Parameterization in large-scale atmospheric modelling

General parameterization problem:

Evaluation of terms involving “averaged” quadratic and higher order products of (unresolved) deviations from

“large-scale” variables and effects of unresolved processes in Large-scale models (e.g. GCMs).

Examples:

(a) Turbulent transfer in the boundary layer

(b) Effects of unresolved wave motions (e.g. gravity-wave drag)

(c) Cumulus parameterization

Other kinds of parameterization problems: radiative transfer, cloud microphysical processes*, chemical processes

$$A_e = (1 - \sum_i \sigma_i) A_L$$

$$(A_c)_i = \sigma_i(z,t) A_L$$

A_L

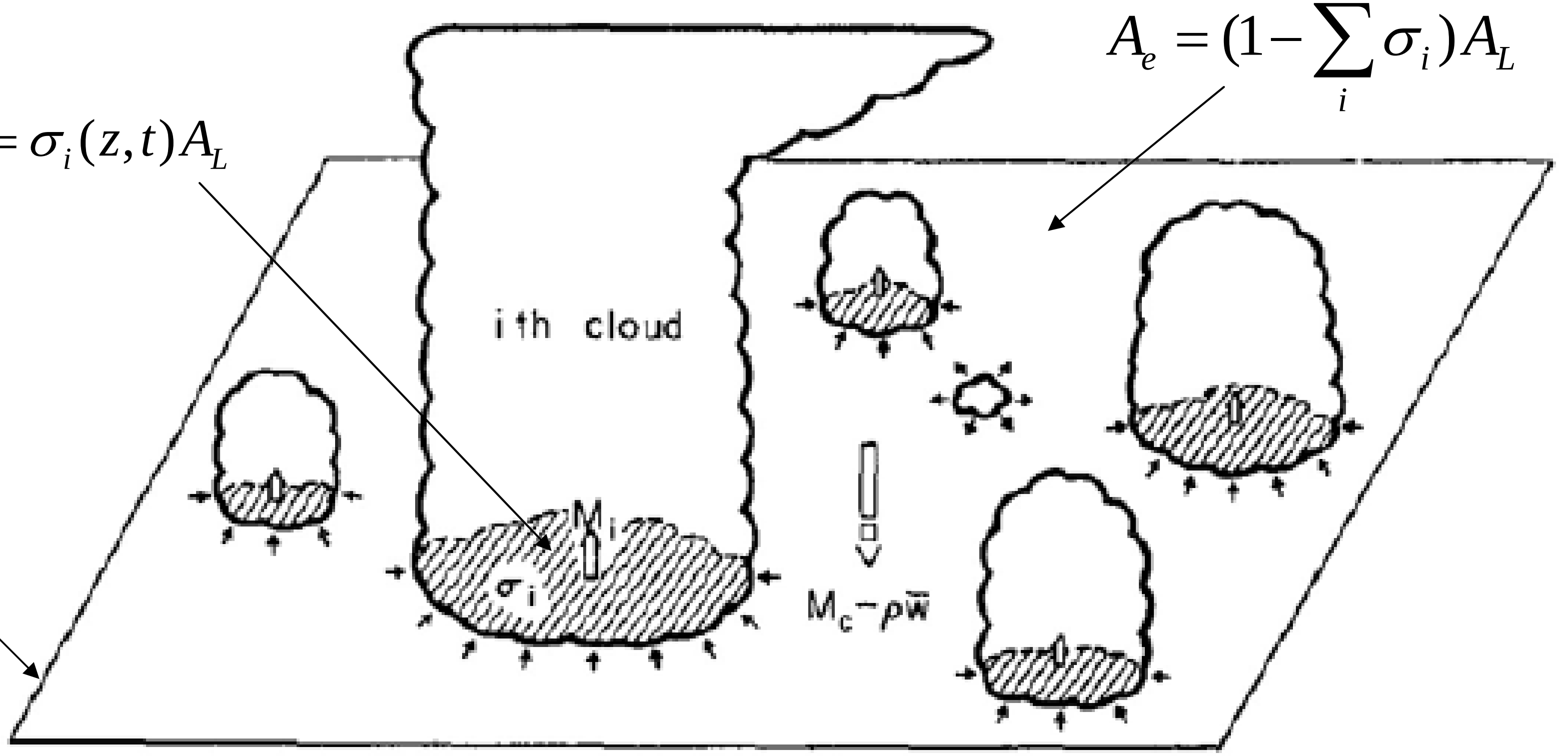


FIG. 1. A unit horizontal area at some level between cloud base and the highest cloud top. The taller clouds are shown penetrating this level and entraining environmental air. A cloud which has lost buoyancy is shown detraining cloud air into the environment.

Large-scale variables and equations

Let an overbar denote the result of an averaging or filtering operation which suppresses fluctuations with temporal and spatial scales smaller than pre-defined limits. e.g. for some appropriately smooth and bounded variable χ after averaging:

$$\overline{\chi}(x,y,z,t) = \frac{1}{\tau L_x L_y L_z} \int_{t-\tau/2}^{t+\tau/2} \int_{x-L_x/2}^{x+L_x/2} \int_{y-L_y/2}^{y+L_y/2} \int_{z-L_z/2}^{z+L_z/2} \chi \, dz' dy' dx' dt'$$

We refer to this as the large-scale variable and assume that our model has sufficient spatial and temporal resolution to represent the variation of this variable once we have determined the equations governing it and an appropriate solution methodology.

Typically, if the variable, χ has the following governing equation:

$$\frac{D\chi}{Dt} = \frac{\partial \chi}{\partial t} + \vec{V} \bullet \nabla \chi = Q, \qquad \chi' = \chi - \overline{\chi}$$

And the mass continuity equation is: $$\frac{\partial \rho}{\partial t} + \nabla \bullet \left(\vec{V} \rho \right) = 0;$$

Then applying the averaging operation gives, approximately:

$$\frac{\partial \overline{\chi}}{\partial t} + \overline{\vec{V}} \bullet \nabla \overline{\chi} + \frac{1}{\overline{\rho}} \nabla \bullet (\overline{\rho} \overline{V'\chi'}) \cong \overline{Q} \qquad \text{if} \qquad \left| \frac{\rho'}{\overline{\rho}} \right| << 1$$

In cases to be considered (e.g. cumulus parameterization) $\nabla \bullet \overline{\rho}\overline{V'\chi'} \cong \frac{\partial (\overline{\rho}\overline{w'\chi'})}{\partial z}$

Determining this term is a goal of the parameterization

Atmospheric Equations $[\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{V} \bullet \nabla]$

Motion $$\frac{D\vec{V}}{Dt} - f\hat{k} \times \vec{V} = -\frac{1}{\rho}\nabla P - g\hat{k} + \nu\nabla^2\vec{V}$$ $$\vec{V} = \hat{i}u + \hat{j}v + \hat{k}w$$

