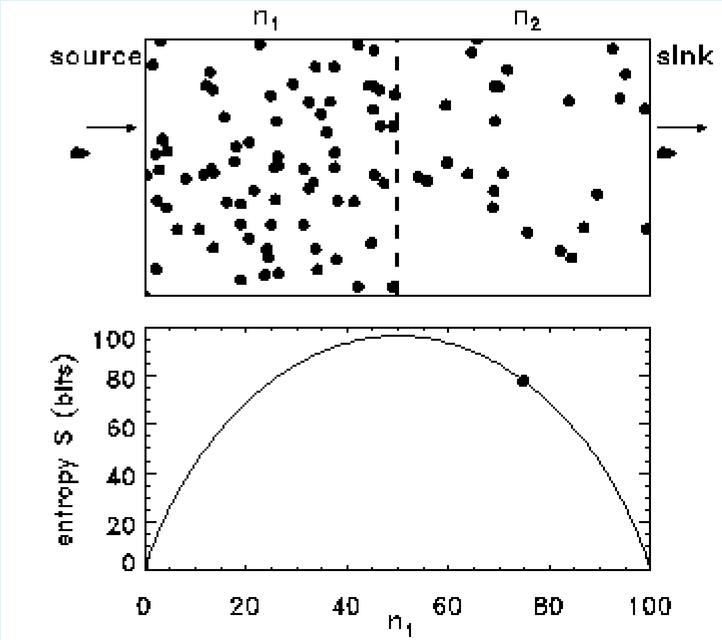
Treguier (*GAFD* 1989)

Merryfield & Holloway (*JFM* 1997)

$\mu\langle\psi\rangle = \langle q\rangle$ **continues to ~hold if eddy timescales < forcing, dissipative timescales**

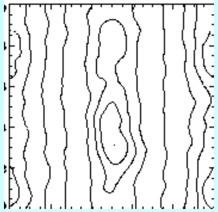
Simple example revisited

- Remove *constant-N constraint* by adding *A* particles s^{-1} to the left compartment, and extracting particles from the right compartment at the same rate
- This external forcing prevents equilibrium from being attained. However, the *tendency* for *S* increase due to random "dynamics" persists



➡ entropy increase tendency acts as "force" which balances applied force in statistical equilibrium

SHH76



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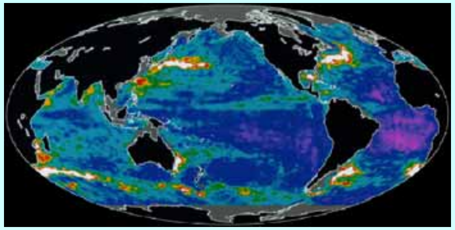
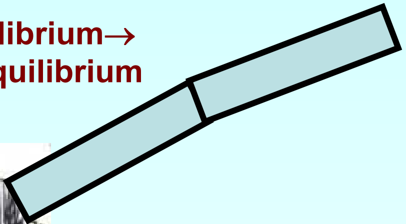
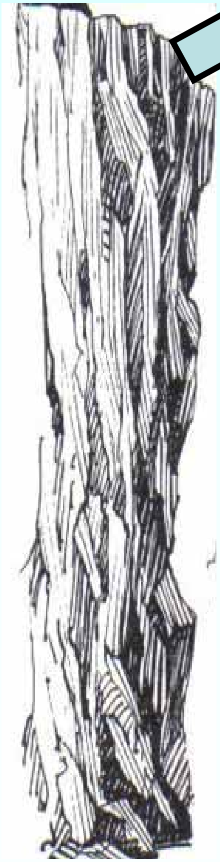


Higher-order invariants $\int q^n \, dA,\ n \geq 3$ **(typically not conserved by numerics)**

Equilibrium→ Disequilibrium





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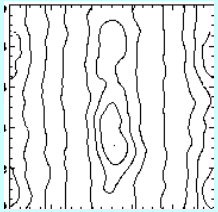

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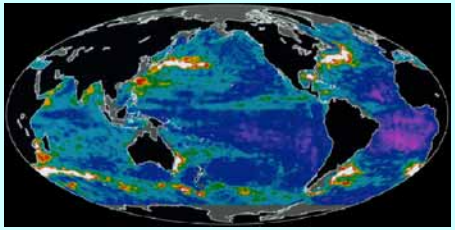

Higher-order invariants $\int q^n \, dA$, $n \geq 3$ **(typically not conserved by numerics)**

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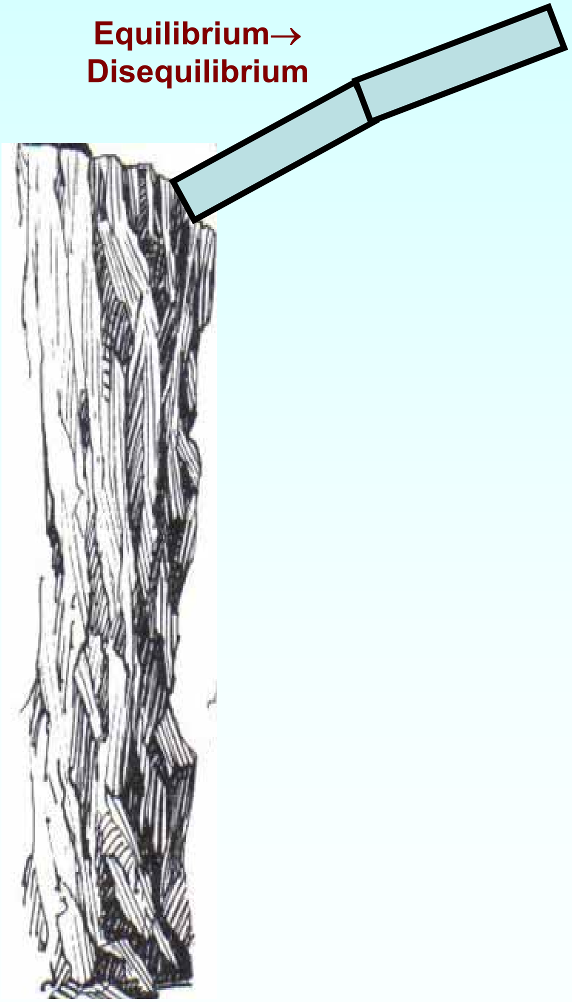
$$\boxed{\langle q \rangle = \mu \langle \psi \rangle \longrightarrow \langle q \rangle = f(\langle \psi \rangle)}$$



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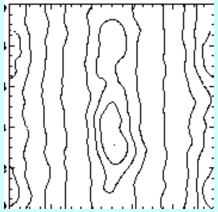

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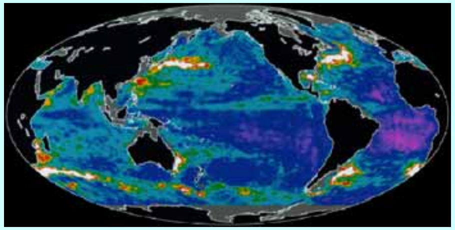

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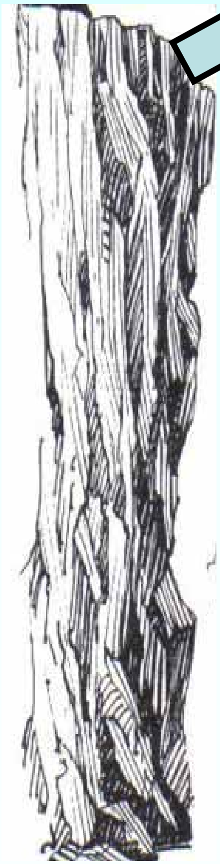
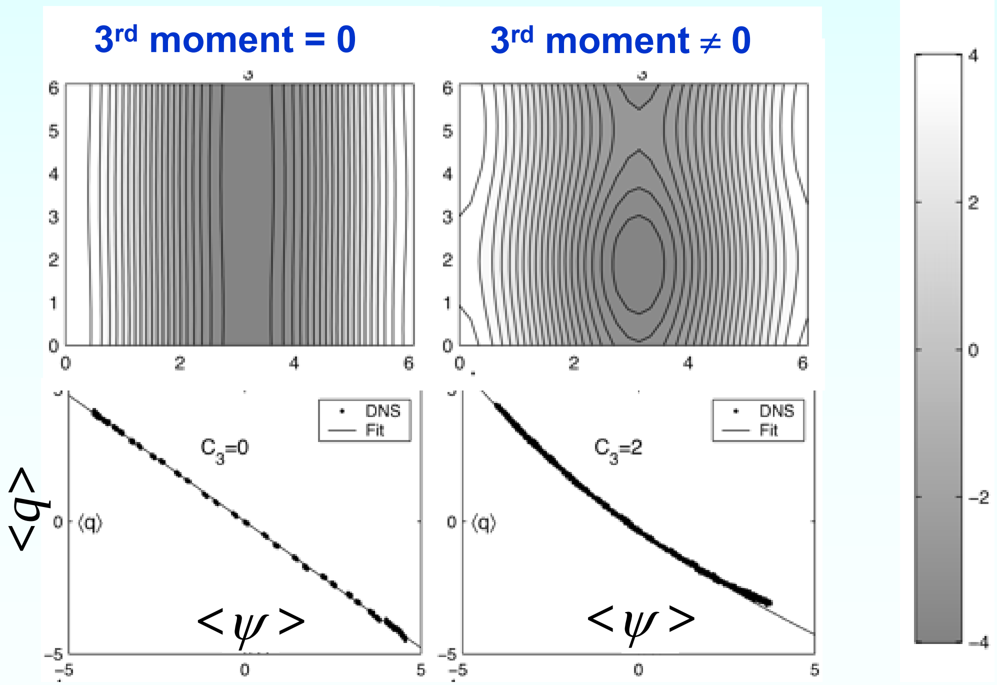


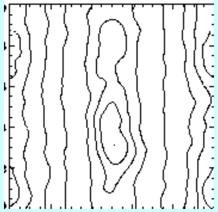
$$\boxed{\langle q \rangle = \mu \langle \psi \rangle \;\longrightarrow\; \langle q \rangle = f(\langle \psi \rangle)}$$

Abramov and Majda (PNAS 2003): **special numerics to conserve many moments**



→ ***n* = 3 statistically relevant, *n* >3 not**

SHH76



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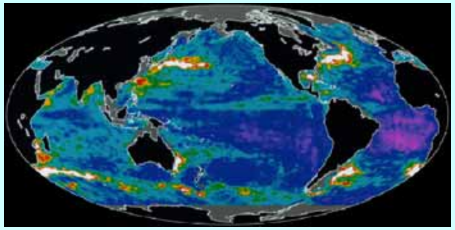
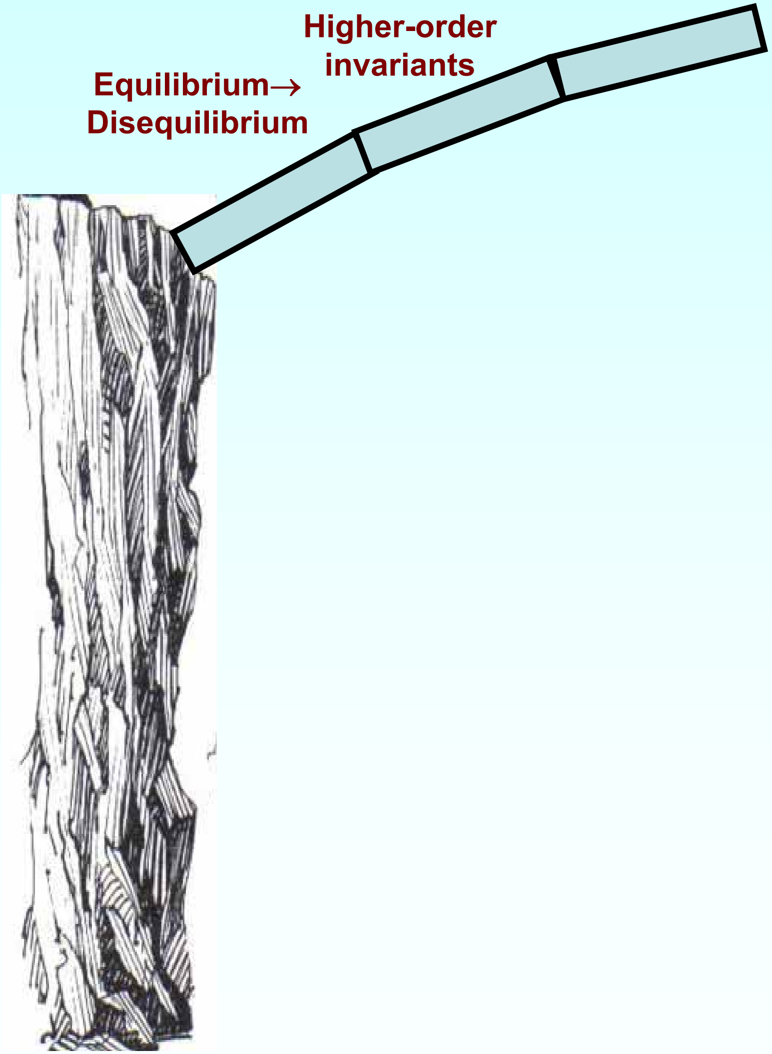


Stratification



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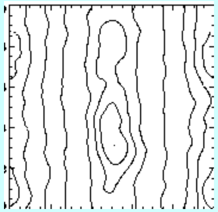

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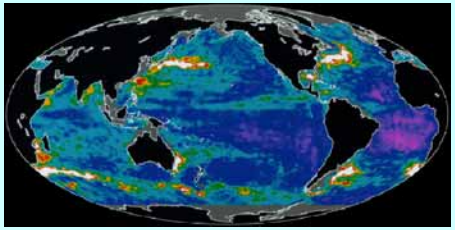
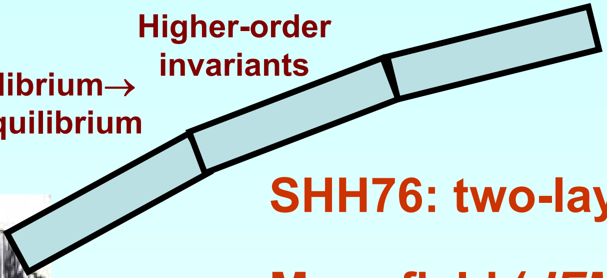
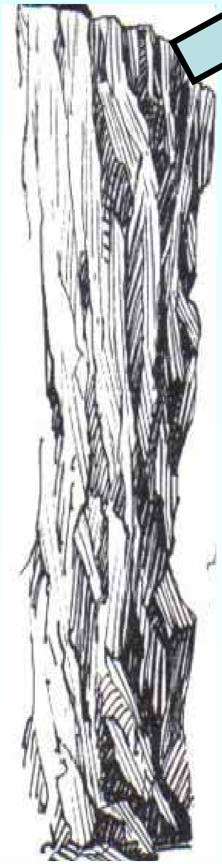


Stratification

Higher-order invariants

Equilibrium→ Disequilibrium

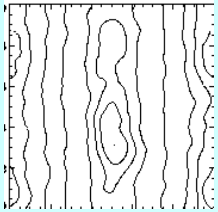




SHH76: two-layer QG

Merryfield (*JFM* 1998): arbitrary $N(z) \rightarrow \mu(z)\overline{\psi}(z) = \overline{q}(z)$

SHH76



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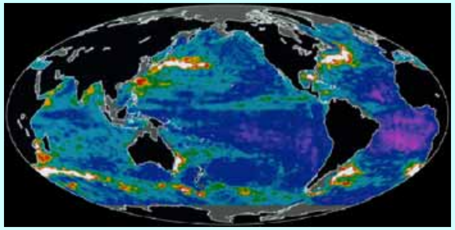

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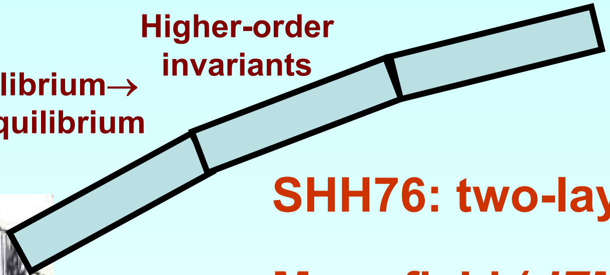

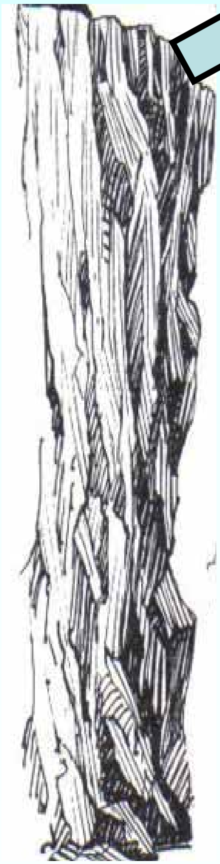

- mean flows **bottom intensified**
- isopycnals ***domed*** over bumps
- barotropic transport ***independent of N(z)***

SHH76: two-layer QG

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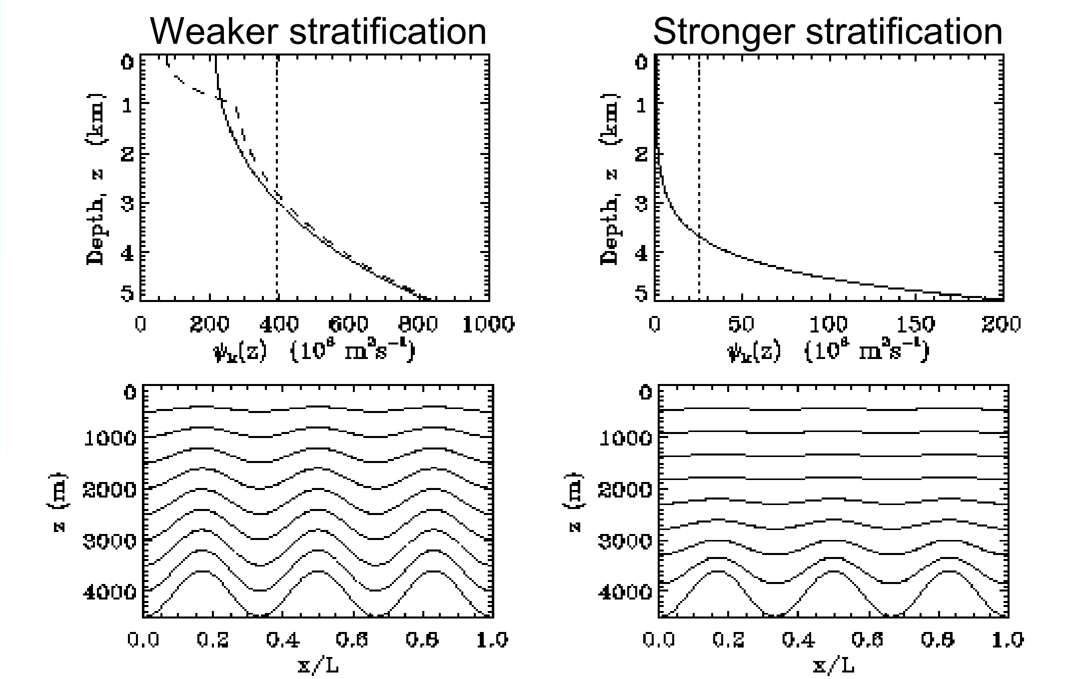
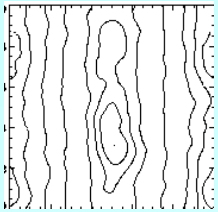