Mass continuity $$\frac{\partial\rho}{\partial t} + \nabla \bullet \left(\rho\vec{V}\right) = 0$$

Thermodynamic $$c_p \frac{DT}{Dt} - \frac{1}{\rho}\frac{Dp}{Dt} = L(c-e) - \nabla \bullet \vec{F}_{rad} + k\nabla^2 c_p T + D$$

Or: $$\frac{DS}{Dt} - gw - \frac{1}{\rho}\frac{Dp}{Dt} = L(c-e) + k\nabla^2 T + \dot{D}$$ $$S = c_p T + \Phi \qquad \frac{D\Phi}{Dt} = gw$$

Vapour $$\frac{Dq_v}{Dt} = e - c$$

Condensed water $$\frac{Dq_w}{Dt} = c - e - \Pr$$

Equation of State (ideal gas) $$P = \rho R T_v \qquad T_v \cong T(1 + .61 q_v - q_w)$$

Energy Conservation (e.g., Gill, 1982, ch. 4)

$$\mathsf{E} = \rho \vec{V} \bullet \vec{V} / 2$$ (kinetic energy)

$$h = c_p T + L q_v + \Phi$$ (moist static energy)

$$\rho \frac{D}{Dt}(\mathsf{E} + h) - \frac{\partial p}{\partial t} = \mu \nabla^2 \mathsf{E} + k \nabla^2 (c_p T) - \nabla \bullet F_{rad}$$

$$\Rightarrow \quad \frac{\partial}{\partial t}[\rho(\mathsf{E} + h) - p] + \nabla \bullet [\rho \vec{V}(\mathsf{E} + h) - \mu \nabla \mathsf{E} - k \nabla (c_p T) + \vec{F}_{rad}] = 0$$

$$\mathsf{D} = \nu \left(\left| \frac{\partial \vec{V}}{\partial x} \right|^2 + \left| \frac{\partial \vec{V}}{\partial y} \right|^2 + \left| \frac{\partial \vec{V}}{\partial z} \right|^2 \right)$$

(μ, ν) Molecular dynamic and kinematic viscosity

For air $\nu \approx 1.4 \times 10^{-5} m^2 / s$ at 15C , 100hPa

Kolmogorov scales (for which viscosity and dissipation are independent parameters): $L_K = \left(\nu^3 / \mathsf{D}\right)^{1/4}; U_K = \left(\nu \mathsf{D}\right)^{1/4}$

These are small for the atmosphere (~ 1mm, .1 m/s) . Therefore it is permissible to neglect viscous terms for parameterization purposes but not to ignore effects/processes that lead to dissipation and associated heating

Quasi-anelastic approximations for AGCM (Atmospheric GCM) parameterization

Background state:

-hydrostatically balanced
- slowly varying (on the smaller, unresolved horizontal and temporal scales - e.g. that of quasi-balanced planetary scale circulation regime).
- deviations from it are small enough to allow linearization of the equation of state (ideal gas law) to determine relationships between key thermodynamic variables:

$$\bar{p} = \bar{\rho} R \bar{T}_v$$

$$\frac{\partial \bar{p}}{\partial z} = -\bar{\rho} g = \bar{p}\left(\frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z} + \frac{1}{\bar{T}_v}\frac{\partial \bar{T}_v}{\partial z}\right) \quad => \quad \frac{1}{\bar{\rho}}\frac{\partial \bar{\rho}}{\partial z} = -\frac{g}{R\bar{T}_v}\left(1 + \frac{R}{g}\frac{\partial \bar{T}_v}{\partial z}\right) \cong -\frac{g}{R\bar{T}_v}$$

$$\frac{p'}{\bar{p}} \cong \frac{\rho'}{\bar{\rho}} + \frac{T_v'}{\bar{T}_v} \quad => \quad \left(\frac{p'}{\bar{\rho}}\right)\frac{\partial \bar{\rho}}{\partial z} + g\left(\rho'\right) \cong -\bar{\rho} g \frac{T_v'}{\bar{T}_v}$$

$$\Rightarrow -\left(\frac{\partial p'}{\partial z} + g\rho'\right) \cong -\bar{\rho}\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right) + \bar{\rho} g \frac{T_v'}{\bar{T}_v}$$

Using these results leads to the following:

$$\overline{\rho}(1+\rho'/\overline{\rho})\frac{D\vec{V}}{Dt} = -\nabla_H\left(p'\right) + \hat{k}\overline{\rho}\left[-\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right) + g\,\frac{T_v'}{\overline{T}_v}\right] - \left\{f\hat{k}\times\vec{V} + \nabla_H\overline{p}\right\}$$

$$\{\frac{\partial}{\partial t}\left(\overline{\rho}(1+\rho'/\overline{\rho})\right)\} + \nabla\bullet\left(\overline{\rho}(1+\rho'/\overline{\rho})\vec{V}\right) = 0$$

Terms in curly brackets: negligible for the parameterized scales but not for the resolved scales

$$\overline{\rho}(1+\rho'/\overline{\rho})\frac{Dh}{Dt} - \left(\frac{\partial p'}{\partial t} + \vec{V}\bullet\nabla_H p'\right) + \overline{\rho}w\left[g\,\frac{T_v'}{\overline{T}_v} - \frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right] - \left\{\frac{\partial\overline{p}}{\partial t} + \vec{V}\bullet\nabla_H\overline{p}\right\} = Q \qquad (h = c_pT + L_vq + \Phi)$$

Terms involving $\rho'/\overline{\rho}$ will also be neglected compared to unity. The approximate mass continuity equation which will be used is:

$$\{\frac{\partial}{\partial t}\overline{\rho}\} + \nabla\bullet\left(\overline{\rho}\vec{V}\right) = 0$$

Upon using this continuity equation to develop the flux-form equations and averaging:

$$\frac{\partial\left(\overline{\rho}\overline{\vec{V}}_H\right)}{\partial t} + \nabla\bullet\left(\overline{\rho}\overline{\vec{V}}\,\overline{\vec{V}}_H\right) + f\hat{k}\times\overline{\vec{V}} = -\nabla_H\overline{p} - \frac{\partial\left(\overline{\rho}\,\overline{w'\vec{V}'}\right)}{\partial z}$$

+ other such (horizontal) terms

$$\frac{\partial\left(\overline{\rho}\overline{h}\right)}{\partial t} + \nabla\bullet\left(\overline{\rho}\overline{\vec{V}}\overline{h}\right) - \left(\frac{\partial\overline{p}}{\partial t} + \overline{\vec{V}}\bullet\nabla_H\overline{p}\right) = \overline{Q} - \frac{\partial\left(\overline{\rho}\,\overline{(w'h')}\right)}{\partial z} + \frac{\partial}{\partial z}(\overline{w'p'}) - \overline{\rho}\,\overline{(w'B)}$$

+ other…

$$\overline{Q} = \overline{Q}_R + \overline{\mathrm{D}} \qquad B = gT_v'/\overline{T}_v$$

Parameterization of the effects of Moist Convection in GCMs

- Mass flux schemes
  - Basic concepts and quantities
  - Quasi-steady Entraining/detraining plumes (Arakawa&Schubert and similar approaches)
  - Buoyancy sorting
    - Raymond-Blythe, Emanuel
    - Kain-Fritsch
  - Closure Conditions, Triggering
- Adjustment Schemes
  - Manabe
  - Betts-Miller

Traditional Assumptions for Cumulus Parameterization:

1. Quasi-steady assumption: effects of averaging over a cumulus life-cycle can be represented in terms of steady-state convective elements .