SHH76



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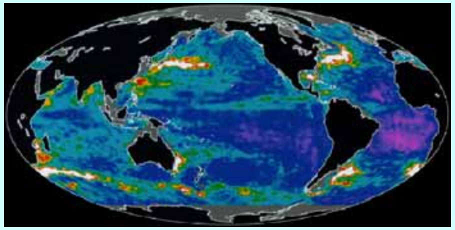
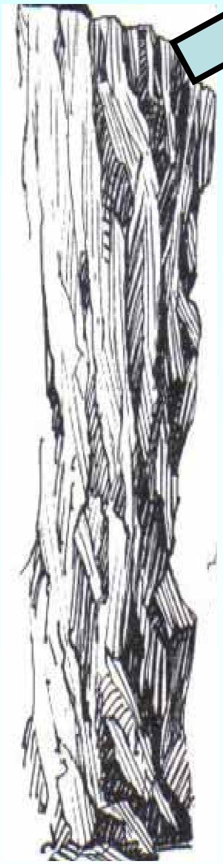
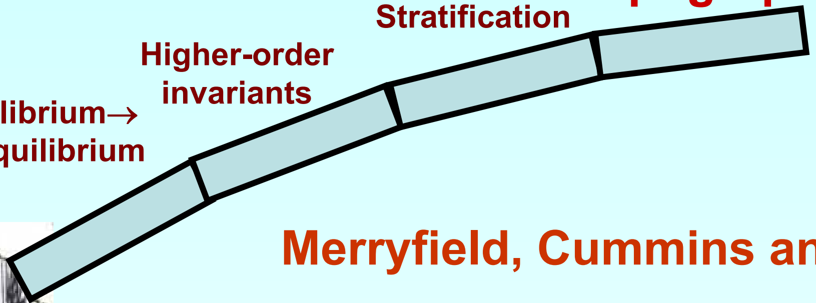
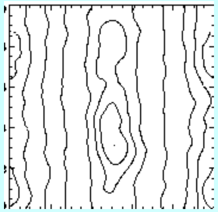






Merryfield, Cummins and Holloway (JPO 2001)

SHH76



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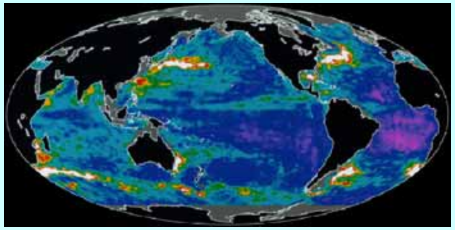
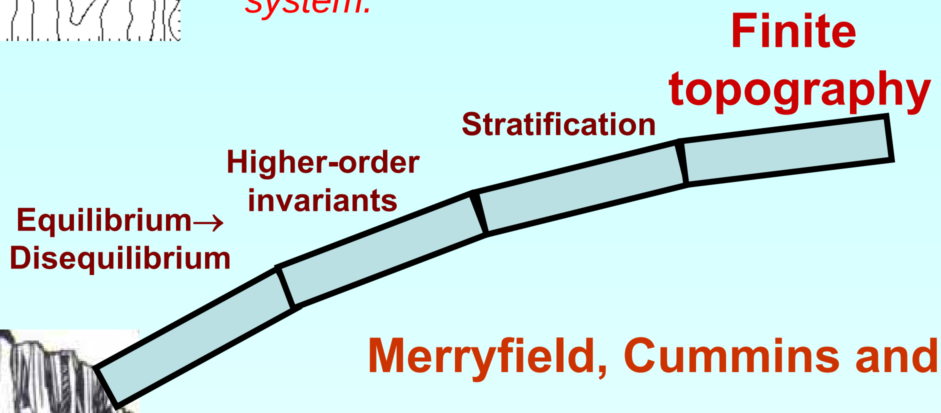
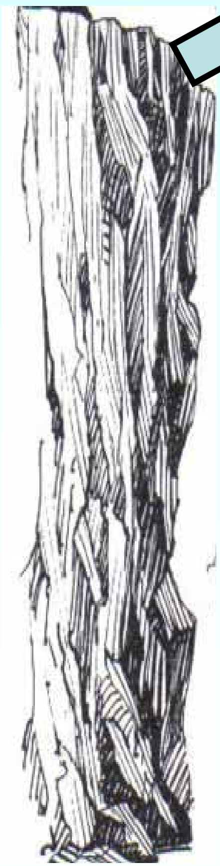






Merryfield, Cummins and Holloway (JPO 2001)

Barotropic vorticity equation:

$$\frac{\partial q}{\partial t} + \mathbf{u} \cdot \nabla q = 0,$$

where

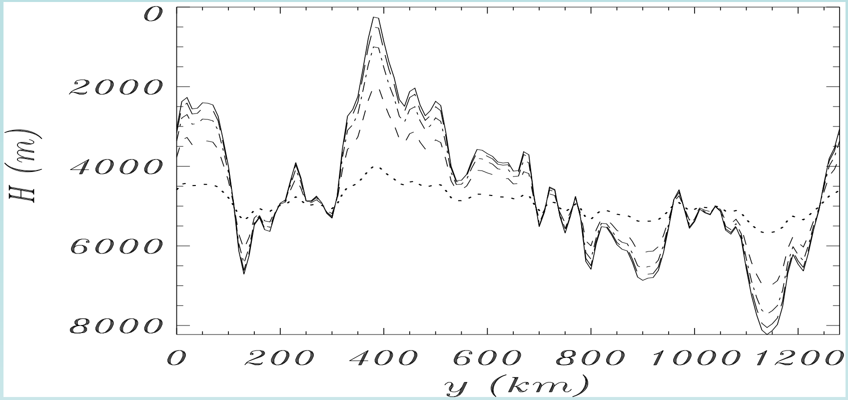
$$q = \frac{\zeta + f}{H}, \qquad \zeta = \nabla \cdot H^{-1} \nabla \Psi$$

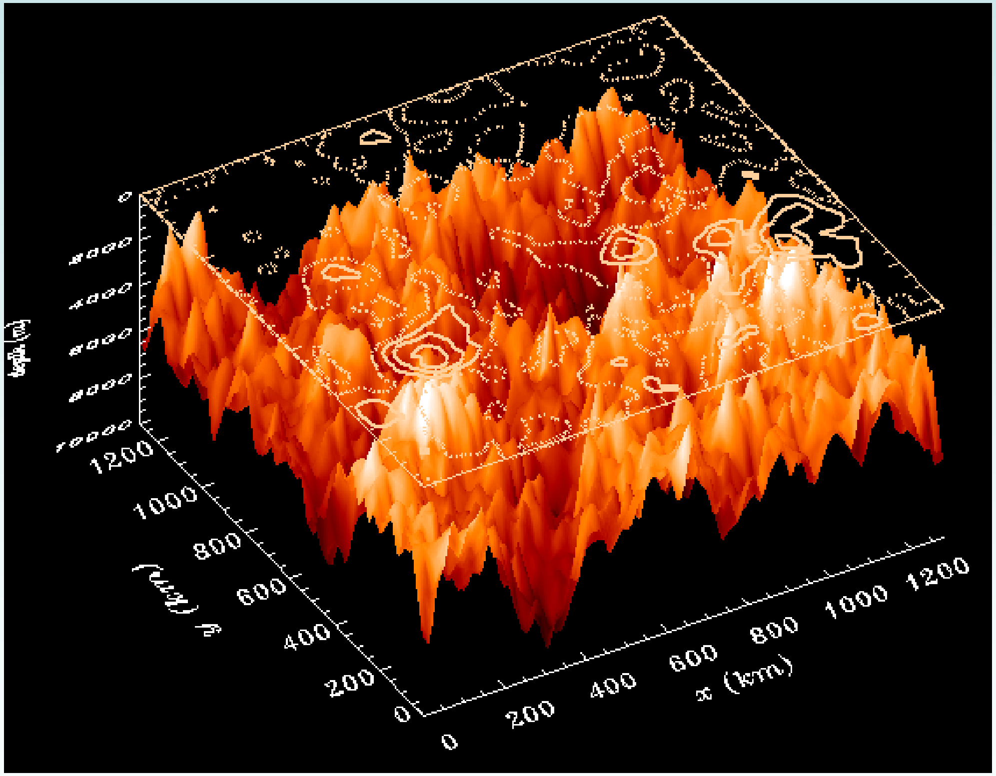
Discretize in space, letting $\Delta x \to 0$

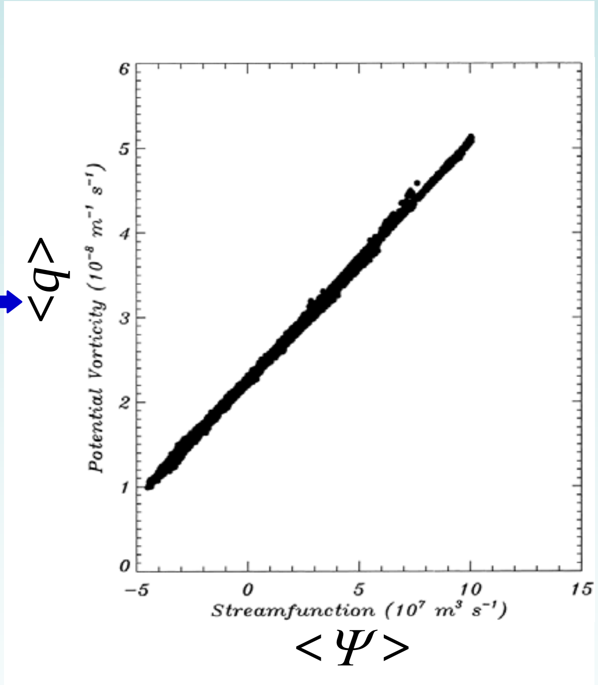
$$\rightarrow \boxed{\mu \langle \Psi \rangle + C = \langle q \rangle}$$

Numerical test

Cross section of random topography





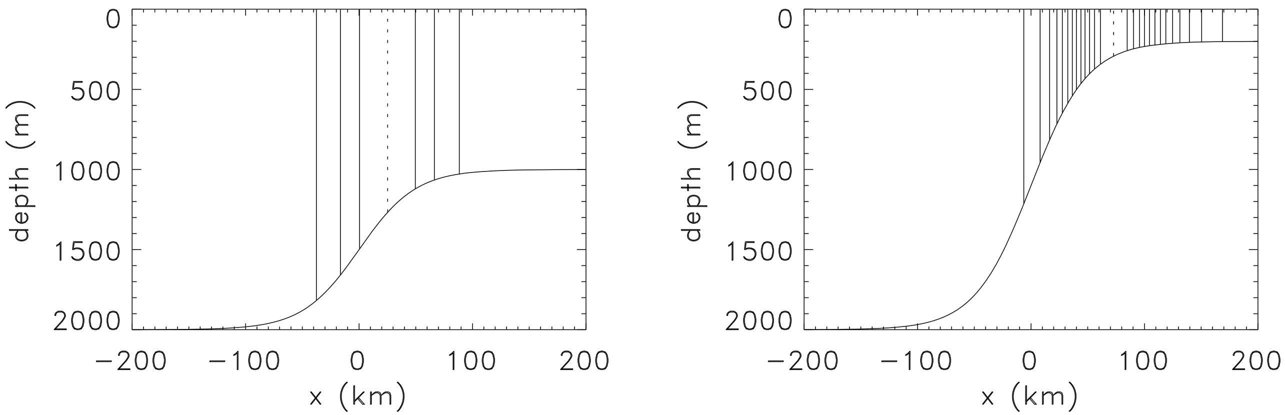


Barotropic streamfunction over topography

Merryfield, Cummins and Holloway (JPO 2001)

Equilibrium flow along a shelf

Unstratified

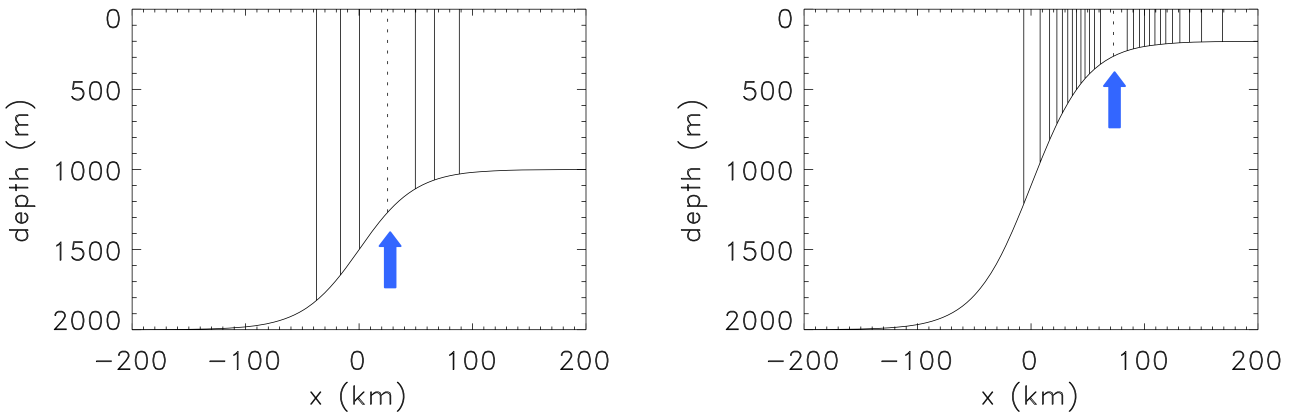


Merryfield, Cummins and Holloway (JPO 2001)

Equilibrium flow along a shelf

➡ **Maximum current *upslope* of maximum slope**

Unstratified



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Equilibrium flow along a shelf

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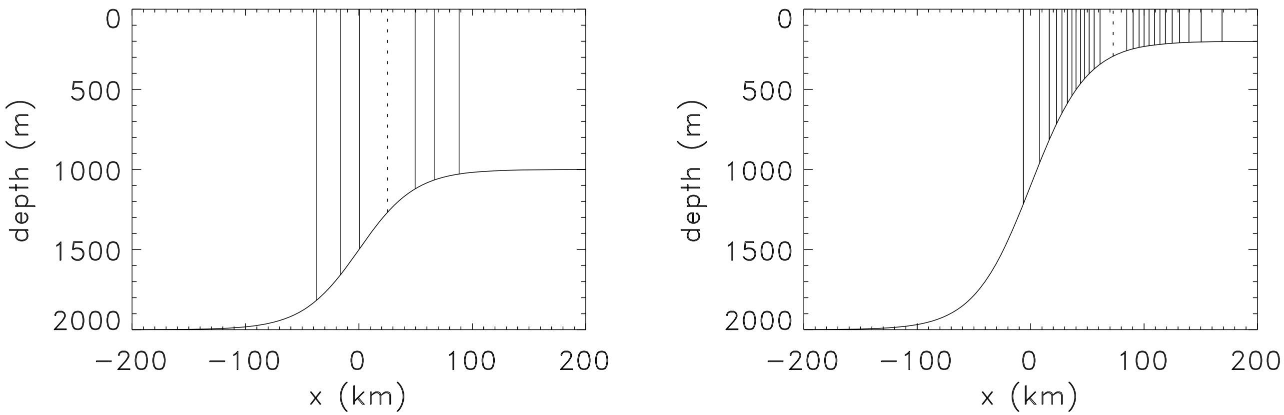
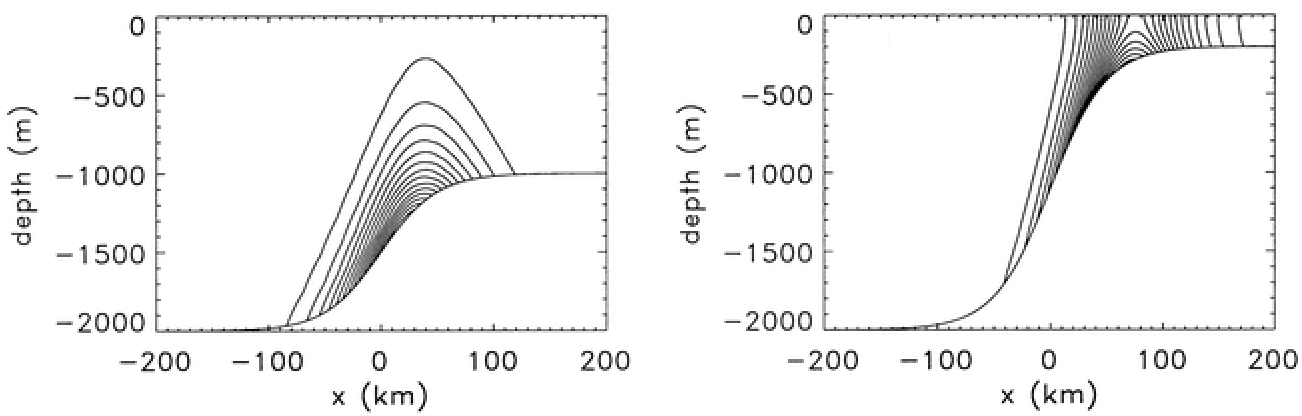
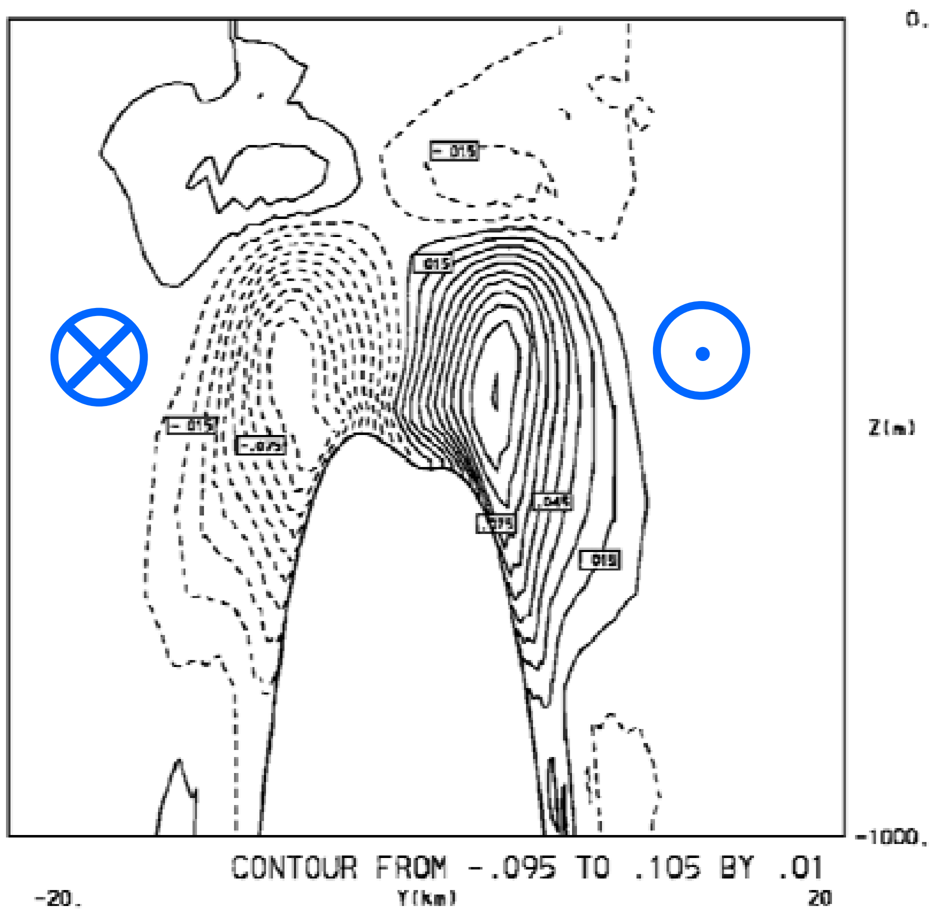