[Transient (cloud life-cycle) formulations: Kuo (1964, 1974); Fraedrich(1974), Betts(1975), Cho(1977), von Salzen&McFarlane (2002).]

2. Pressure perturbations and effects on momentum ignored

[Some of these effects have been reintroduced in more recent work, but not necessarily in an energetically consistent manner]

Parameterization of Moist Convection

Starting equations (neglect terms in curly and other small terms brackets and assume implicitly that the background state is slowly varying on the parameterized scales):

$$\frac{\partial(\overline{\rho}\vec{V})}{\partial t} + \nabla \bullet \left(\overline{\rho}\vec{V}\vec{V}\right) = -\nabla_H\left(p'\right) + \hat{k}\overline{\rho}\left[-\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right) + g\frac{T_v'}{\overline{T}_v}\right]$$

$$\frac{\partial \rho}{\partial t} + \nabla_H \bullet \rho\vec{V} + \frac{\partial(\rho w)}{\partial z} \cong 0 \cong \overline{\rho}\nabla_H \bullet \vec{V} + \frac{\partial(\overline{\rho}w)}{\partial z}$$

$$\frac{\partial\left[\overline{\rho}h\right]}{\partial t} + \nabla \bullet \left(\overline{\rho}h\vec{V}\right) - \left(\frac{\partial p'}{\partial t} + \vec{V} \bullet \nabla_H p'\right) + \overline{\rho}w\left[g\frac{T_v'}{\overline{T}_v} - \frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right] = Q$$

plus similar equations for vapour, condensed water, and other scalar quantities

For the traditional formulation ignore crossed-out terms

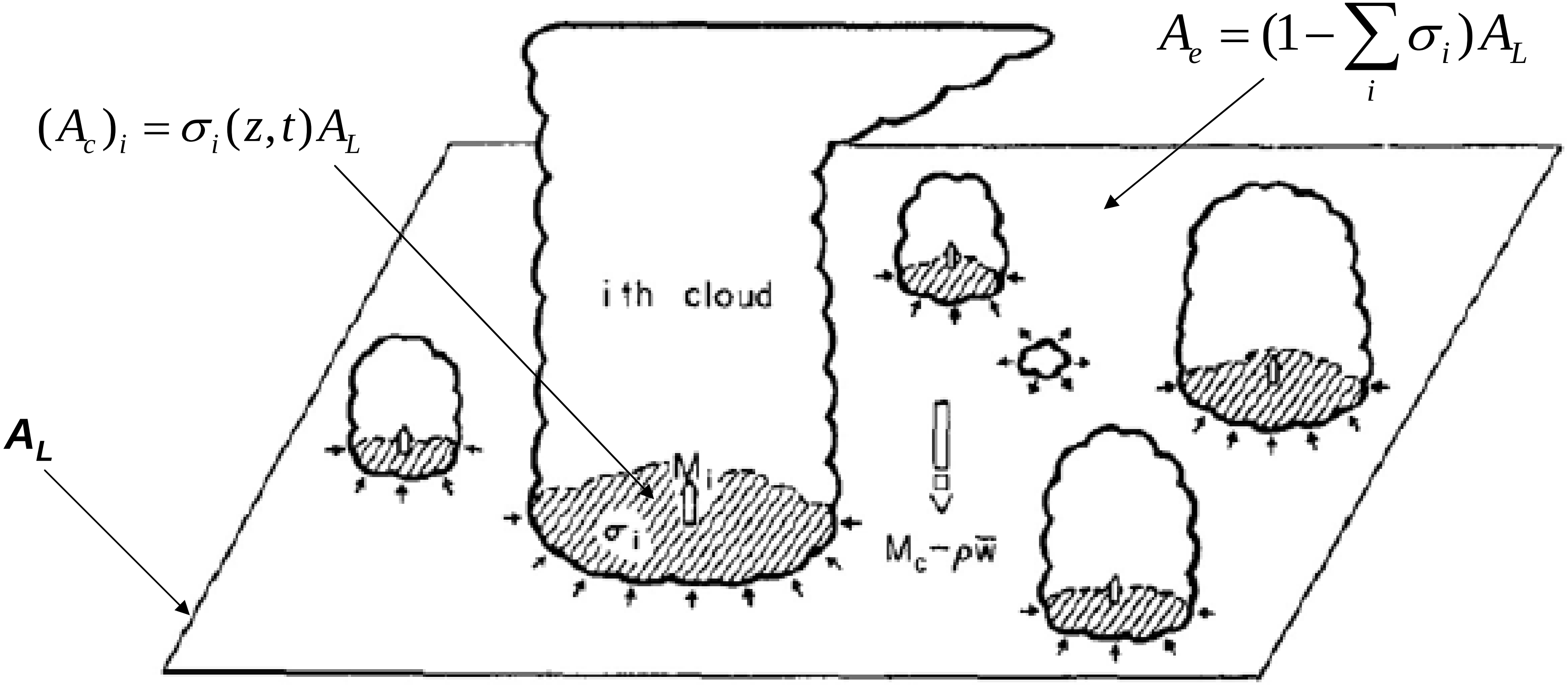


FIG. 1. A unit horizontal area at some level between cloud base and the highest cloud top. The taller clouds are shown penetrating this level and entraining environmental air. A cloud which has lost buoyancy is shown detraining cloud air into the environment.