With depth dependence as in QG



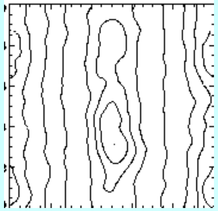
Merryfield, Cummins and Holloway (JPO 2001)

Rectified flow over a seamount



Beckmann and Haidvogel (JGR 1997)

SHH76



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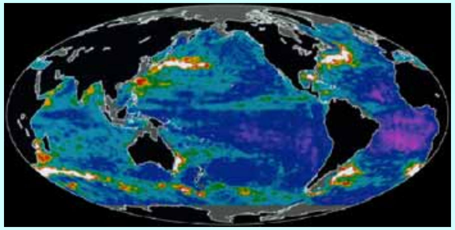
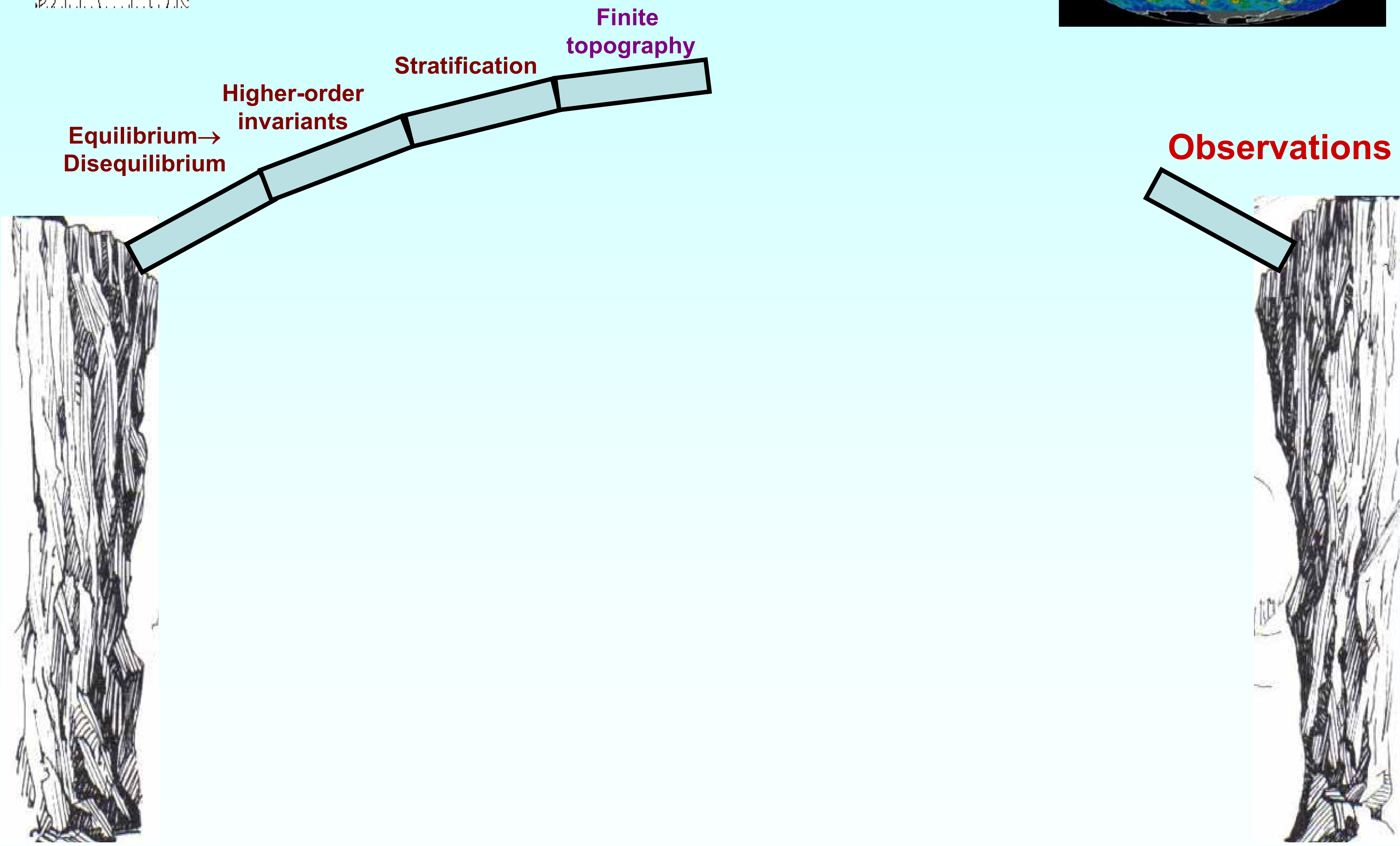
World Ocean





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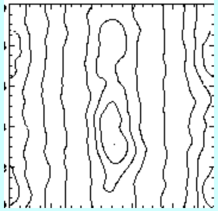

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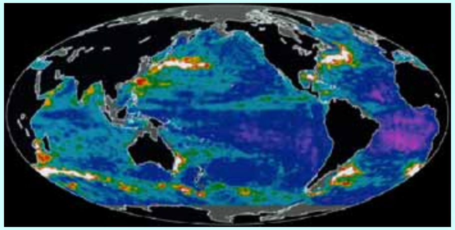
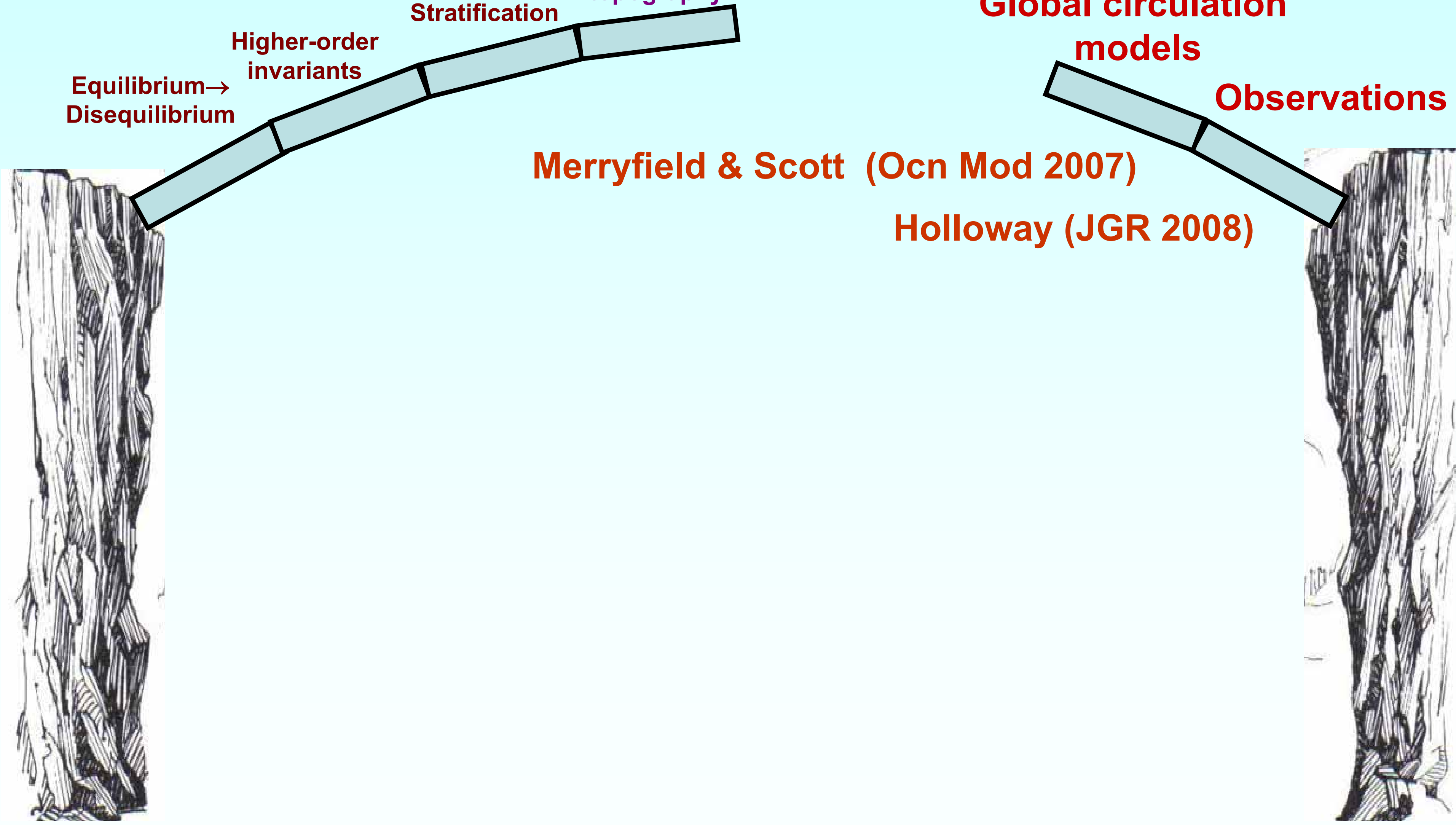
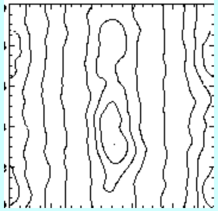




Merryfield & Scott (Ocn Mod 2007)

Holloway (JGR 2008)

SHH76



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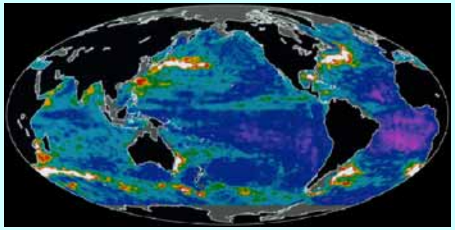
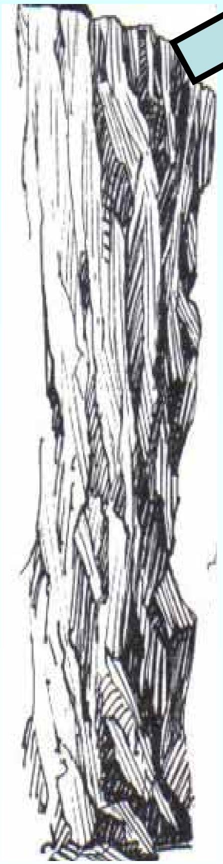
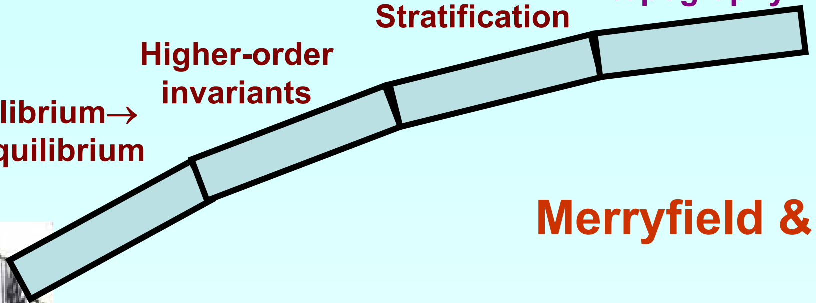
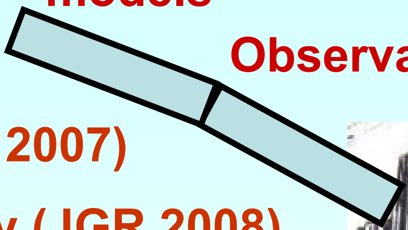






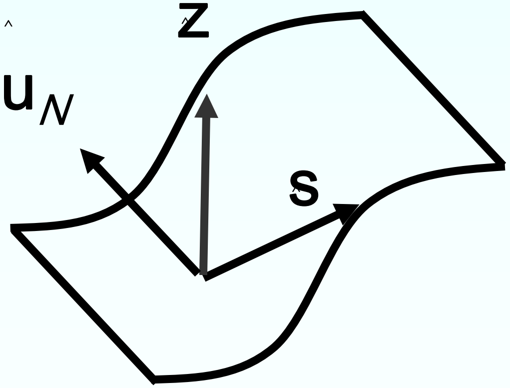


Merryfield & Scott (Ocn Mod 2007)

Holloway (JGR 2008)

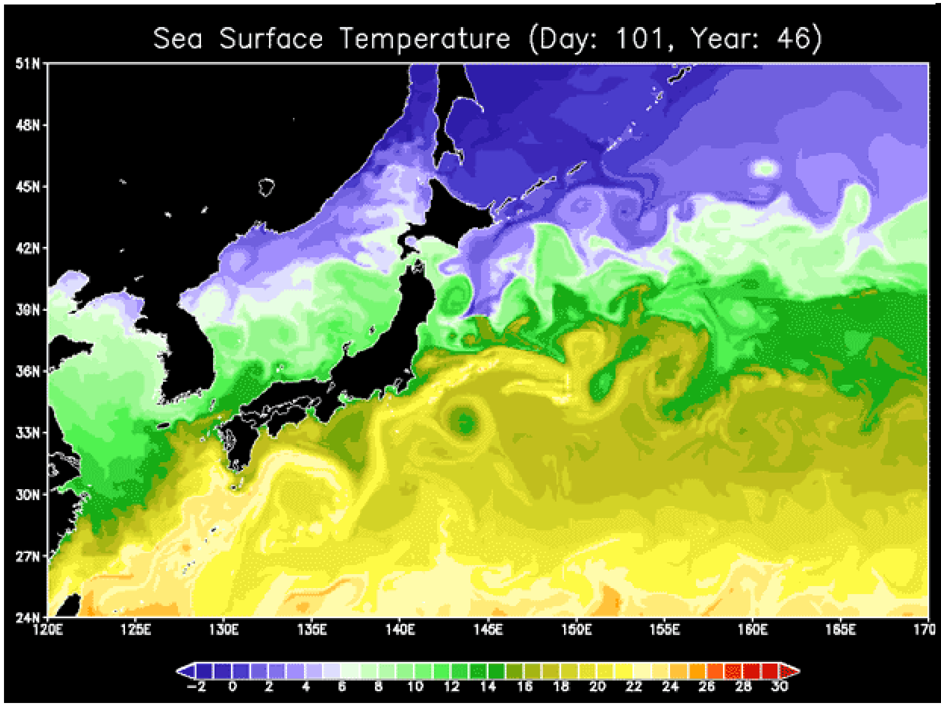
Diagnose pseudo-westward tendency using *topostrophy*

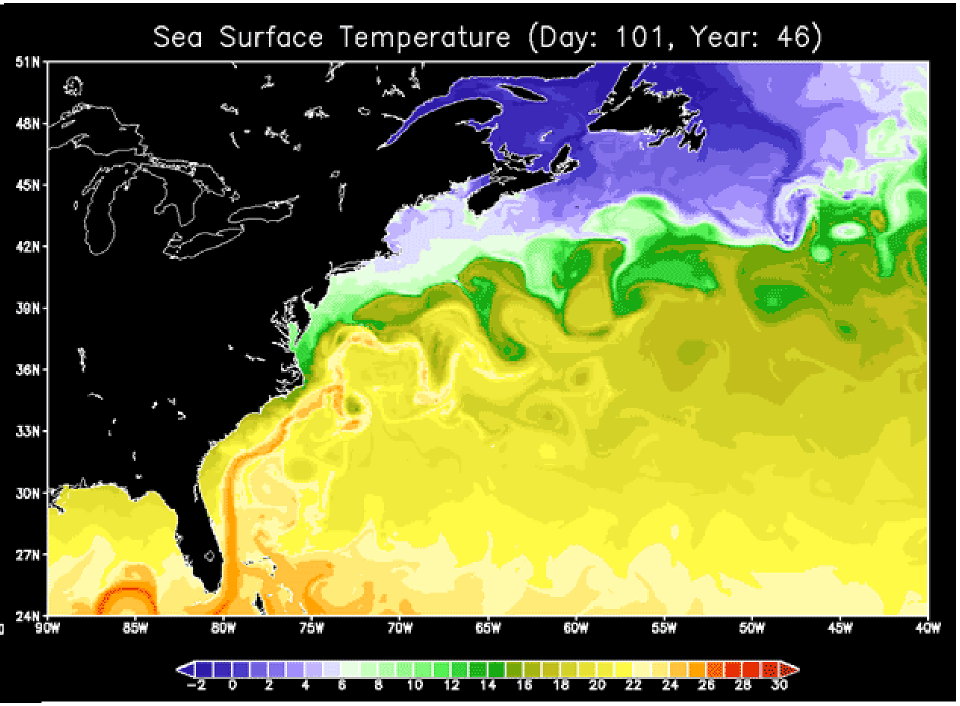
$$T = \frac{\int \hat{\mathbf{u}}\cdot\hat{\mathbf{u}}_N |\mathbf{s}||f|\,dV}{\int |\mathbf{s}||f|\,dV}$$

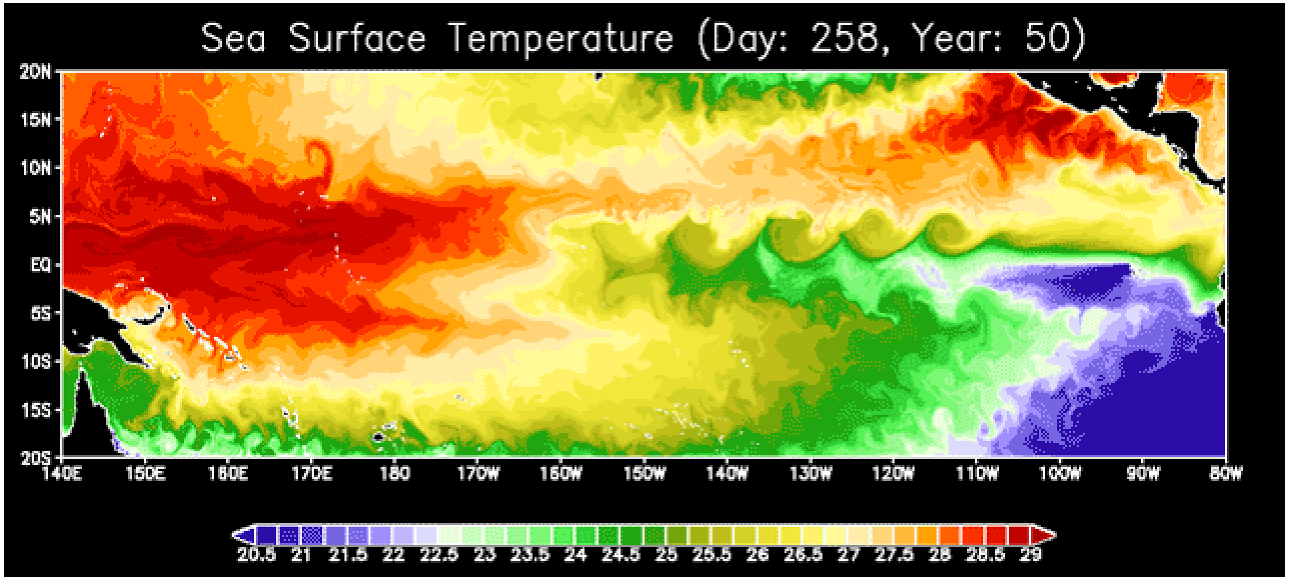


$$\boxed{-1 \le T \le 1}$$

Example of eddy-active ocean model: Ocean Model for Earth Simulator (OFES)

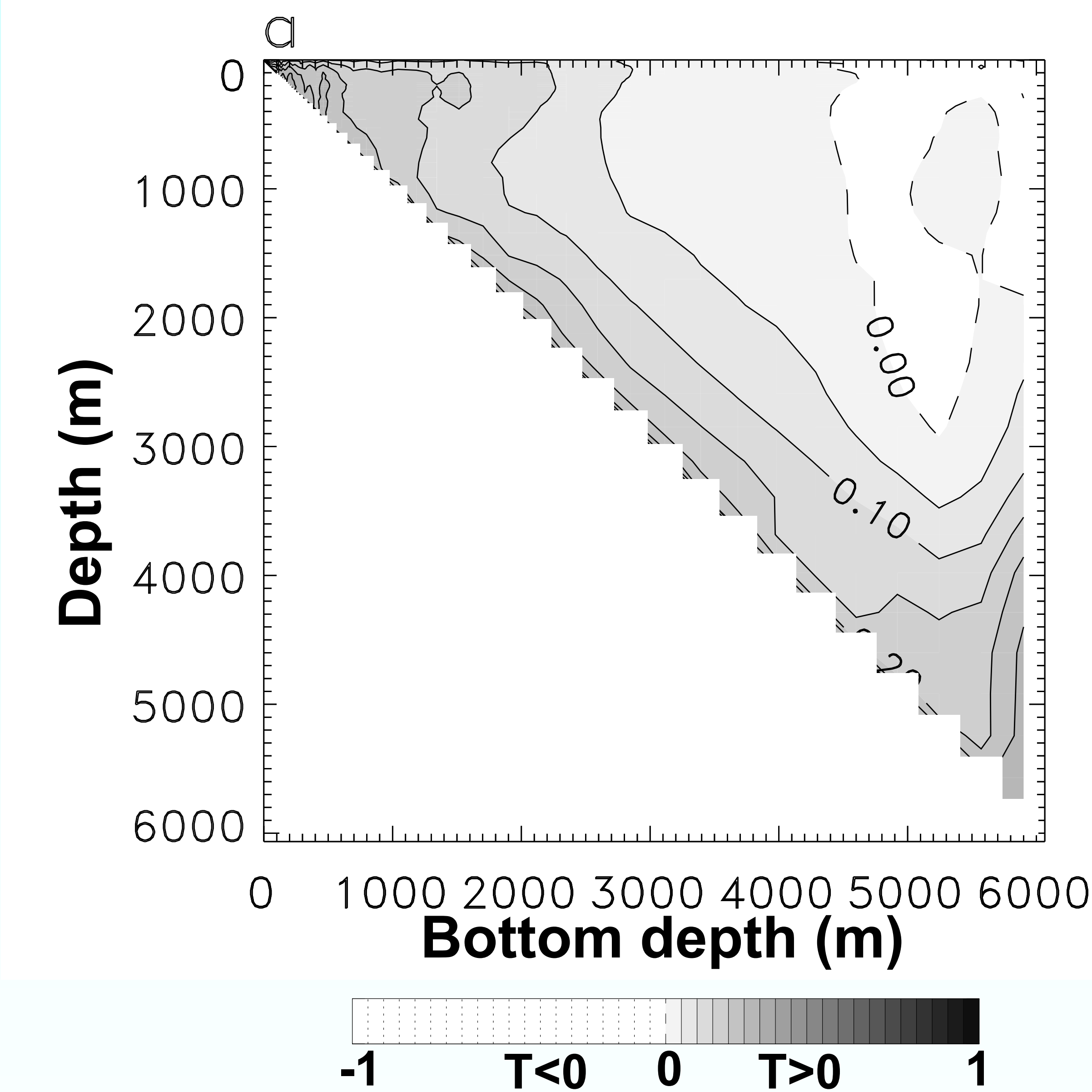






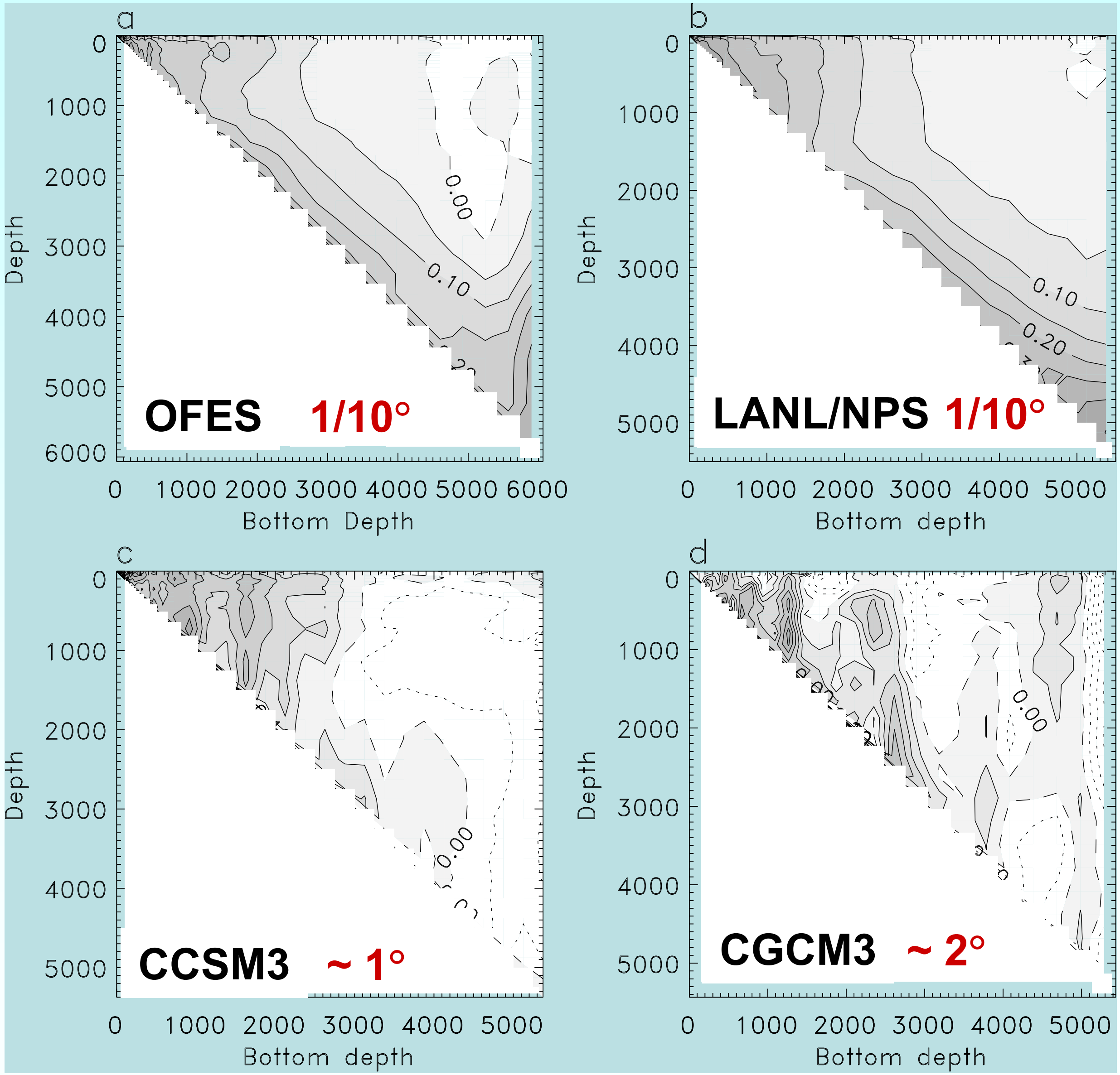
Δx=1/10°

Topostrophy in OFES 1/10° model

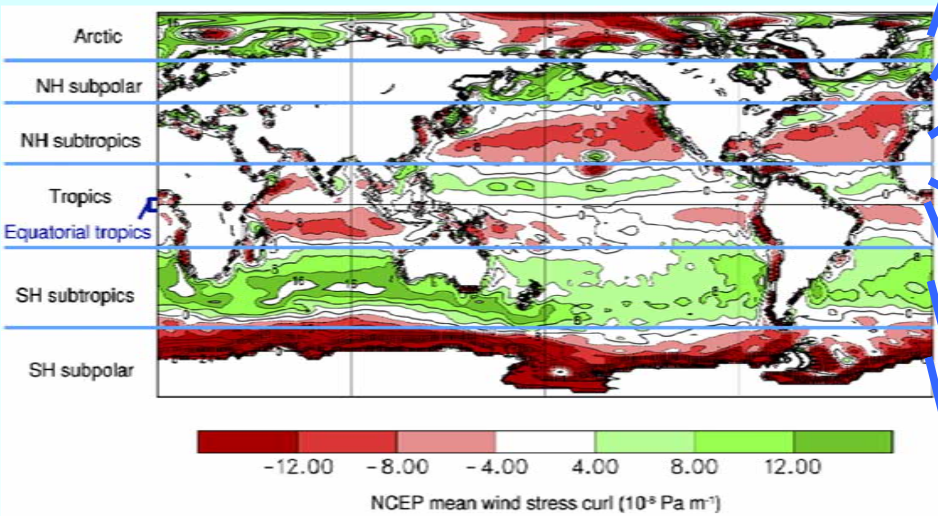


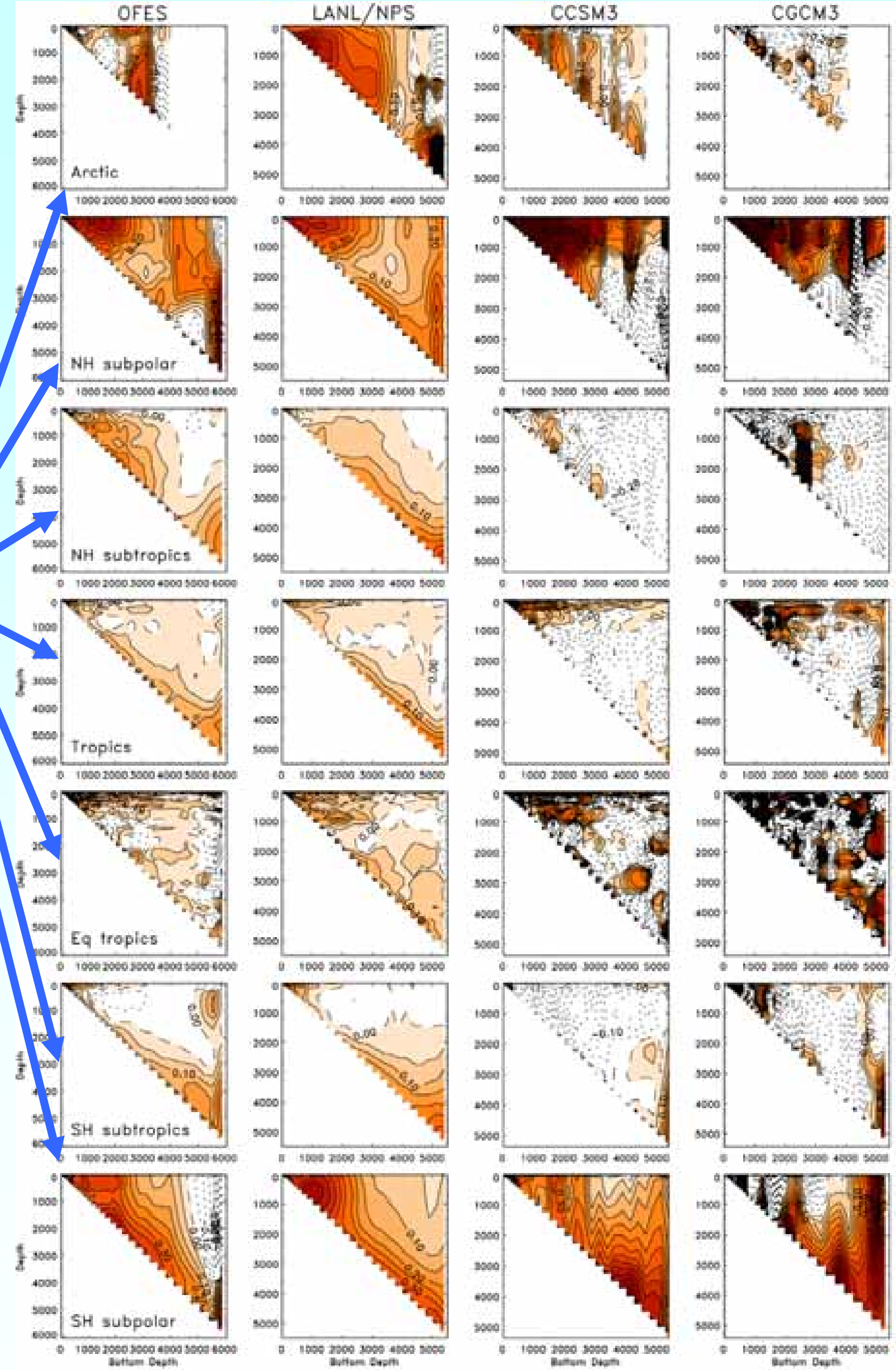
Merryfield & Scott (Ocn Mod 2007)

Topostrophy in eddying vs non-eddying global ocean models



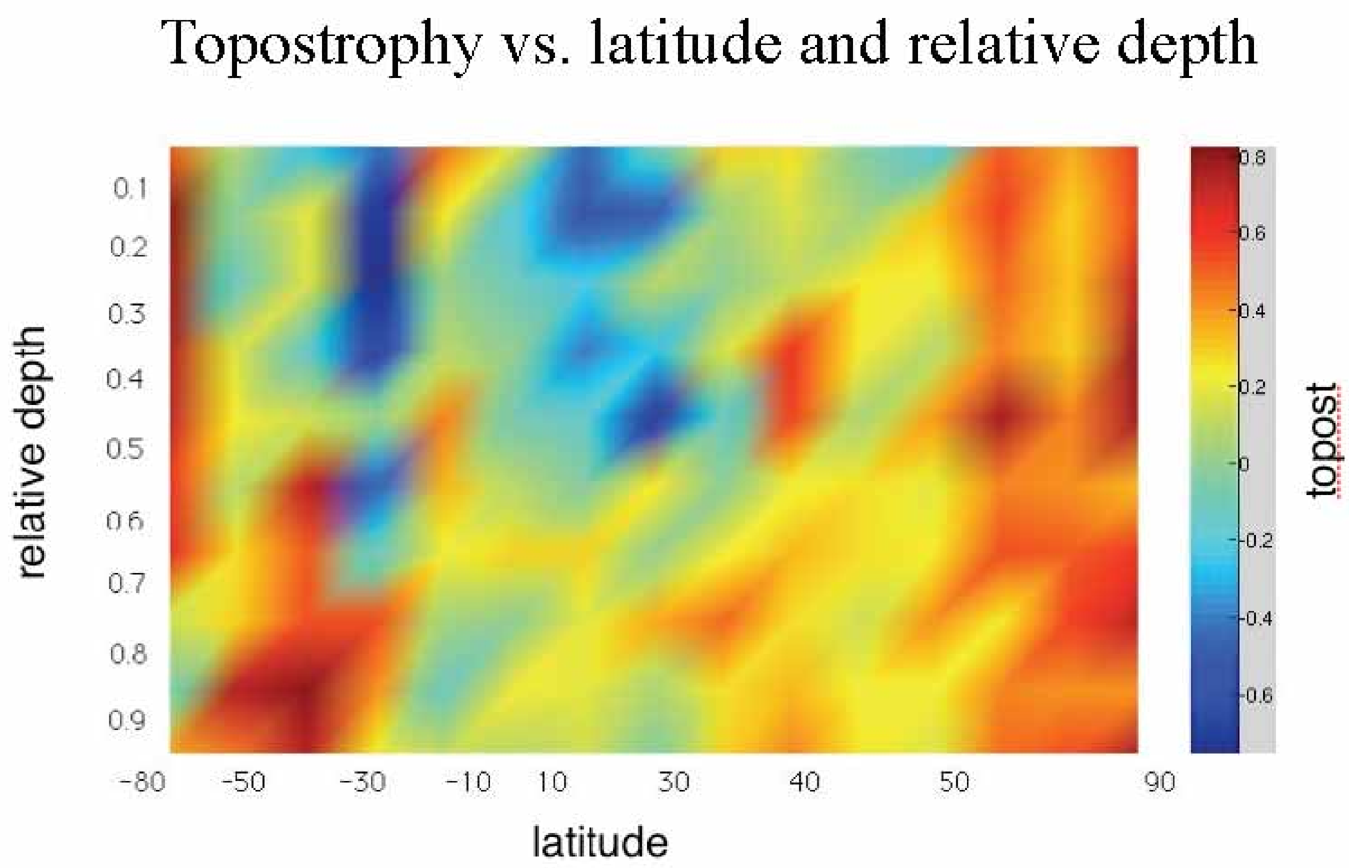
Topostrophy in eddying vs non-eddying global ocean models





Merryfield & Scott (Ocn Mod 2007)

Observed topostrophy from current meter archive



Holloway (JGR 2008)

Observed topostrophy vs models sampled at current meter locations

Model	Number of points	Observed topostrophy	Modeled topostrophy
CCSM3	6921	0.302	0.116
CGCM3	5663	0.292	-0.026
OFES	6526	0.261	0.139
POP	6975	0.314	0.147

Eddy active models

Holloway (JGR 2008)

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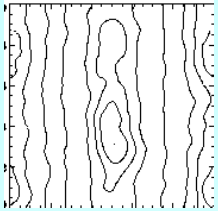
Eddy active models

How much resolution is "enough"?