Spatial Averages

For a generic scalar variable, χ:

Large-scale average: $$\overline{\chi} = \left(\frac{1}{A_L}\right)\iint_{A_L} \chi \, dA$$

Convective-scale average
(for a singlecumulus up/downdraft) : $$\chi_c = \left(\frac{1}{A_c}\right)\iint_{A_c} \chi \, dA$$

Environment average
(single convective element): $$\chi_e = \left(\frac{1}{A_L - A_c}\right)\iint_{Ae} \chi \, dA$$

Where $\sigma = A_c / A_L << 1$ $\qquad |\chi_{c,e}| / |\overline{\chi}| = O(1)$

$$\overline{\chi} = \sigma\chi_c + (1-\sigma)\chi_e \qquad \chi' = \chi - \overline{\chi} = \sigma\chi^* + (1-\sigma)\hat{\chi}$$

Vertical velocity: $\overline{w} = \sigma w_c + (1-\sigma) w_e$ $\qquad |w_c| >> |\overline{w}|, |w_e|$

Ensemble of cumulus clouds: $\sigma = \sum_i \sigma_i \qquad \overline{\chi} = \sum_i \sigma_i \chi_i + (1-\sigma)\chi_e$

Lecture 2

- Cumulus Friction and Energetics
- Parameterization of Boundary Layer Processes

Cumulus effects on the larger-scales

Start with a general conservation equation for χ

$$\frac{\partial(\rho\chi)}{\partial t} + \nabla_H \bullet (\rho\vec{V}\chi) + \frac{\partial(\rho w \chi)}{\partial z} = Q_\chi$$

Plus the assumption: $\rho \cong \overline{\rho}$

(similar to using anelastic assumption for convective-scale motions)

(i) *Average over the large-scale area (assuming fixed boundaries):*

$$\frac{\partial(\overline{\rho}\overline{\chi})}{\partial t} + \nabla \bullet (\overline{\rho}\overline{\vec{V}}\overline{\chi}) + \frac{\partial(\overline{\rho}\overline{w}\,\overline{\chi})}{\partial z} = \overline{Q}_\chi - \frac{\partial(\overline{\rho}\sigma w_c(\chi_c - \overline{\chi}))}{\partial z} - \frac{\partial(\overline{\rho}\sigma(w^*\chi^*)_c)}{\partial z} + others$$

Mass flux (positive for updrafts): $M_c = \overline{\rho}\sigma w_c$

Also: $\overline{Q}_\chi = \sigma(Q_\chi)_c + (1-\sigma)(Q_\chi)_e$; "Top hat" assumption: $(w^*\chi^*)_c = 0$

In practice (e.g. in a GCM) the prognostic variables are also implicitly time averages over convective cloud life-cycles

(ii) *Apply cumulus scale sub-average to the general conservation equation, accounting for temporally and spatially varying boundaries (Leibnitz rule):*

$$\frac{\partial(\bar{\rho}\sigma\chi_c)}{\partial t} + \frac{\bar{\rho}\sigma}{A_c}\oint_{\sigma} v_n \chi_b dl + \frac{\partial(\bar{\rho}\sigma[w_c\chi_c + (w^*\chi^*)_c])}{\partial z} = \sigma(Q_\chi)_c$$

Mass continuity gives:

$$\frac{\partial\bar{\rho}\sigma}{\partial t} + \frac{\bar{\rho}\sigma}{A_c}\oint_{\sigma} v_n dl + \frac{\partial(\bar{\rho}a w_c)}{\partial z} = 0$$

; $v_n =$ the outward directed normal flow velocity (relative to the cloud boundary)

Entrainment (inflow)/detrainment (outflow):

$$E = -\frac{\bar{\rho}\sigma}{A_c}\oint_{\sigma} v_n[1 - H(v_n)]dl \qquad D = \frac{\bar{\rho}\sigma}{A_c}\oint_{\sigma} v_n H(v_n)dl \qquad H(f) = \begin{cases} 1; & f \geq 0 \\ 0; & f < 0 \end{cases}$$

Define: $\chi_E = \frac{\bar{\rho}\sigma}{EA_c}\left|\oint_{c} v_n \chi_b [1 - H(v_n)]dl\right|$ $\qquad \chi_D = \frac{\bar{\rho}\sigma}{DA_c}\left|\oint_{c} v_n \chi_b H(v_n)dl\right|$

Top hat: $\chi_E = \chi_e \cong \bar{\chi}$; $\chi_D = \chi_c$;

Summary for a generic scalar, χ :
(steady and top hat in cloud drafts: neglect crossed-out terms)

$$\frac{\partial(\overline{\rho}\overline{\chi})}{\partial t} + \nabla \bullet \left(\overline{\rho}\overline{\vec{V}}\overline{\chi}\right) + \frac{\partial(\overline{\rho}\overline{w}\overline{\chi})}{\partial z} = -\frac{[M_c(\chi_c - \overline{\chi}) + \overline{\rho}\sigma\cancel{\left(w^*\chi^*\right)_c}]}{\partial z} + \overline{Q}_\chi + other$$

$$\cancel{\frac{\partial(\overline{\rho}\sigma)}{\partial t}} + D - E + \frac{\partial(M_c)}{\partial z} = 0$$

$$\cancel{\frac{\partial(\overline{\rho}\sigma\chi_c)}{\partial t}} + D\chi_D - E\overline{\chi} + \frac{\partial\left(M_c\chi_c + \overline{\rho}\sigma\cancel{(w^*\chi^*)_c}\right)}{\partial z} = \sigma\left(Q_\chi\right)_c$$

$$\therefore \frac{\partial(\overline{\rho}\overline{\chi})}{\partial t} + \nabla \bullet \left(\overline{\rho}\overline{\vec{V}}\overline{\chi}\right) + \frac{\partial(\overline{\rho}\overline{w}\overline{\chi})}{\partial z} = M_c\frac{\partial\overline{\chi}}{\partial z} + D\left(\chi_D - \overline{\chi}\right) + (1-\sigma)\left(Q_\chi\right)_e + other$$

When both updrafts and downdrafts are present, both entraining environmental air:

$$M_c = \rho\sigma w_c = M_u + M_d ; E = E_u + E_d ; D = D_u + D_d$$

$$M_c\chi_c = M_u\chi_u + M_d\chi_d ; D\chi_c = D_u\chi_u + D_d\chi_d$$

Basic cumulus updraft equations (top-hat, traditional)

{Dry static energy: $s=C_pT+gz$; Moist static energy : $h=s+Lq$; $M_u = \overline{\rho}\sigma w_u$ }

$$D_u - E_u + \frac{\partial M_u}{\partial z} = 0$$ mass conservation

$$D_u s_u - E_u \overline{s} + \frac{\partial (M_u s_u)}{\partial z} = Lc_u$$ dry Static Energy

$$D_u q_u - E_u \overline{q} + \frac{\partial (M_u q_u)}{\partial z} = -c_u$$ vapour

$$D_u l_u + \frac{\partial (M_u l_u)}{\partial z} = c_u - P_u$$ condensate

$$D_u h_u - E \overline{h} + \frac{\partial (M_u h_u)}{\partial z} = 0$$ moist Static Energy

$$\theta_v = T_v (p_o / p)^\kappa \;;\; \kappa = R/c_p \;;\; T_v \cong T(1+.608q-l)$$ (virtual temperature)

Entrainment/Detrainment

Traditional organized (e.g.plume) entrainment assumption:

$$E = -\oint_c v_n[1 - H(v_n)]dl = \langle v_{in} \rangle \mathrm{P}_c = \alpha w_c \mathrm{P}_c \qquad \mathrm{P}_c = \oint_c dl \quad \text{(draft perimeter)}$$

$$\Longrightarrow \quad E = \left(\alpha \frac{P_c}{A_c} \right) \bar{\rho} \sigma w_c = \lambda M_c \approx \frac{2\alpha}{R_c} \bar{\rho} \sigma w_c$$

Arakawa & Schubert (1974) (and descendants, e.g. RAS, Z-M):
- λ is a constant for each updraft [saturated homogeneous (top-hat) entraining plumes]
- detrainment is confined to a narrow region near the top of the updraft, which is located at the level of zero buoyancy (determines λ)

Kain & Fritsch (1990) (and descendants, e.g. Bretherton et al, 2004):
- Rc is specified (constant or varying with height) for a given cumulus
- entrainment/detrainment controlled by bouyancy sorting (i.e. the effective value of α is constrained by buoyancy sorting)

Episodic Entrainment and non-homogeneous mixing
(Raymond&Blythe, Emanuel, Emanuel&Zivkovic-Rothman):
-Not based on organized entrainment/detrainment
- entrainment at a given level gives rise to an ensemble of mixtures of undiluted and environmental air which ascend/descend to levels of neutral buoyancy and detrain

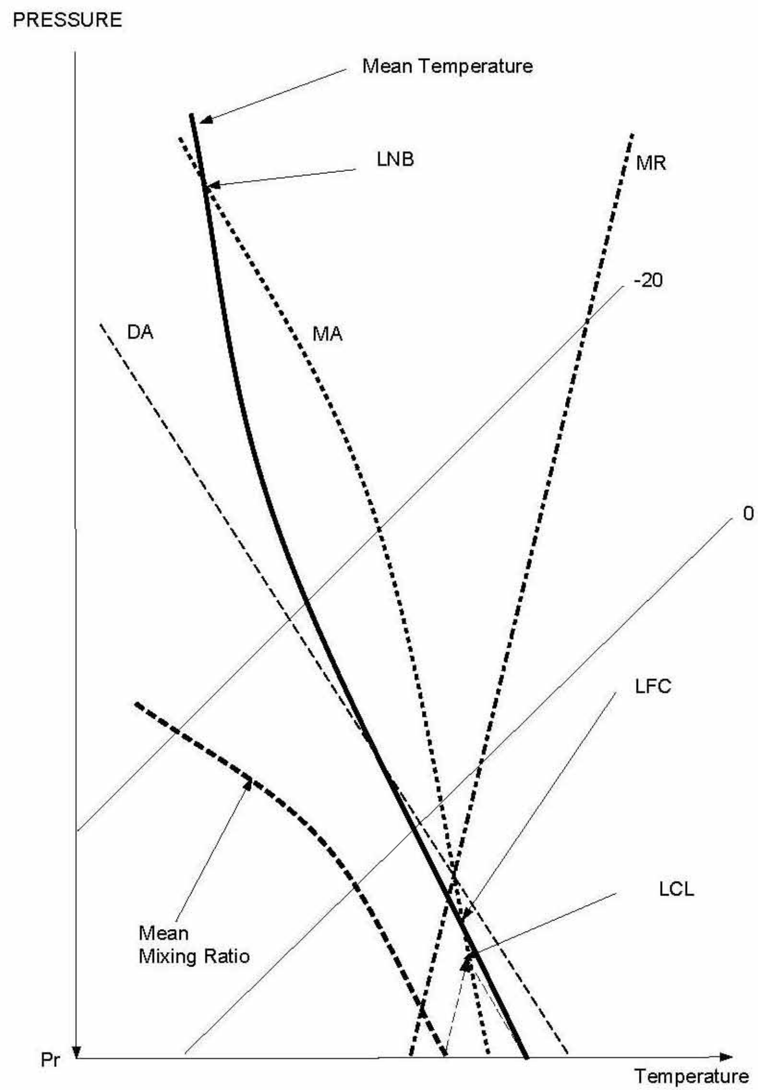


Figure 1. A schematic view of a conditionally unstable atmospheric sounding showing the mean temperature and water vapor mixing ratio profiles , typical orientations of dry (DA) and moist (MA) adiabats, and isopleths of mixing ratio (MR). Also shown are the 0C and -20C isotherms, locations of the lifting condensation level (LCL) and the levels of free convection (LFC), and neutral buoyancy (LNB) for an undiluted parcel that does not contain liquid water.