Holloway (JGR 2008)

SHH76



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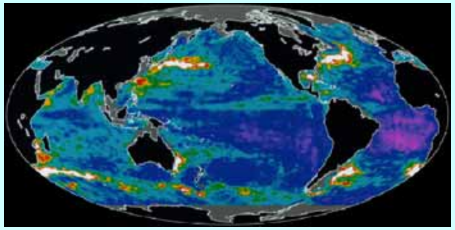
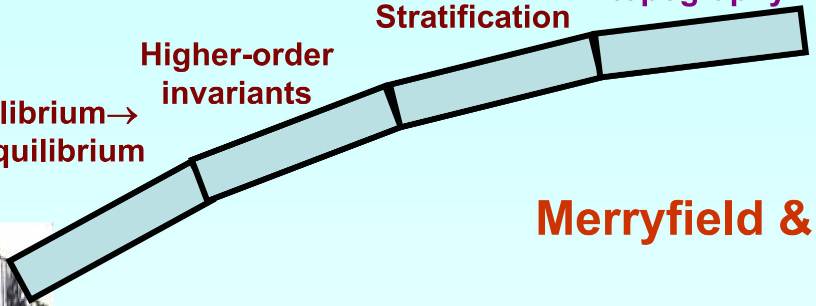
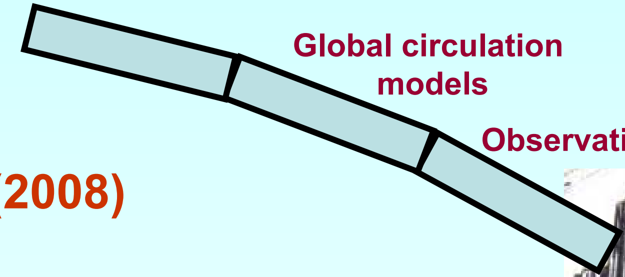
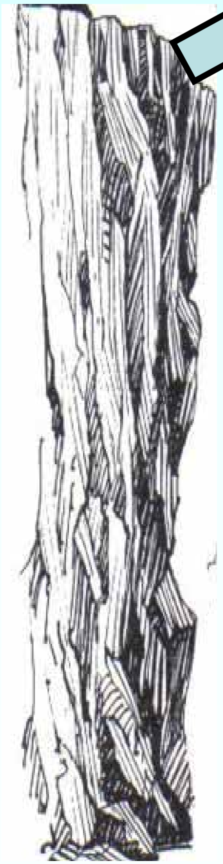
World Ocean





Ultra-high res basin model

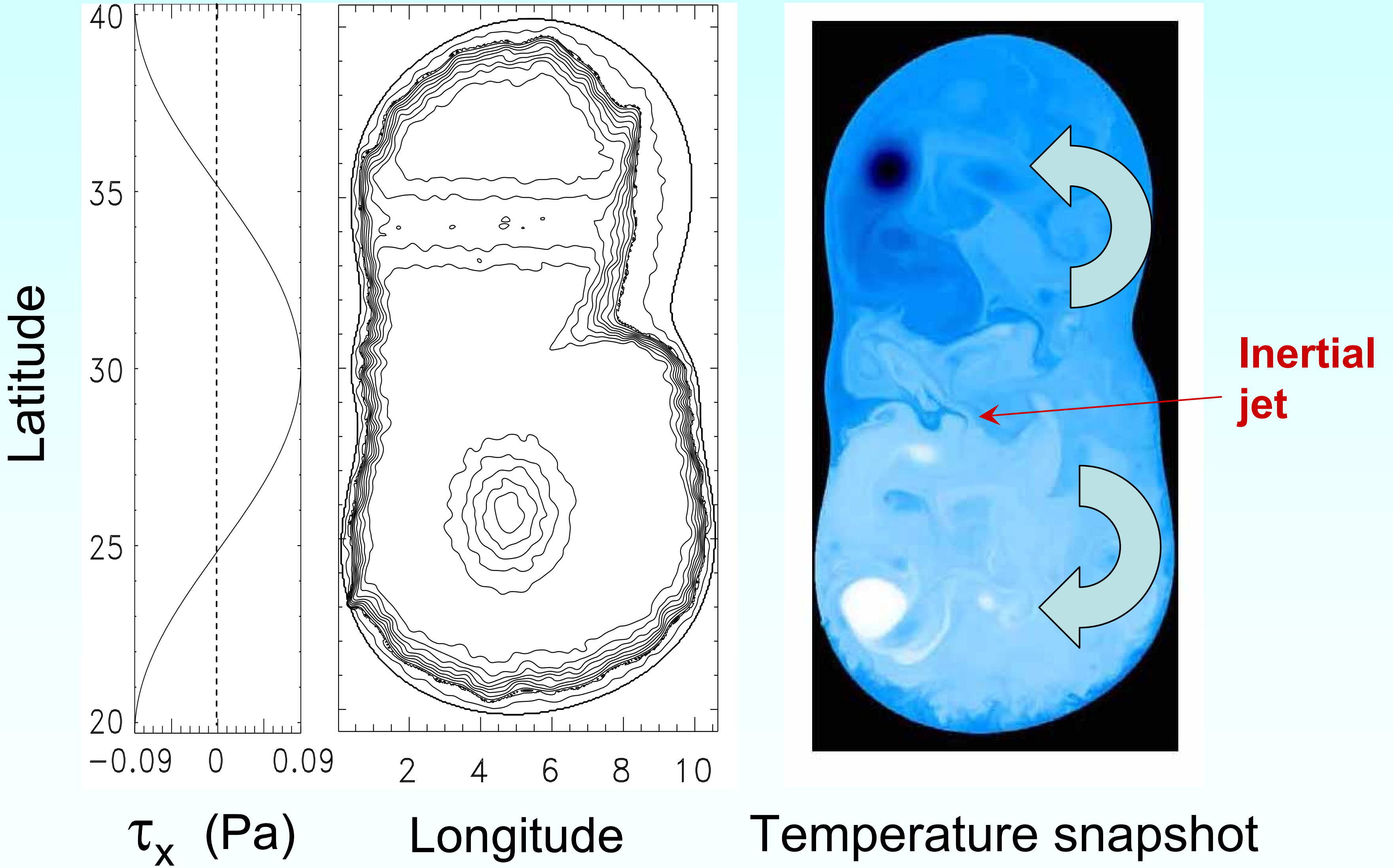




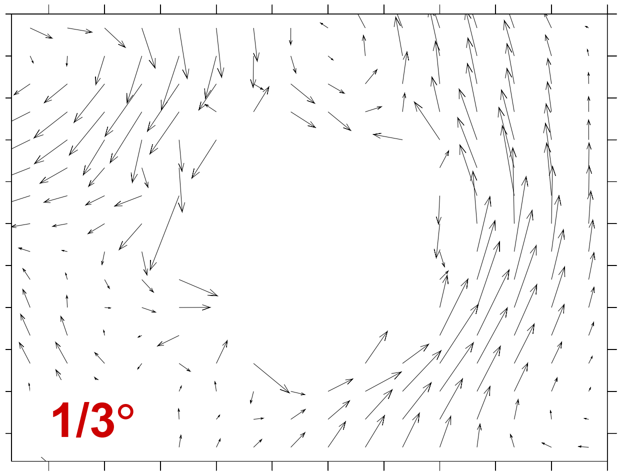
Merryfield & Scott (2008)

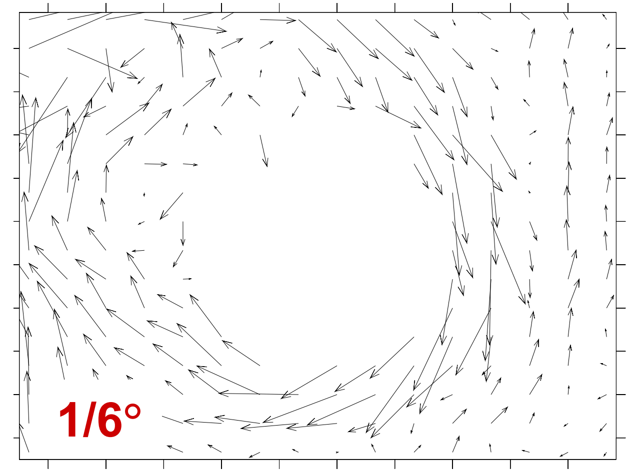
- Model fictitious basin using MOM4
- Resolutions 1/3° to 1/48°
- Simplified physics:
  - Stratified by temperature
  - Linear equation of state
  - Steady zonal wind stress
- Laplacian + biharmonic viscosity, coefficients varied to
  - Preserve resolution of Munk layer
  - Preserve grid Reynolds number