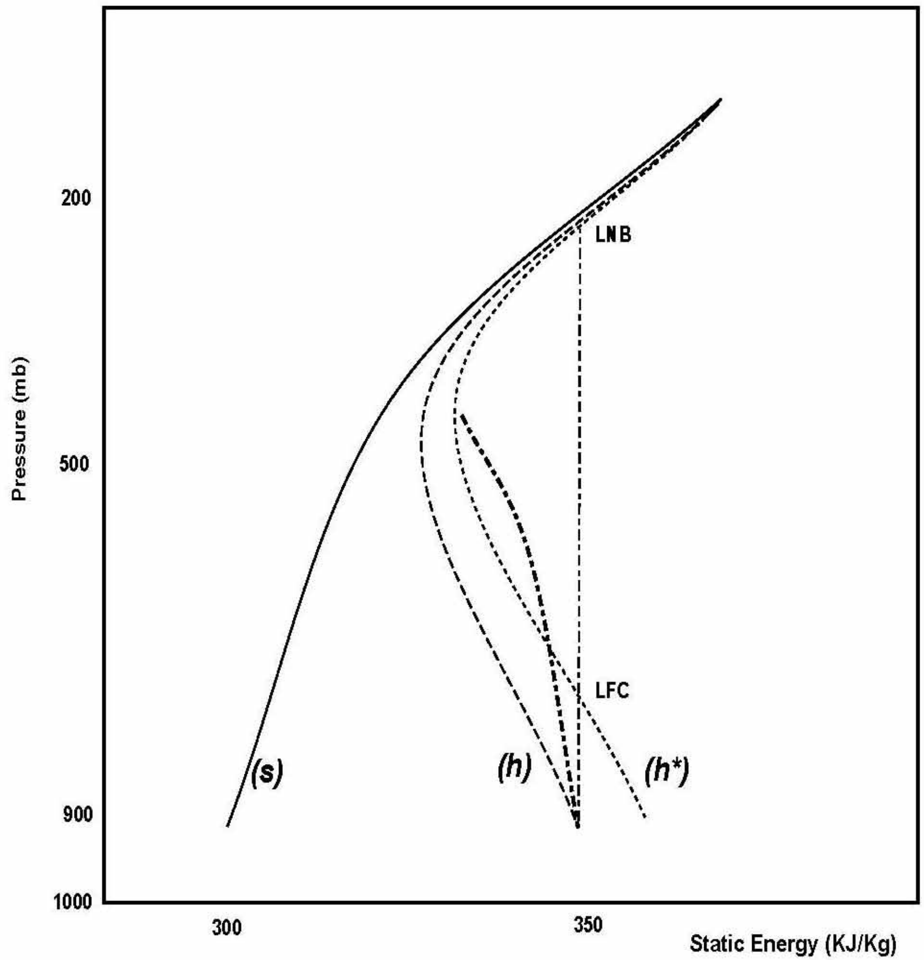
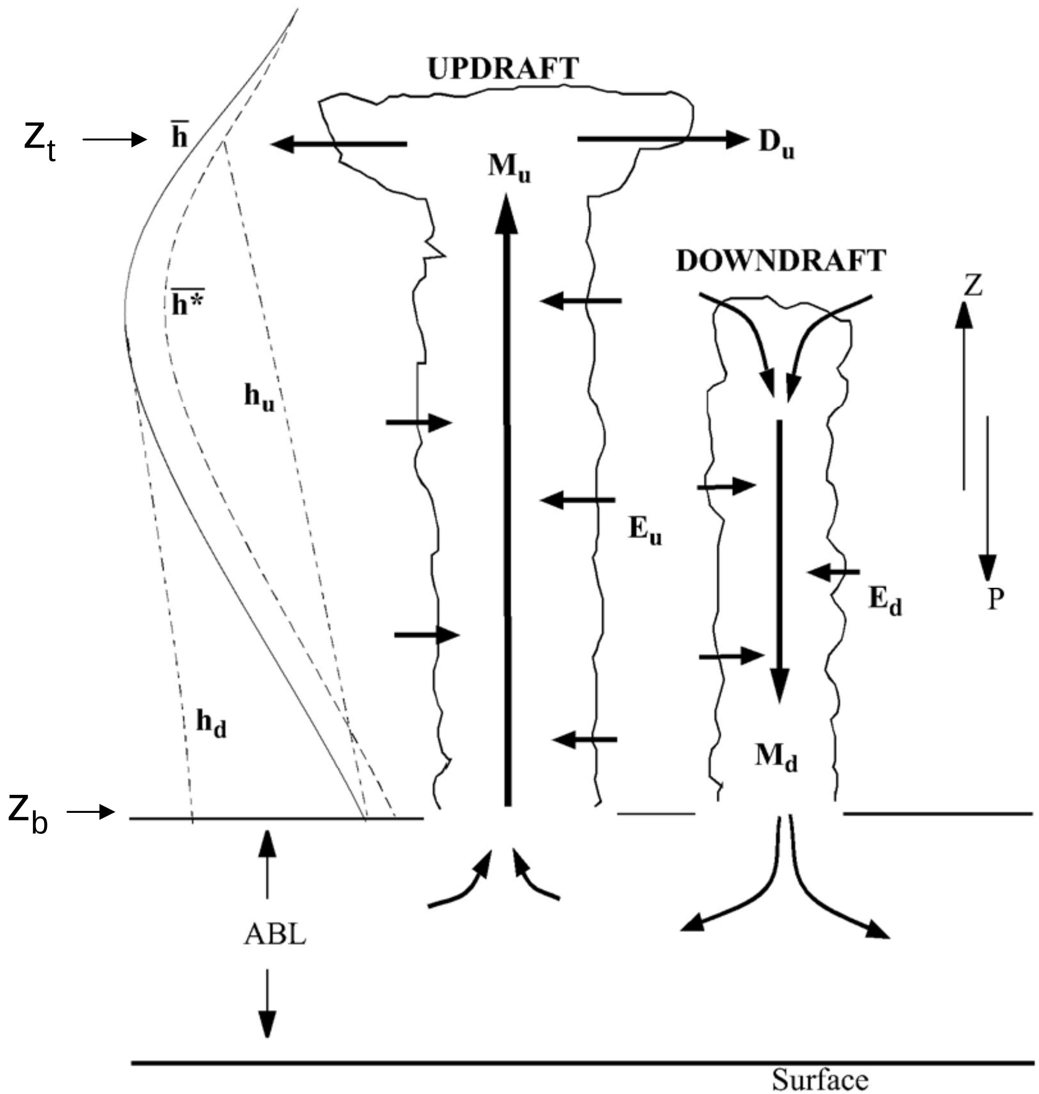


Figure 2. Schematic view of a conditionally unstable atmospheric sounding in terms of the mean dry (s) and moist (h) static energy profiles. Also shown are the moist static energy profiles for saturated air with the same temperature as the mean sounding (h^*), the moist static energy profiles for undiluted (vertical long-short dashed) and diluted (bold long-short dashed) cumulus cloud soundings, the levels of free convection (LFC) and neutral buoyancy (LNB) for an undiluted parcel which ascends pseudo-adiabatically (i.e. does not retain condensed water).

environmental temperature and pressure. In this figure an undiluted parcel ascending from the atmospheric boundary layer (ABL) follows the vertical straight (long-short dashed) line. Its temperature exceeds that of the environment between the points where its moist static energy exceeds the saturated value for



UPDRAFT
z_t
$\bar{h}$
D_u
M_u
DOWNDRAFT
Z
$\overline{h^*}$
h_u
E_u
E_d
P
h_d
M_d
z_b
ABL
Surface

Determining fractional entrainment rates (e.g. when $T_c \cong T_e$ at the top of an updraft)

$$\frac{\partial h_i}{\partial z} = \lambda_i \left(\overline{h} - h_i \right) \qquad h_i(z_b) = \overline{h}(z_b) \qquad h_i((z_t)_i, \lambda_i) = h^*((z_t)_i)$$

$$h^*((z_t)_i) = \overline{h}(z_b)\exp[-\lambda_i((z_t)_i - z_b)] + \lambda_i \int_{z_b}^{(z_t)_i} \overline{h}(z')\exp[\lambda_i(z' - (z_t)_i)]dz'$$

Note that since updrafts are saturated with respect to water vapour above the LCL:

$$\frac{h_i - h^*}{c_p} = T_i - \overline{T} + \left(\frac{L}{c_p} \right)(q_i - q^*(\overline{T}, p)) \cong (T_i - \overline{T})(1 + \frac{L}{c_p}\frac{\partial q^*}{\partial \overline{T}}) + O(T_i - \overline{T})^2$$

This determines the updraft temperature and w.v. mixing ratio given its mse.