Model configuration



u over seamount

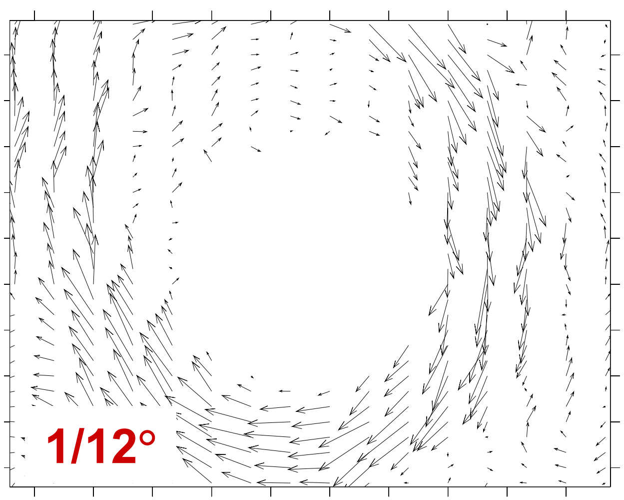


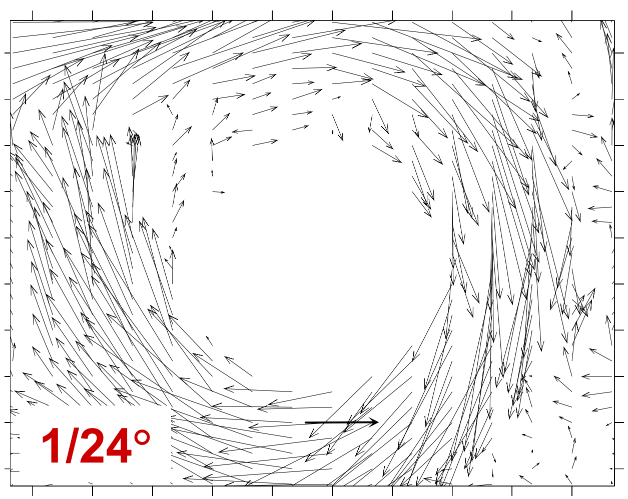


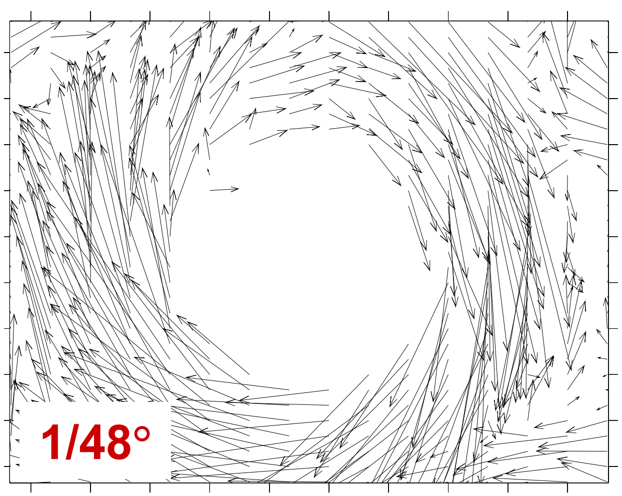
→ .0004 ms^{-1}

→ .004 ms^{-1}

Note changes of scale

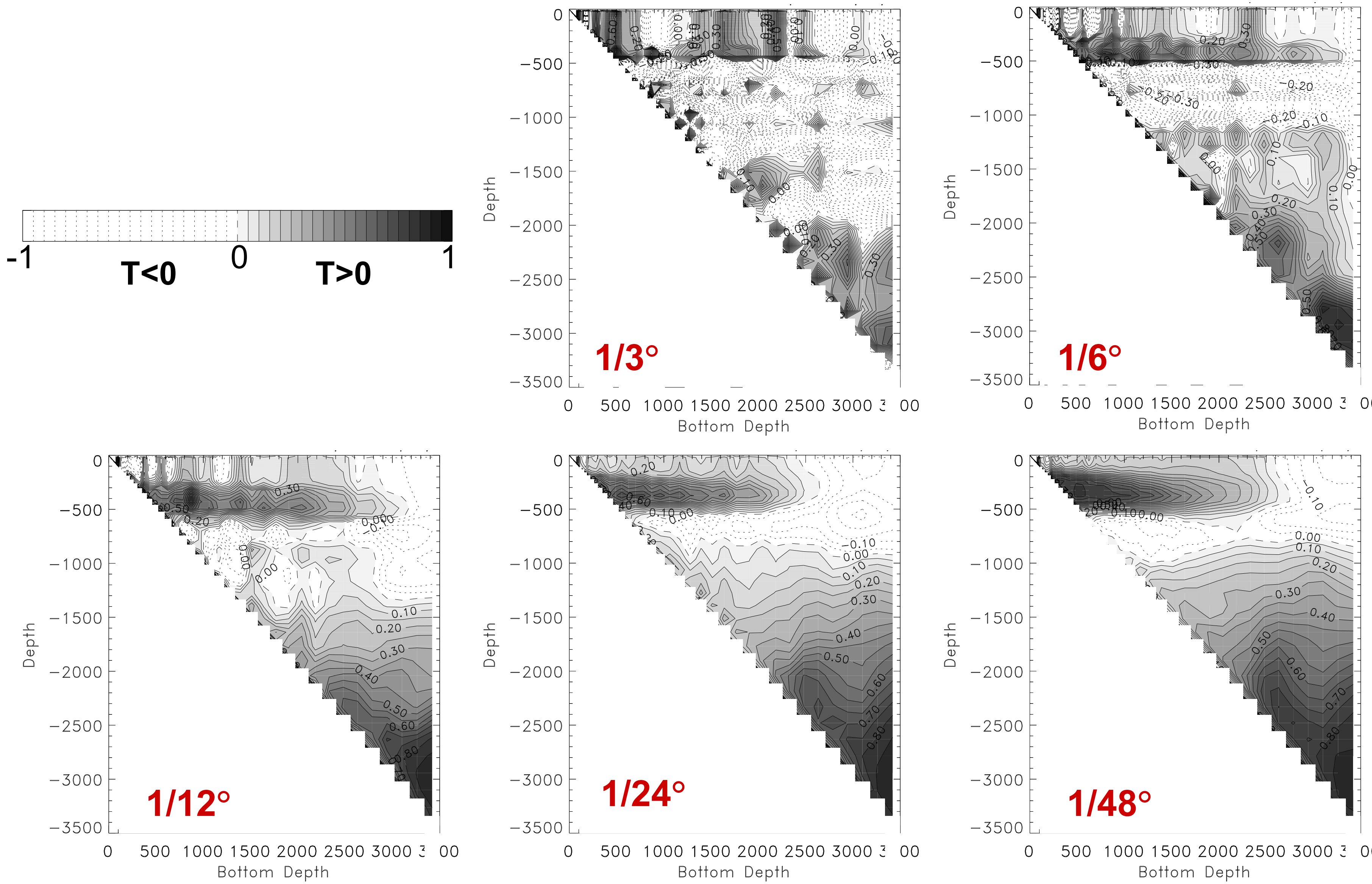




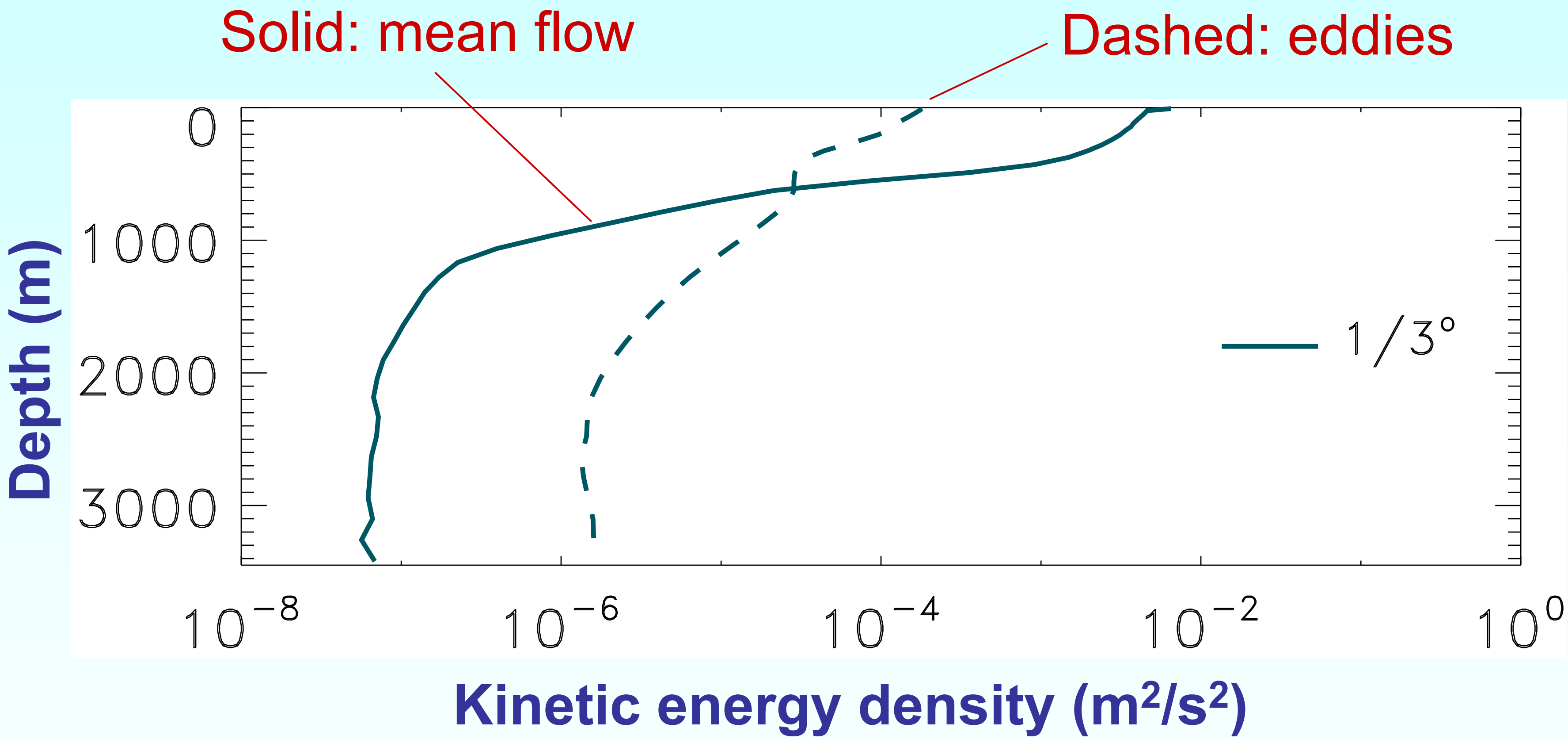


→ .04 ms^{-1} (all)

Whole-domain topostrophy



Kinetic Energy vs Depth



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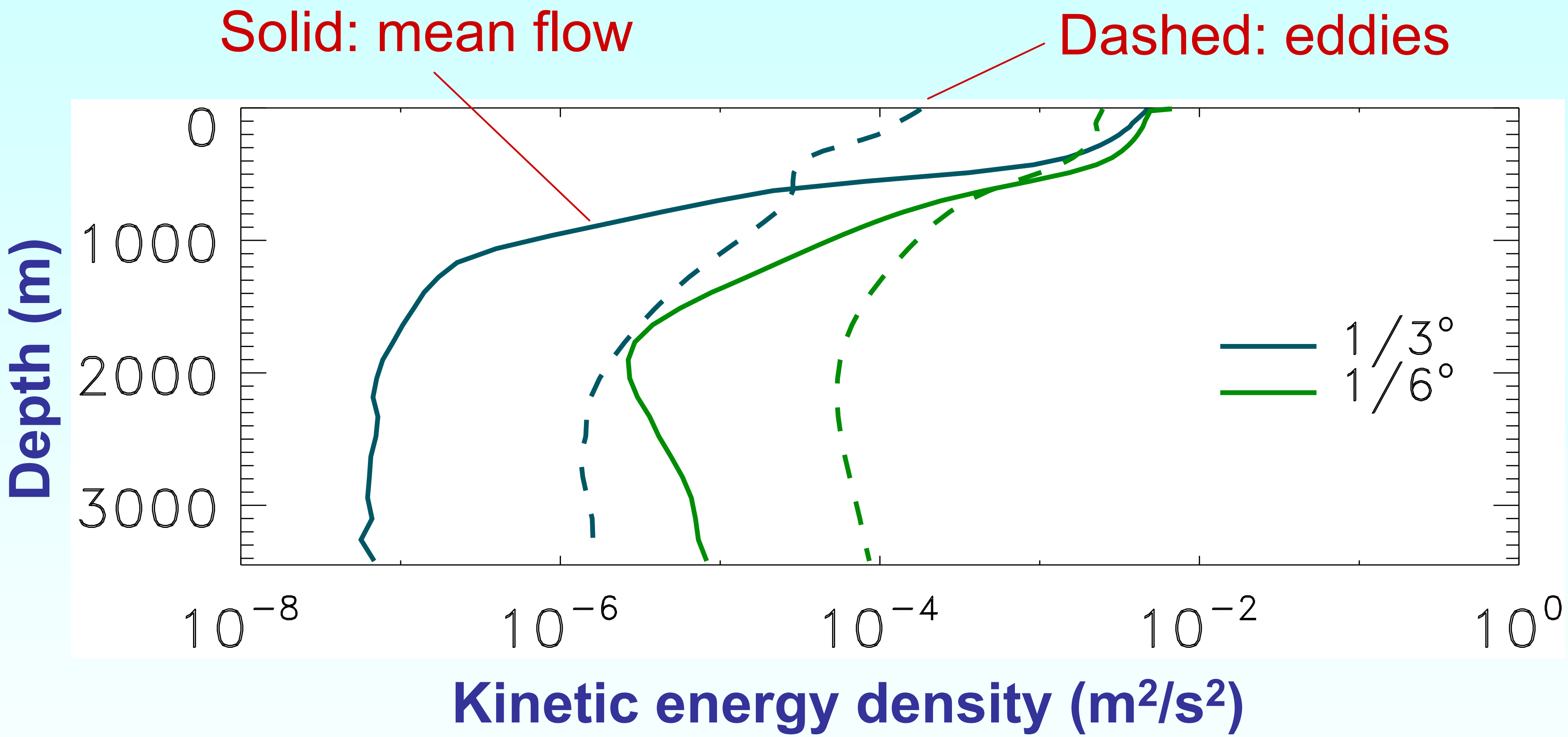

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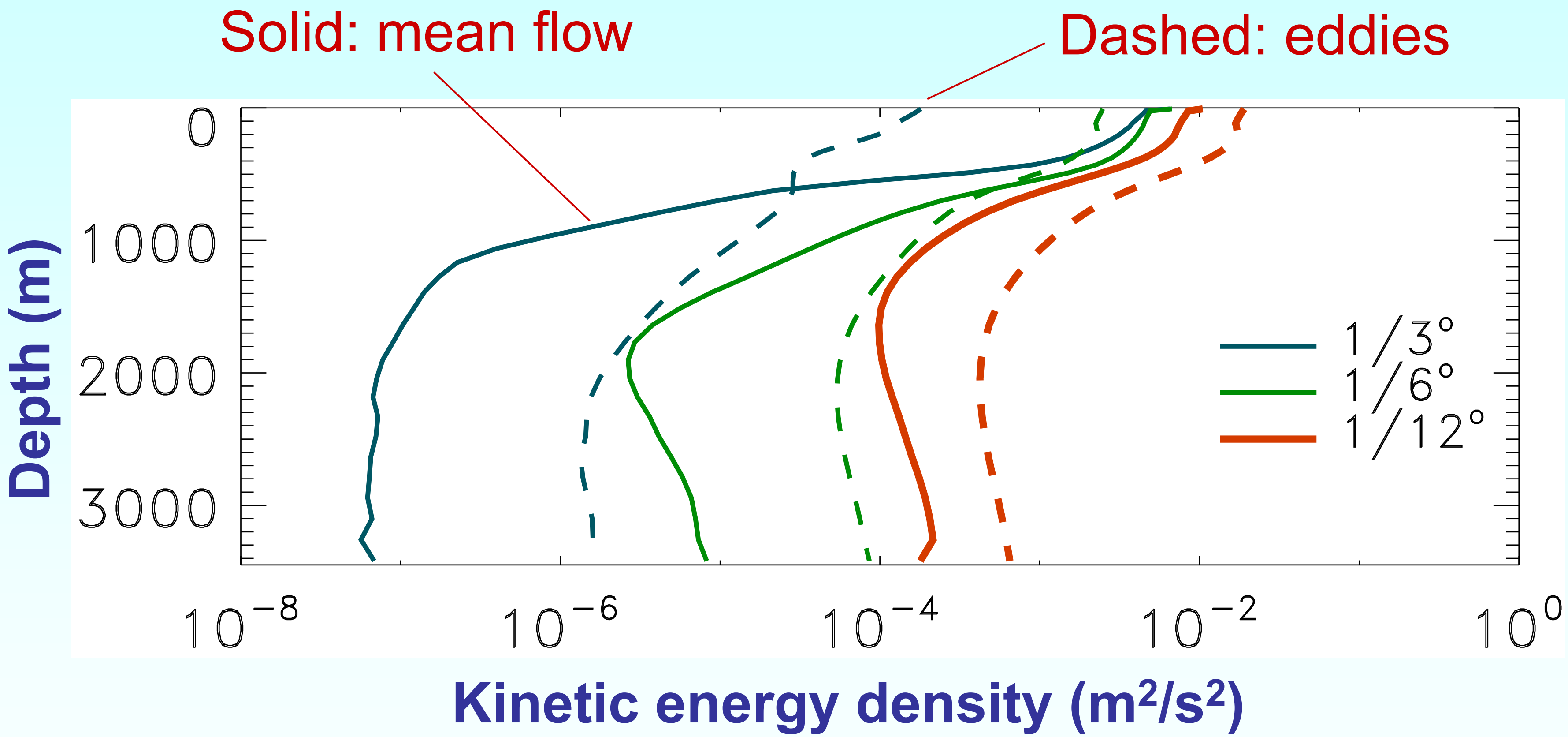

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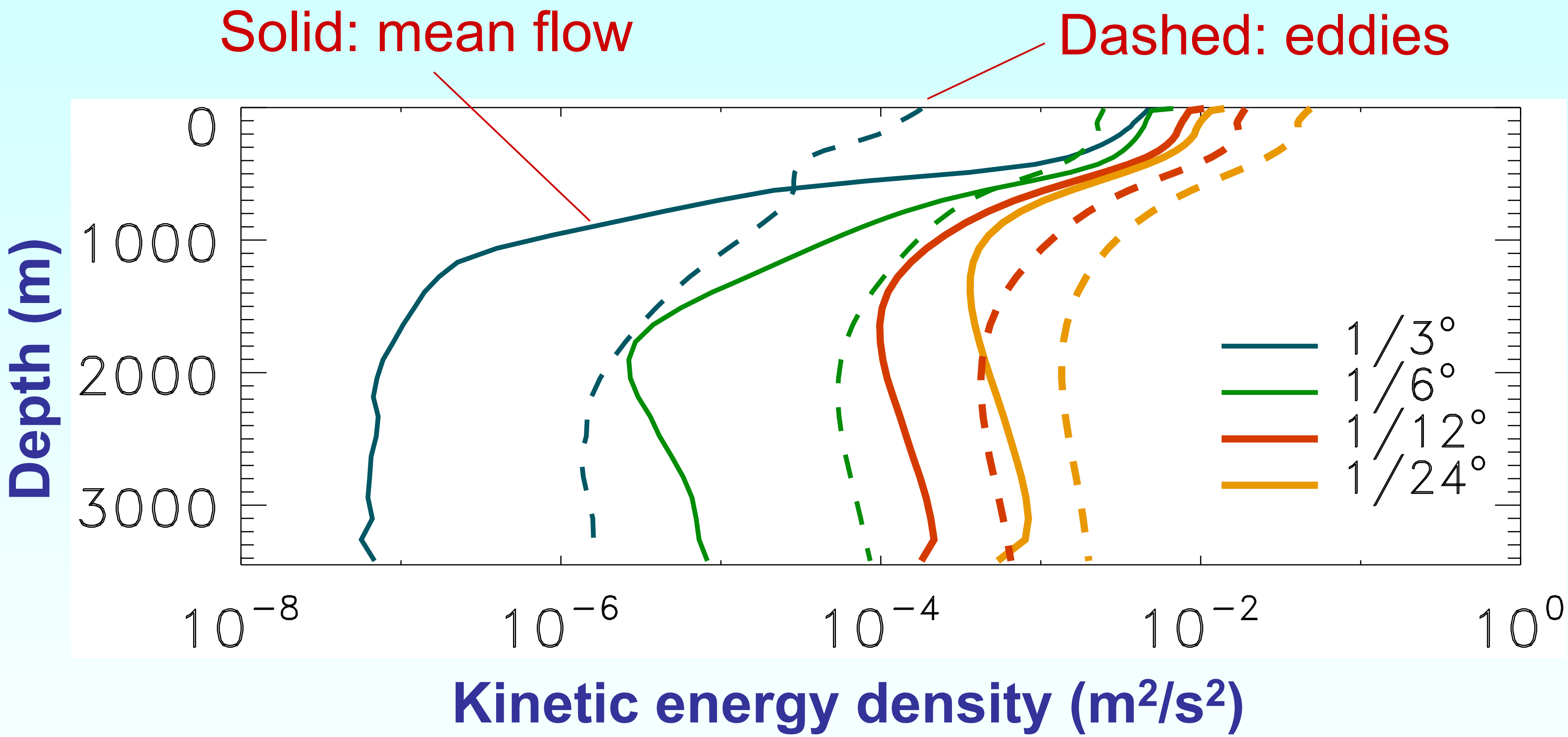

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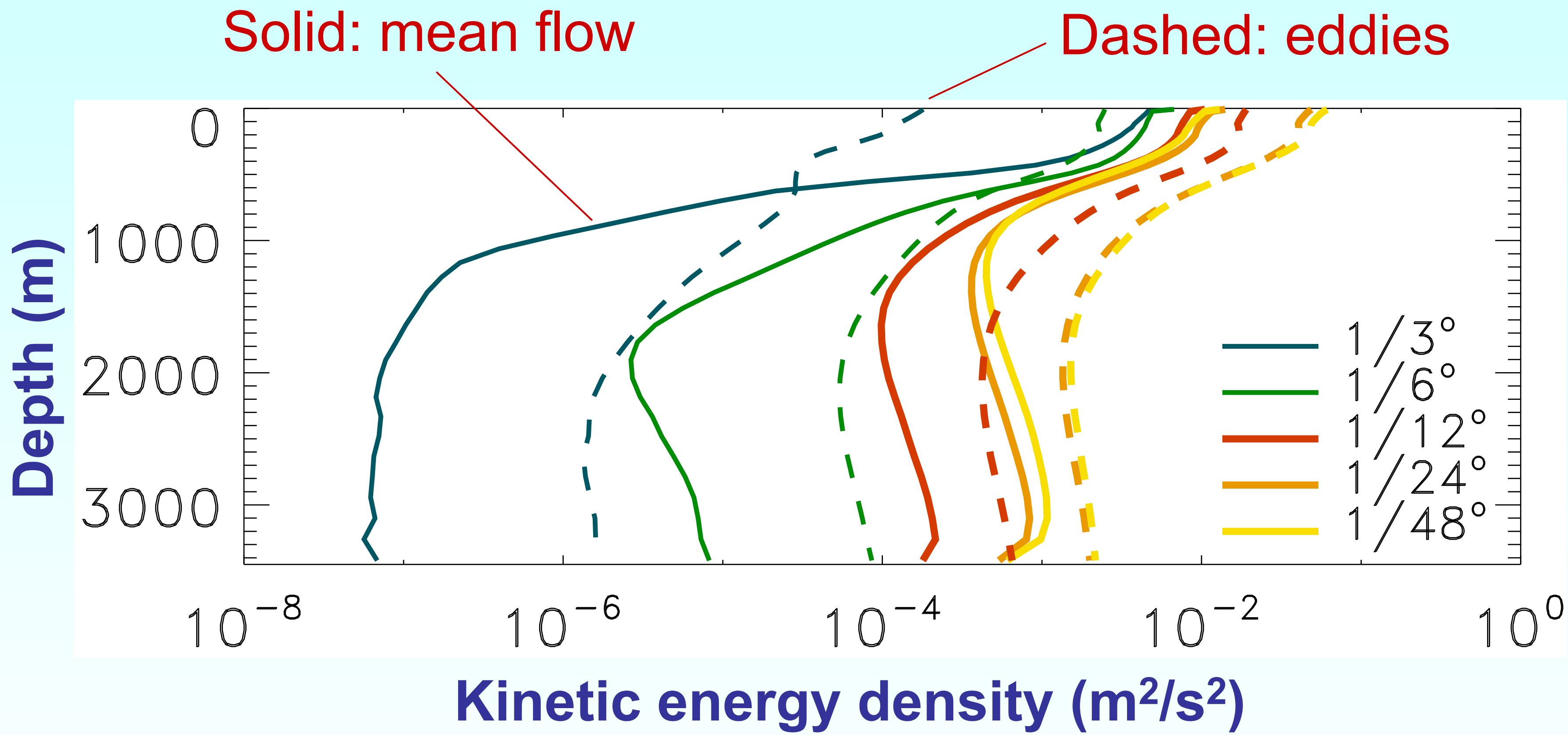
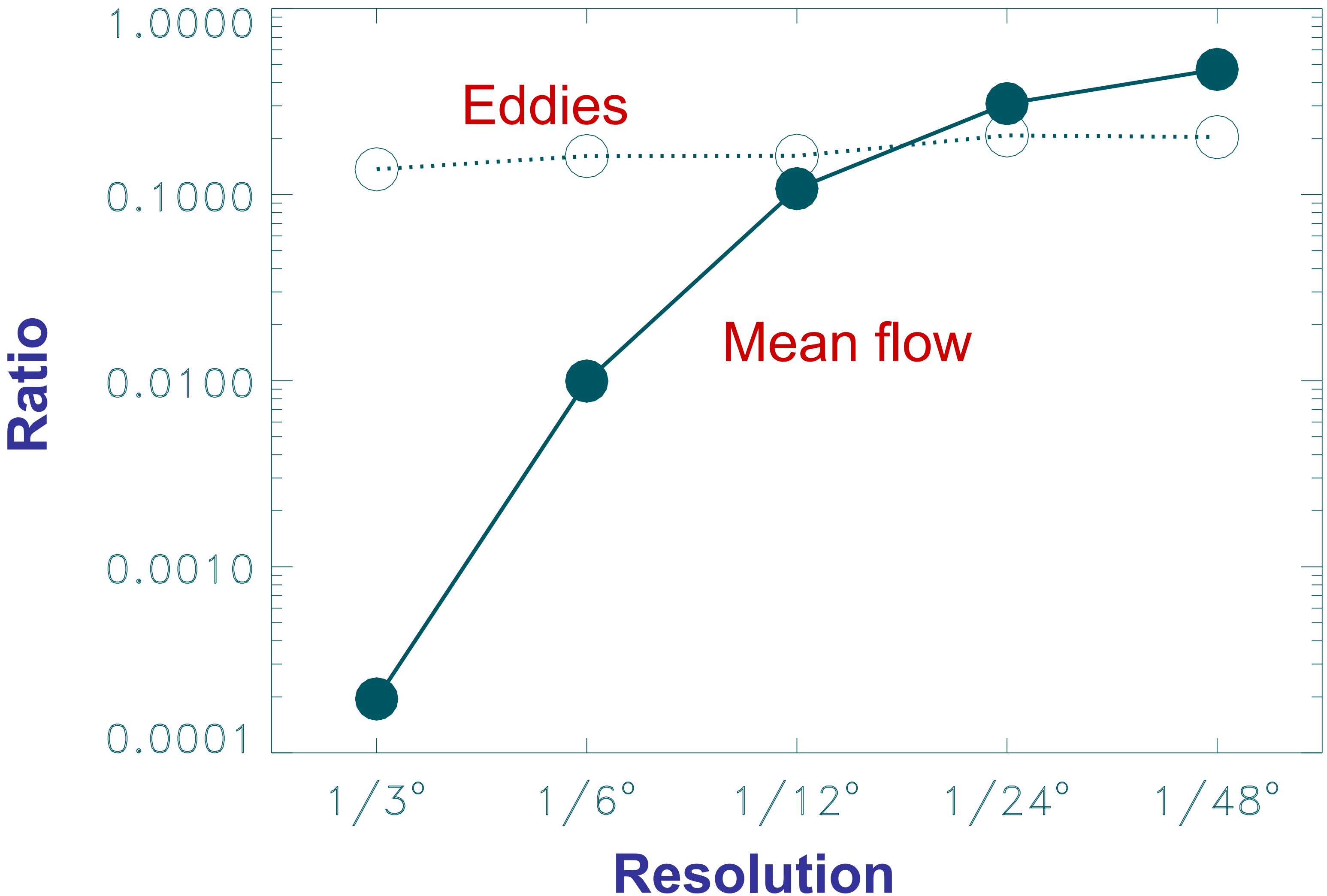
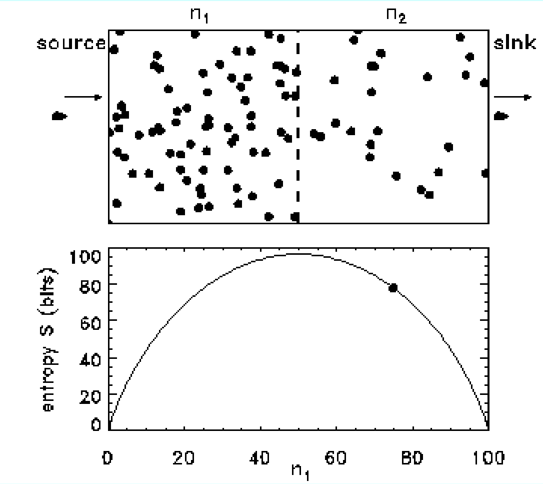


Ratio of abyssal to upper ocean KE

$KE_{>1000m}$ / $KE_{<1000m}$



What about parameterization?



- Back to *arrow of time*

- Holloway (JPO 1992): replace viscous operator with forcing toward higher entropy state

- Consequences of entropy-gradient forcing are not always obvious!

Two-layer flow with surface forcing and bottom friction

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = \mathscr{F}\delta_{i1} - D_i, \quad i = 1, 2$$

$\nu \nabla^2 \psi_2 \, \delta_2$

Zero-mean forcing **Rayleigh friction**

Two-layer flow with surface forcing and bottom friction

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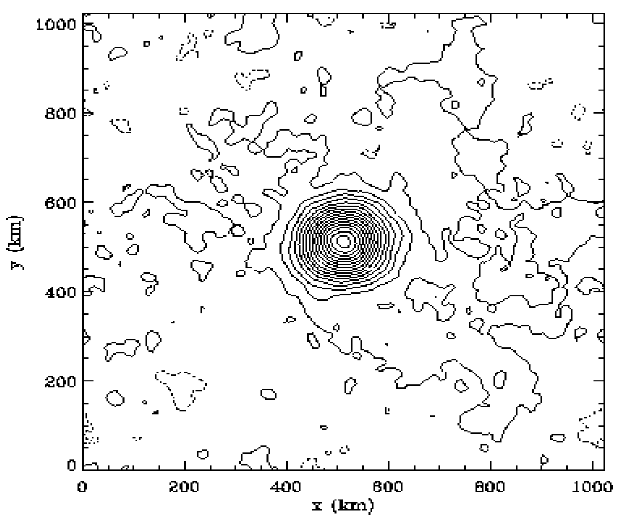
Zero-mean forcing

$\nu\nabla^2\psi_2\,\delta_2$

Rayleigh friction

lower layer



topography

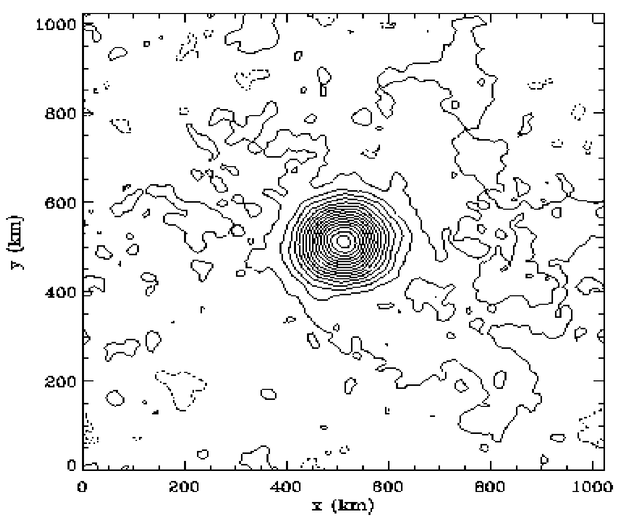
low bottom friction

Merryfield and Holloway (JFM 1999)

Two-layer flow with surface forcing and bottom friction

upper layer

lower layer



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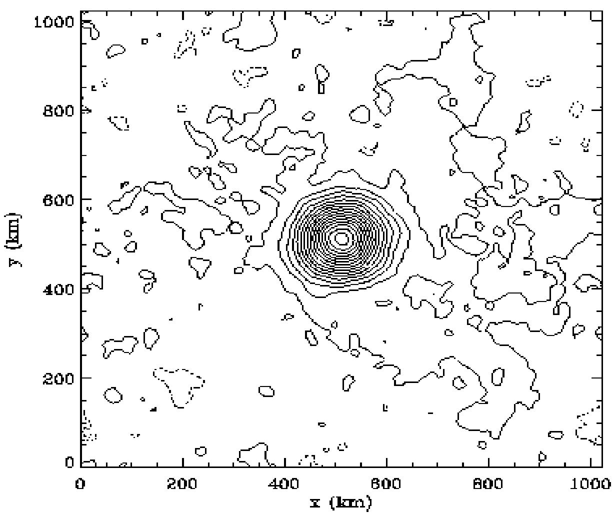

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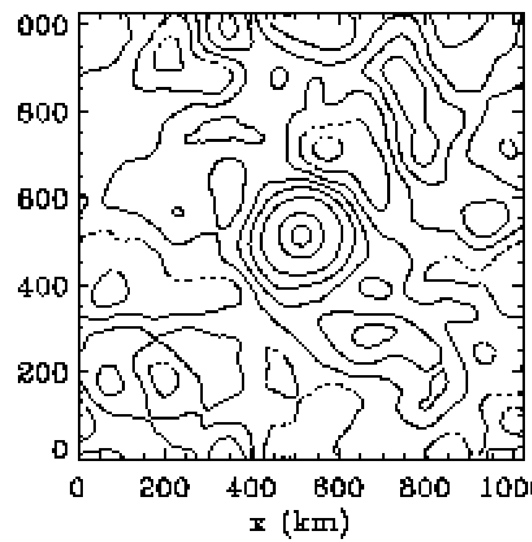
low bottom friction



high bottom friction

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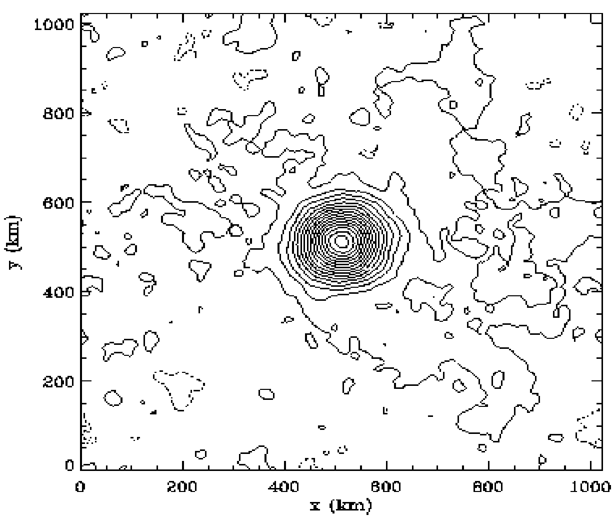
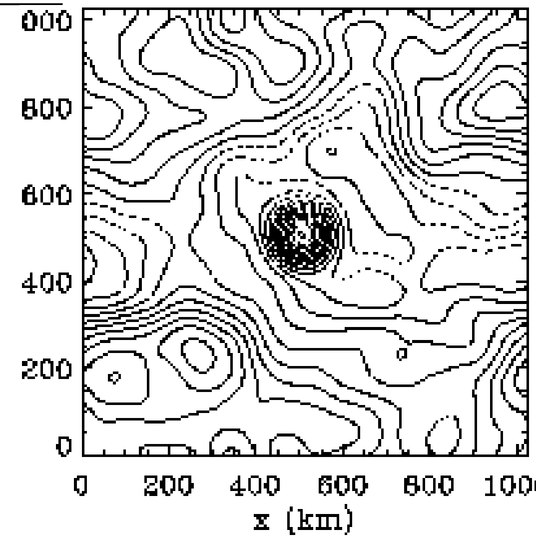
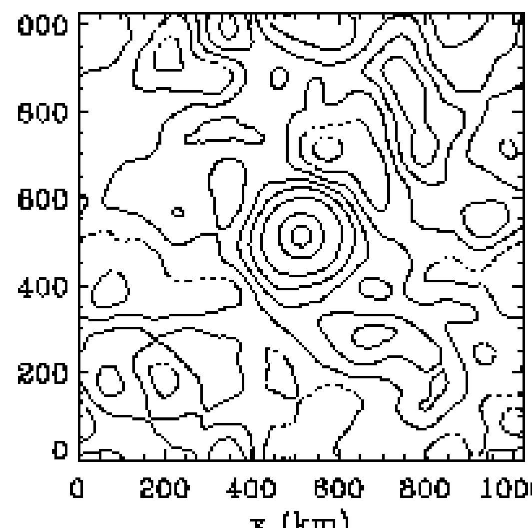


topography

low bottom friction



!



high bottom friction

Merryfield and Holloway (JFM 1999)

Intrepretation

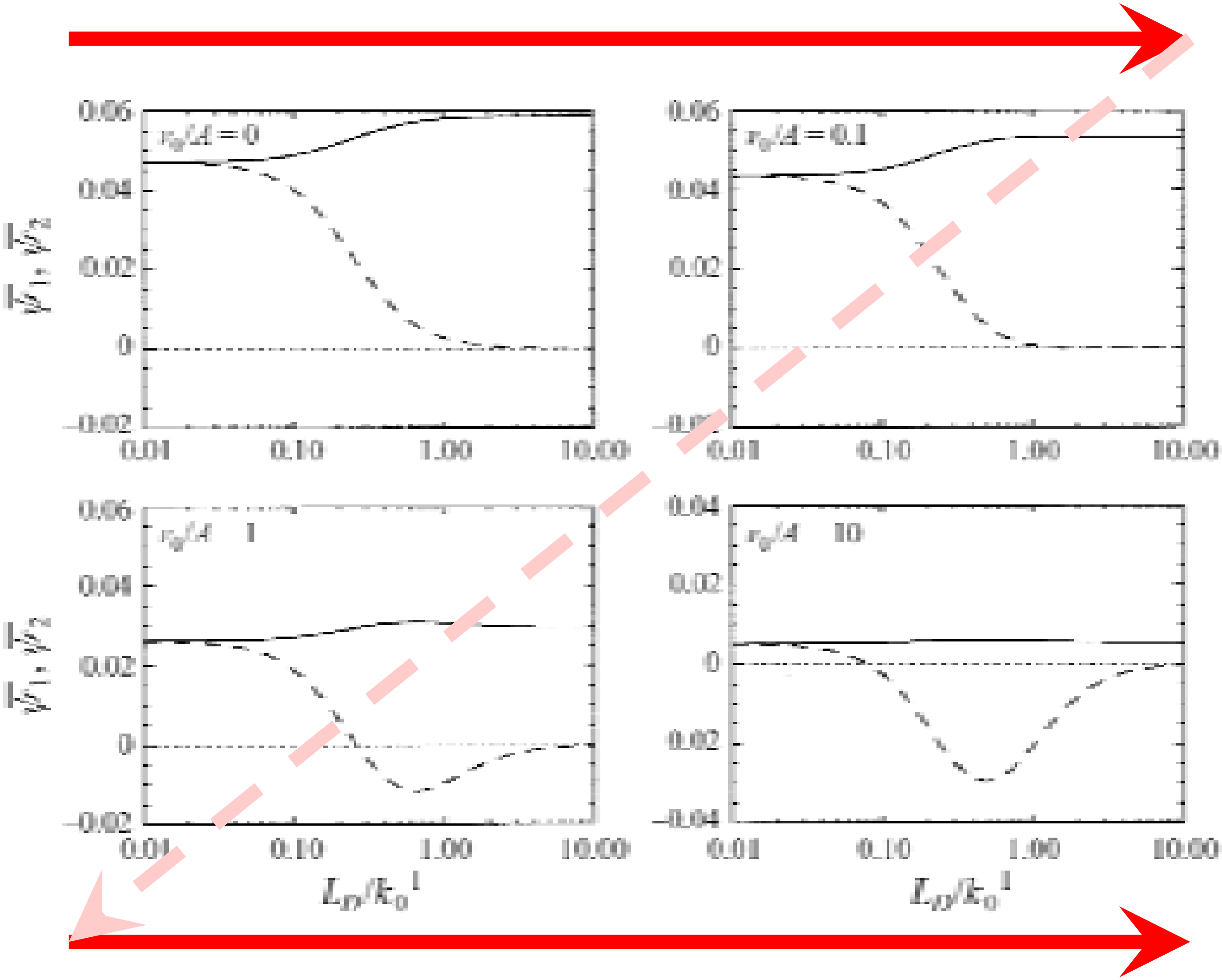
- Consider ensemble mean
- Heuristically replace nonlinear terms by tendency toward equilibrium $q^*_{i,k}$:

$$\frac{\partial \overline{q}_{1,k}}{\partial t} = -A_k(\overline{q}_{1,k} - \overline{q}^*_{1,k}),$$

$$\frac{\partial \overline{q}_{2,k}}{\partial t} = -A_k(\overline{q}_{2,k} - \overline{q}^*_{2,k}) + \nu_0 k^2 \overline{\psi}_{2,k}.$$

- Substitute equilibrium $q^*_{i,k}$
- Find ψ_i as a function of k, ν_0 / A

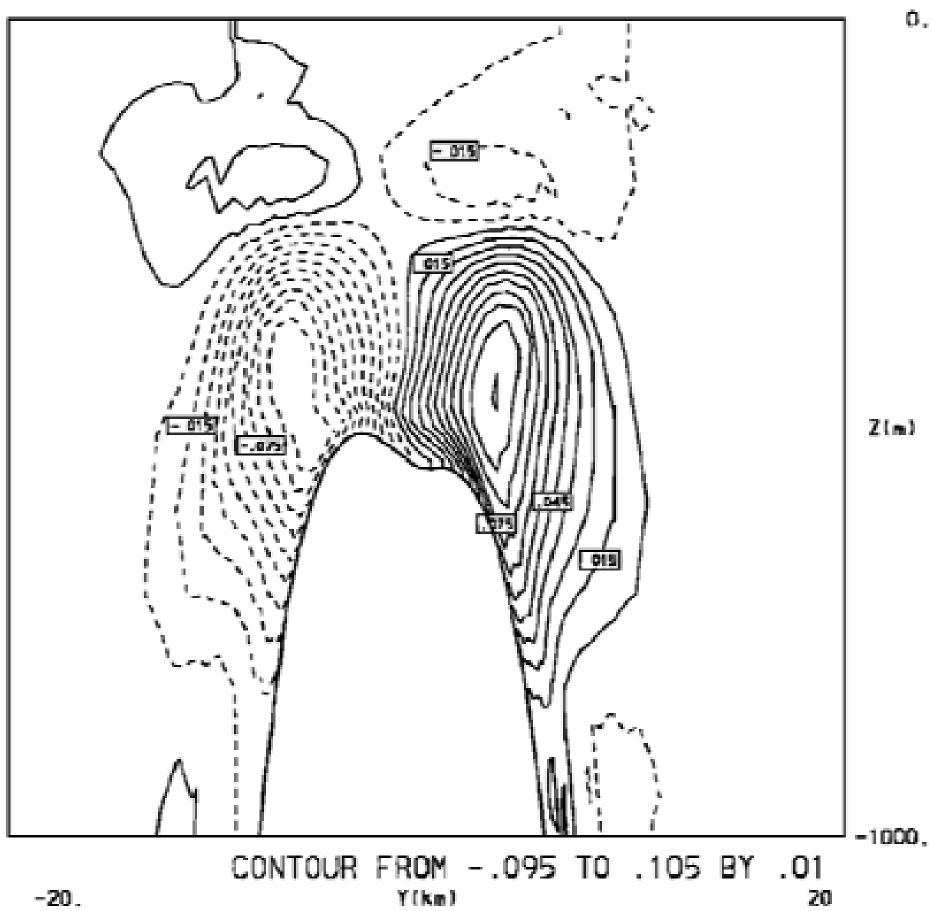
Increasing bottom friction



Increasing bottom friction

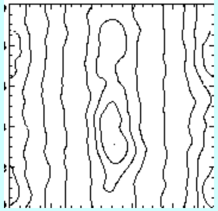
Merryfield and Holloway (JFM 1999)