Fractional entrainment rates for updraft ensembles

(a) Single ensemble member detraining at $z=z_t$

$$E_u = \lambda(z_t) M_u ; D_u = 0 \qquad (z_b \le z < z_t)$$

$$M_u = M_b \exp\left[\lambda(z_t)(z - z_b)\right]$$

Detrainment over a finite depth Δz_t : $D(z_t) = M_u(z_t) / \Delta z_t$

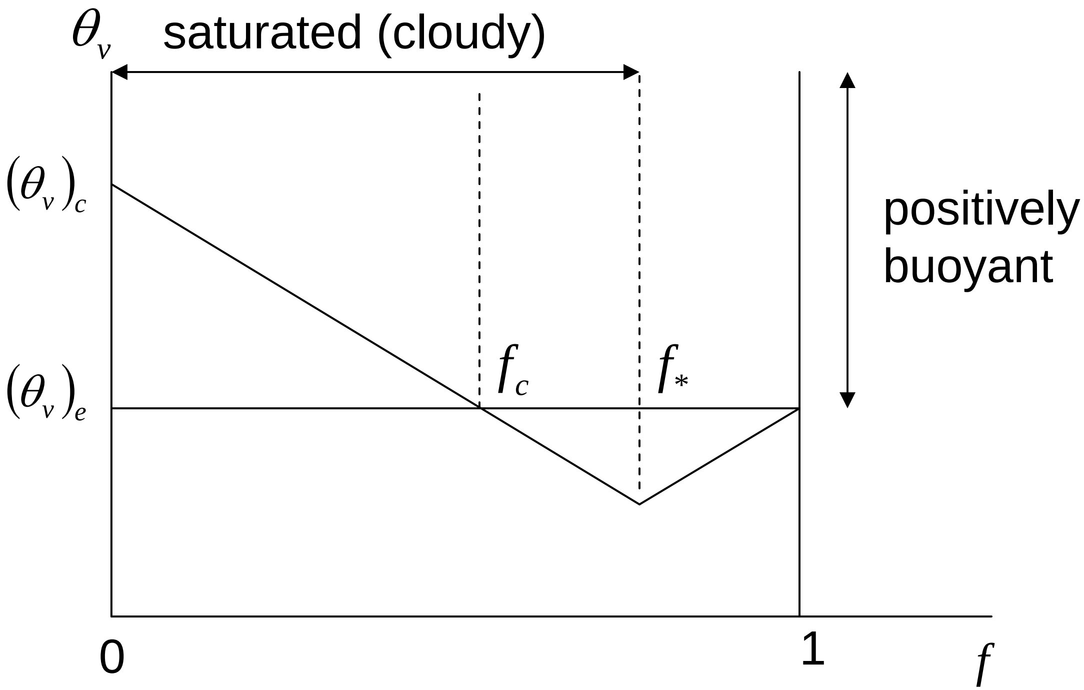
(b) Discrete ensemble based on a range of tops

$$M_u = \sum_i M(z, \lambda_i); \lambda_i = \lambda((z_t)_i) \qquad M_u h_u = \sum_i M_i h_i$$

$$E_u = \sum_i \lambda_i M_i \qquad D_u = -\sum_i \frac{\partial M_i}{\partial \lambda_i} \frac{\Delta \lambda}{\Delta z_t}$$

Buoyancy Sorting

Entrainment produces mixtures of a fraction, f, of environmental air and (1-f) of cloudy (saturated cumulus updraft) air. Some of the mixtures may be positively buoyant with respect to the environment, some negegatively buoyant, some saturated with respect to water, some unsaturated



Kain-Fritsch (1990) (see also Bretherton et al, 2003):

Suppose that entrainment into a cumulus updraft in a layer of thickness δz leads to mixing of $\lambda M_c dz$ of environmental air with an equal amount of cloudy air. K-F assumed that all of the negatively buoyant mixtures ($f>f_c$) will be rejected from the updraft immediately while positively buoyant mixtures will be incorporated into the updraft. Let *P(f)* be the pdf of mixing fractions. Then:

$$E = 2\lambda_o M_u \int_0^{f_c} fP(f)df \qquad D = 2\lambda_0 M_u \int_{f_c}^{1} (1-f)P(f)df$$

This assumes that negatively buoyant air detrains back to the environment without requiring it to descend to a level of nuetral bouyancy first).

Emanuel:

Mixtures are all combinations of environement air and undiluted cloud-base air. Each mixture ascends(positively buoyant)/descends (negatively buoyant), typically without further mixing to a level of nuetral buoyancy where it detrains.

Closure and Triggering

- Triggering:
  - It is frequently observed that moist convection does not occur even when there is a positive amount of CAPE. Processes which overcome convective inhibition must also occur.
- Closure:
  - The simple cloud models used in mass flux schemes do not fully determine the mass flux. Typically an additional constraint is needed to close the formulation.
  - The closure problem is currently still poorly constrained by theory.

Both may involve stochastic processes

Closure Schemes In Use (typically to determine the net mass flux at the base of the convective layer)

- Moisture convergence~ Precipitation (Kuo, 1974- for deep precipitating convection)
- Quasi-equilibrium [Arakawa and Schubert, 1974 and descendants (RAS, Z-M, Zhang&Mu, 2005)]
- Prognostic mass-flux closures (Pan & Randall, 1998;Scinocca&McFarlane, 2004)
- Closures based on boundary-layer forcing (Emanuel&Zivkovic-Rothman, 1998; Bretherton et al., 2004)
- Stochastic closures may combine one of the above with a stochastic formulation for cumulus ensemble properties (e.g. Craig&Cohen papers, Plant&Craig)

Zonally averaged variance of latent heating for different convective closures and downdraft evaporation efficiency parameters

(Scinocca & McFarlane, 2004)