Rectified flow over a seamount



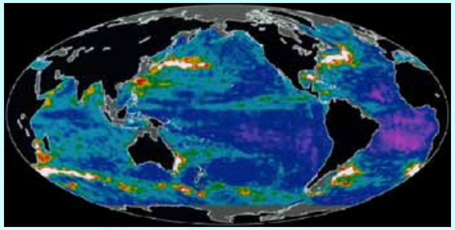
Beckmann and Haidvogel (JGR 1997)

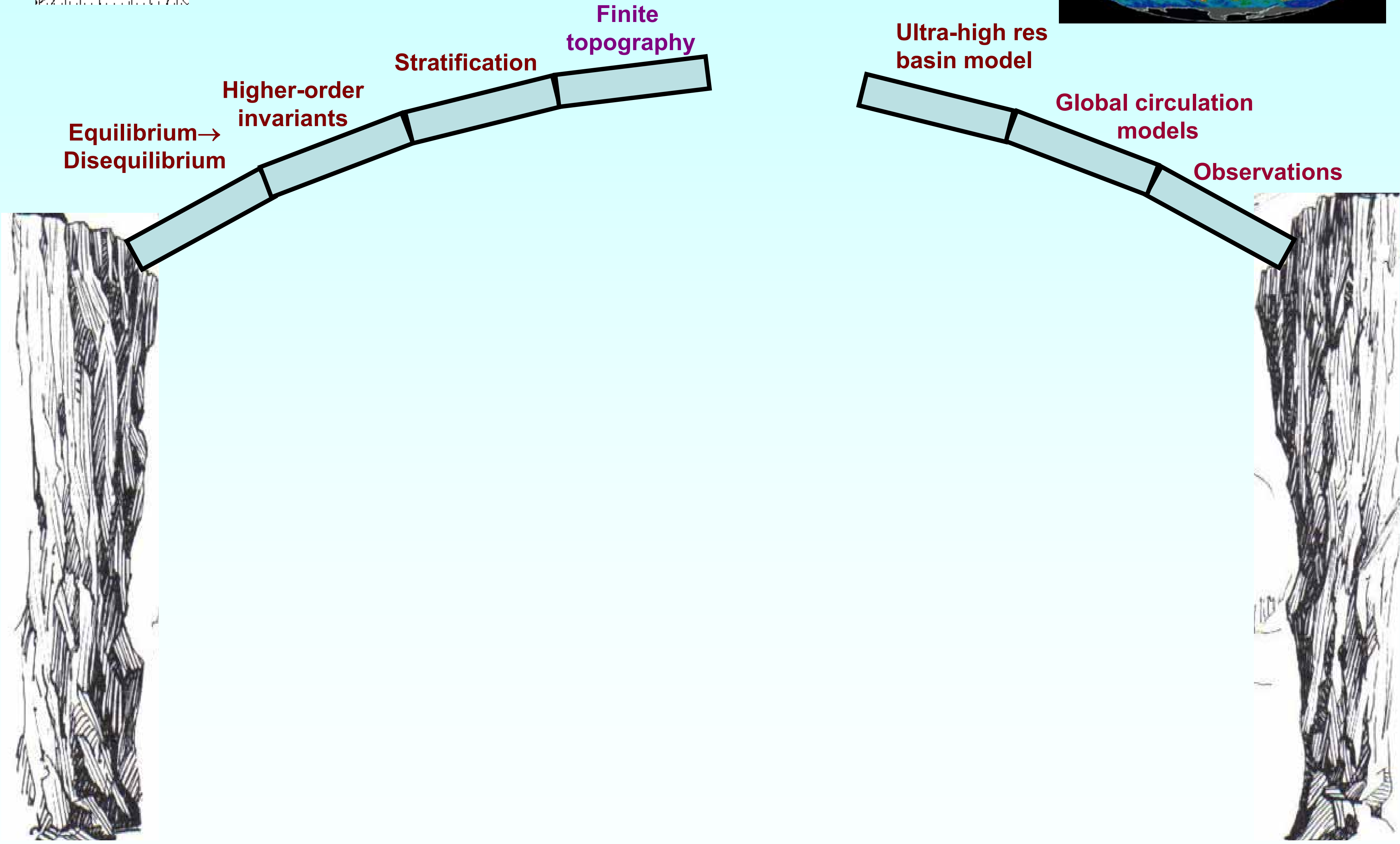
SHH76



"We suggest that some of the statistical trends observed in non-equilibrium flows may be looked on as manifestations of the tendency for turbulent interactions to maximize the entropy of the system."

World Ocean

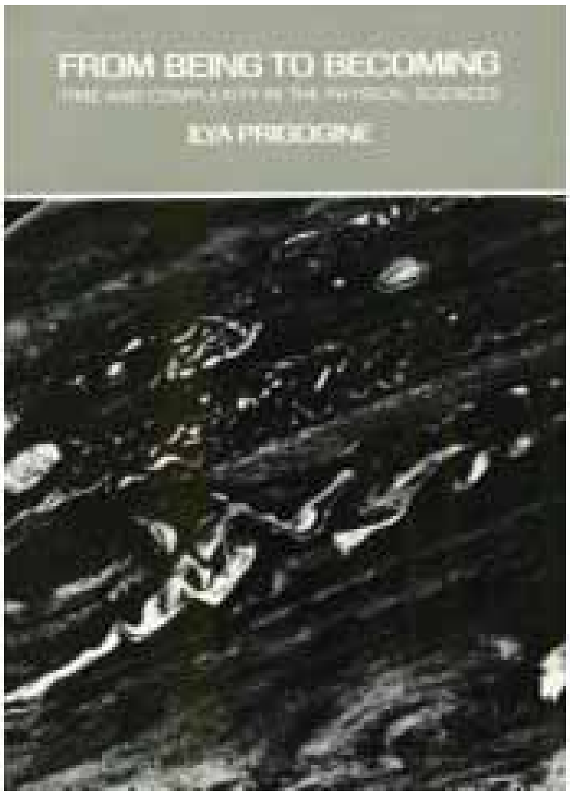


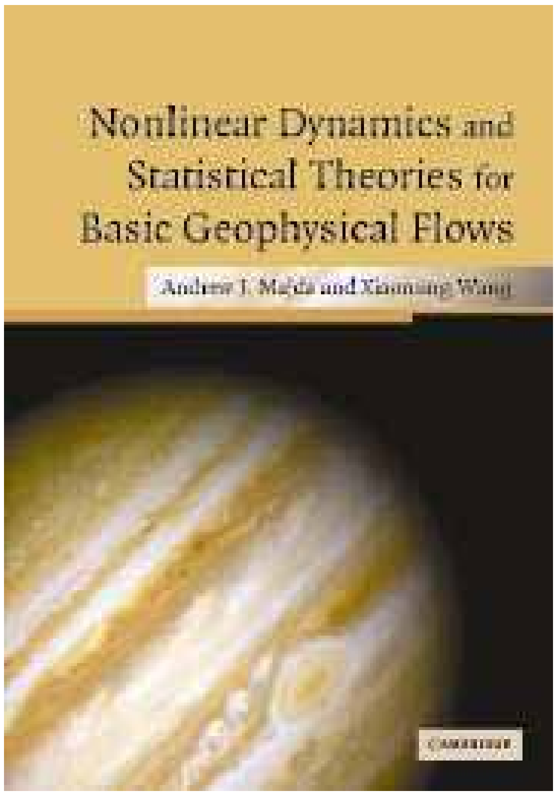


Summary

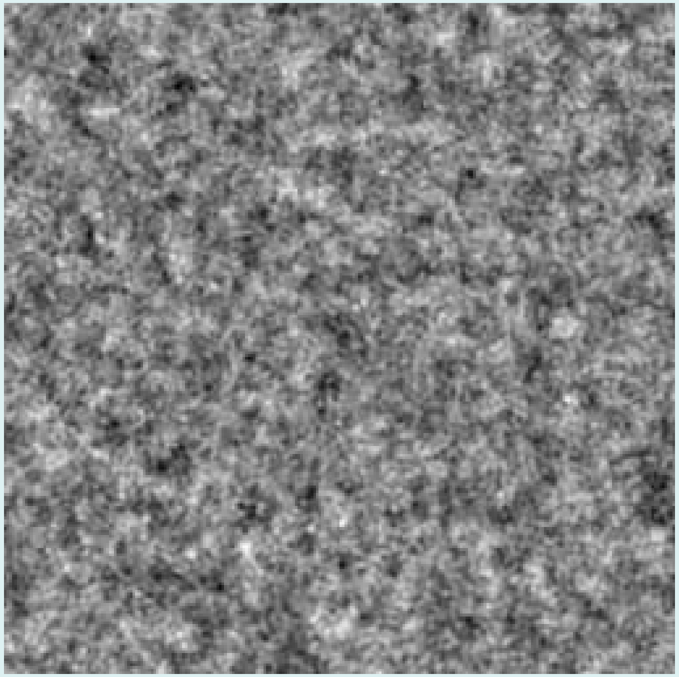
- Equilibrium flow over topography generally in pseudowestward direction
- Non-ideal fluids experience "force" toward higher entropy
- Numerical and observational evidence is mounting that this effect has strong influence on (deep) ocean circulation
- OGCMs not converged, even at 1/10°

Statistical dynamics in the cosmos





2d turbulence and stellar systems obey similar statistical mechanics



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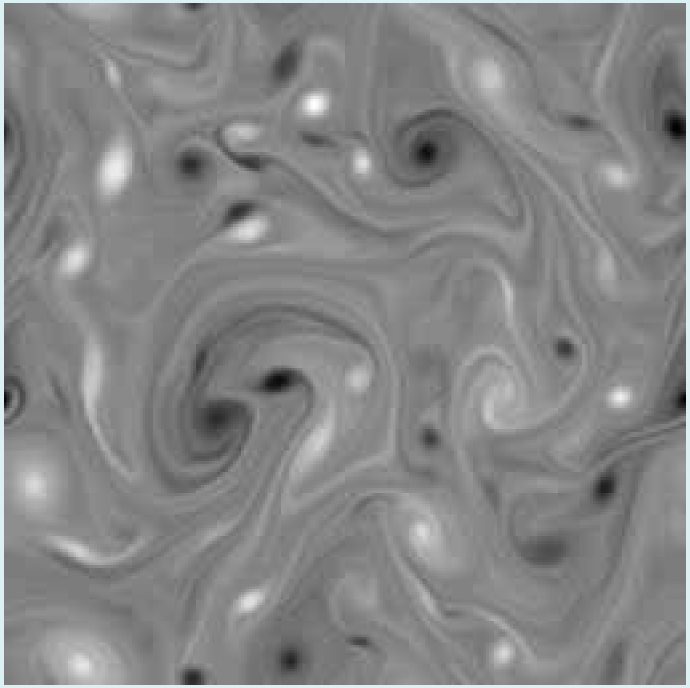
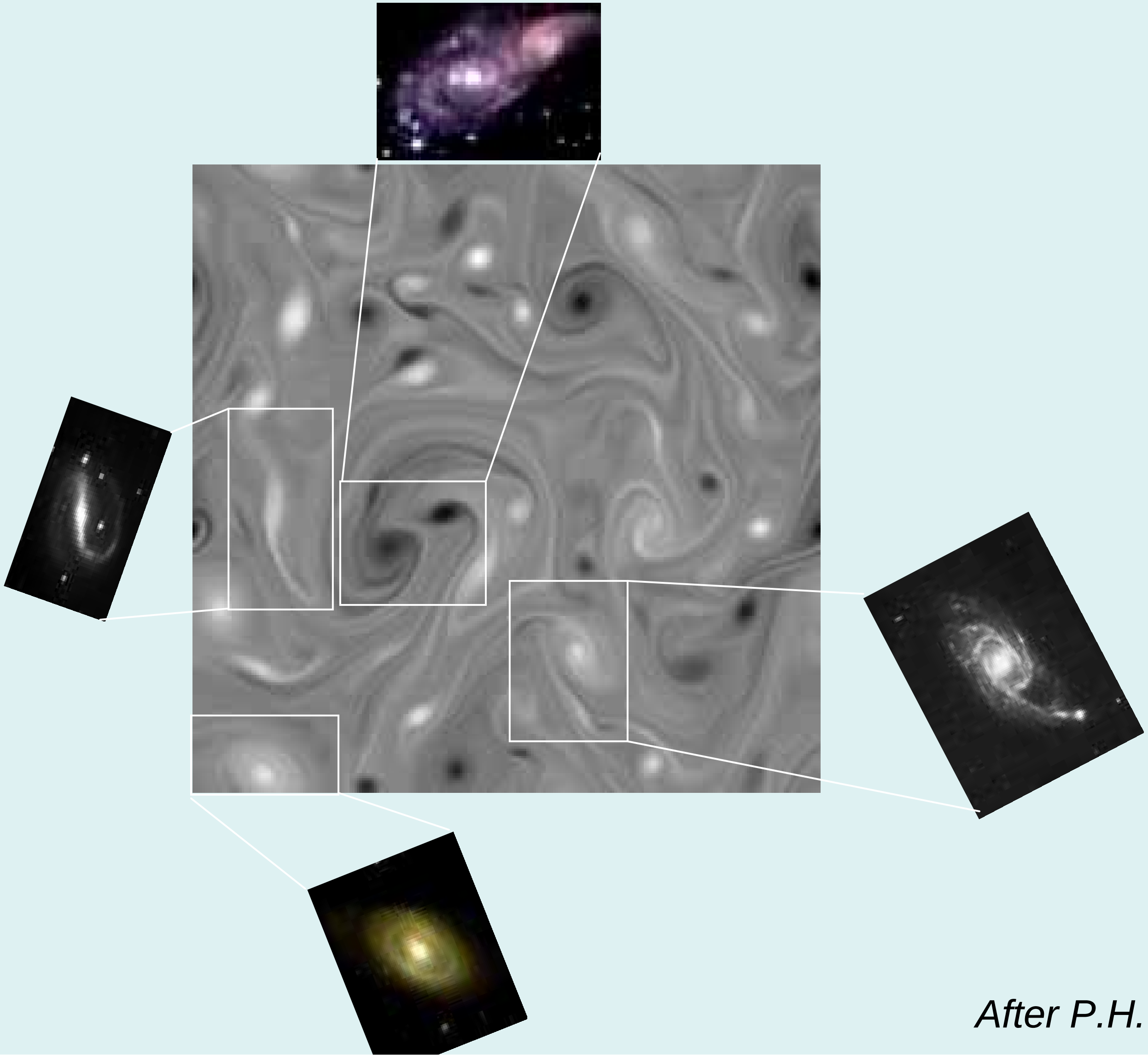


2d turbulence and stellar systems obey similar statistical mechanics

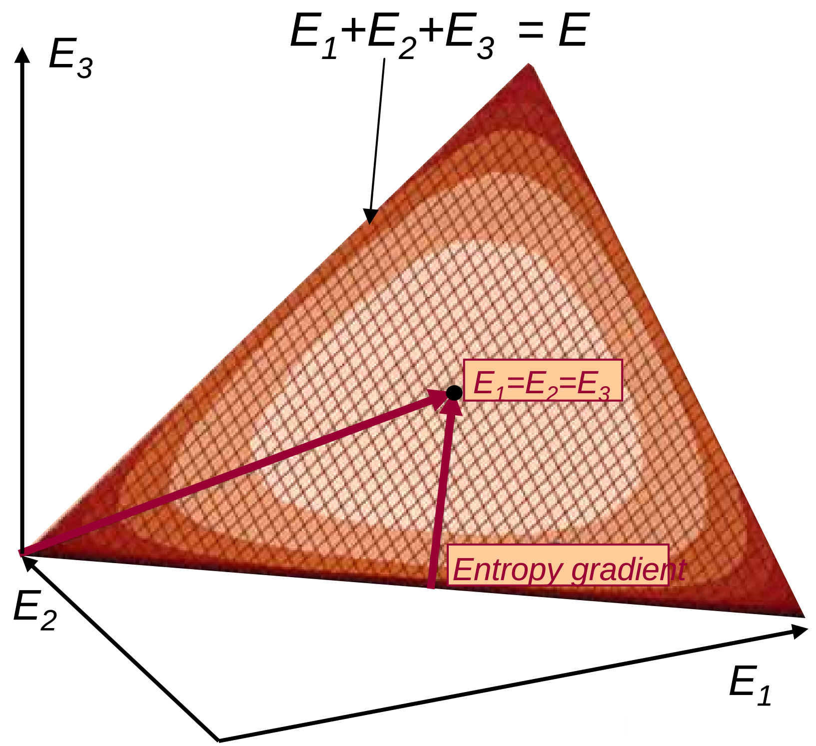


After P.H. Chavanis

Picturing entropy

$$S = \langle \log P \rangle = \frac{1}{2}\sum_{\mathbf{k}} \log E(\mathbf{k})$$

3d turbulence: *E conserved*



2d turbulence: *E and Q_2 conserved*