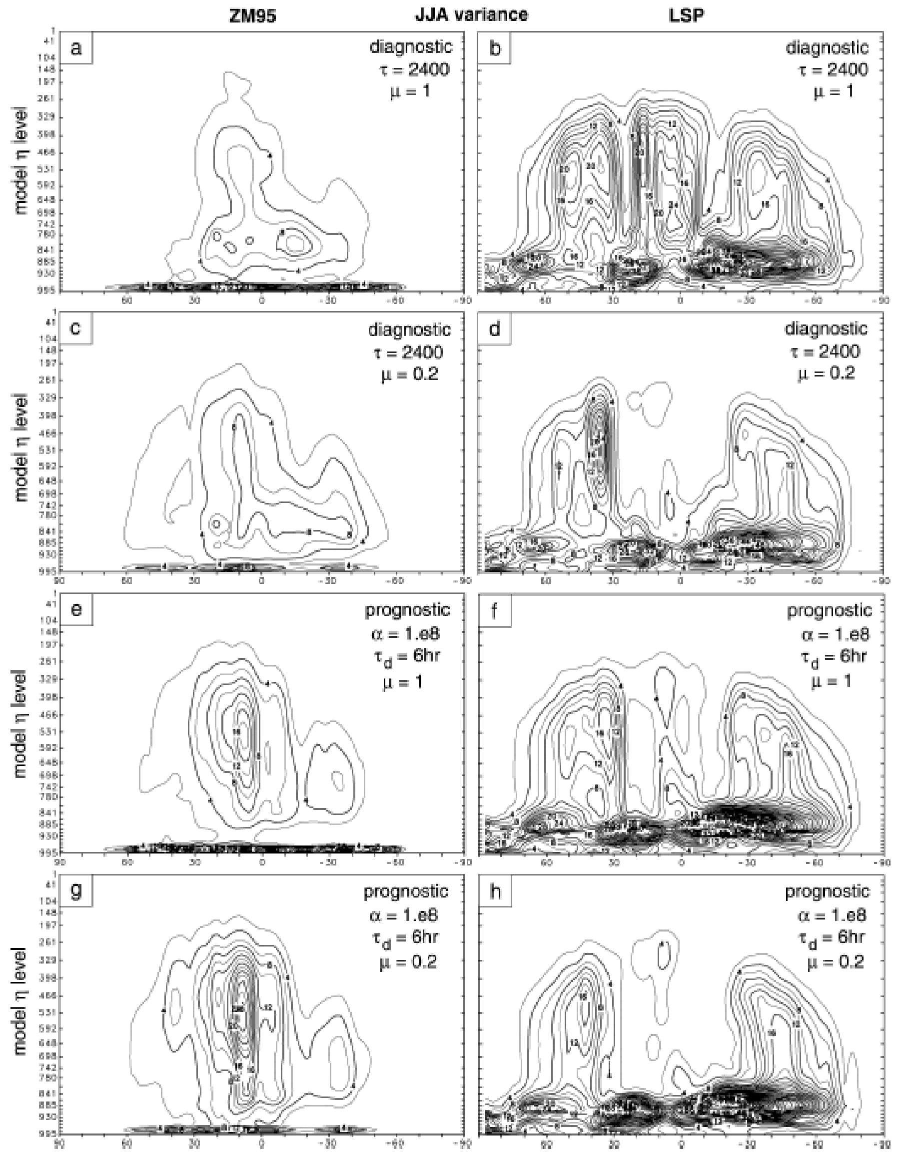


FIG. 9. As in Fig. 5 except for the sensitivity experiments presented in Figs. 7 and 8. Heating rates are displayed in units of 0.1 K day^{-1} with a contour interval of 2 K day^{-1} for both ZM and LSP.

Cumulus friction and energetics

$$M_c = \bar{\rho}\sigma w_c \qquad \frac{\partial M_c}{\partial z} + D - E = 0$$

(usually parameterized)

$$\frac{\partial (M_c \vec{V}_c)}{\partial z} + D\vec{V}_c - E\vec{V}_E = -\sigma\left(\nabla_H p'\right)_c + \bar{\rho}\sigma\left[B_c - \left(\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)_c\right]\hat{k}$$

$$\frac{\partial\left(M_c h_c\right)}{\partial z} + Dh_c - Eh_E + \bar{\rho}M_c B_c - \sigma\left(\vec{V} \bullet \nabla_H p' + \bar{\rho}w\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)_c = \mathrm{D}_c$$

Take the dot product of $\vec{V}_c$ with the momentum eq. It can be shown that

$$\frac{\partial\left[M_c K_c\right]}{\partial z} + DK_c - EK_E + E\frac{\left|\vec{V}_c - \vec{V}_E\right|^2}{2} = -\sigma V_c \bullet \left(\nabla_H p' + \bar{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)_c + \bar{\rho}M_c B_c$$

Total energy:

$$\frac{\partial}{\partial z}\left[M_c\left(K_c + h_c\right)\right] + D\left(K + h\right)_D - E(K + h)_E + E\frac{\left|\vec{V}_c - \vec{V}_E\right|^2}{2} = \sigma\left((\vec{V} - \vec{V}_c) \bullet \left(\nabla_H p' + \bar{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)\right)_c + \mathrm{D}_c$$

The dissipation heating term is intrinsically positive. Choose it as

$$D_c = E\frac{\left|\vec{V}_c - \vec{V}_E\right|^2}{2}$$

Assume , consistent with top-hat that

$$\sigma\left[\left(\vec{V} - \vec{V}_c\right)\bullet\left(\nabla_H p' + \overline{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)\right]_c$$ is negligible

$$\therefore \frac{\partial\left(M_c h_c\right)}{\partial z} + Dh_c - Eh_E + M_c B_c \cong \sigma\vec{V}_c \bullet \left(\nabla_H p' + \overline{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c + E\frac{\left|\vec{V}_c - \vec{V}_E\right|^2}{2}$$

↑

All parameterized

From the mean equations (ignoring, for simplicity, non-cumulus contributions to prime terms):

$$\frac{\partial\left(\overline{\rho}\overline{\vec{V}}_H\right)}{\partial t}+\nabla\bullet\left(\overline{\rho}\overline{\vec{V}}\overline{\vec{V}}_H\right)+f\hat{k}\times\overline{\vec{V}}=-\nabla_H\overline{p}-\frac{\partial\left(M_c(\vec{V}_c-\overline{\vec{V}})_H\right)}{\partial z} \qquad (1)$$

$$\frac{\partial\left(\overline{\rho}\overline{h}\right)}{\partial t}+\nabla\bullet\left(\overline{\rho}\left(\overline{\vec{V}}\right)\overline{h}\right)-\left(\frac{\partial\overline{p}}{\partial t}+\overline{\vec{V}}\bullet\nabla_H\overline{p}\right)=\overline{Q}-\frac{\partial\left(M_c\left[(h_c)-(\overline{h})\right]\right)}{\partial z}+\left[\overline{\left(\vec{V}-\overline{\vec{V}}\right)\bullet\left(\nabla_H p'+\overline{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)}\right]_c-M_cB_c \qquad (2)$$

$$\overline{Q}=\overline{Q}_R+\overline{D}$$

$$\left[\overline{\left(\vec{V}-\overline{\vec{V}}\right)\bullet\left(\nabla_H p'+\overline{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)}\right]_c\cong\sigma\left(\vec{V}_c-\overline{\vec{V}}\right)\left(\nabla_H p'+\overline{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c$$ (for top-hat profiles)

Kinetic energy: $\overline{\vec{V}}_H\bullet eq(1)$ $+\ cumulus\ k.e.\ eq.$

$$\frac{\partial\left(\overline{\rho}\overline{K}_H\right)}{\partial t}+\nabla\bullet\left(\overline{\rho}\overline{\vec{V}}\overline{K}_H\right)=-\overline{\vec{V}}\bullet\nabla_H\overline{p}-\frac{\partial}{\partial z}\left[M_c(K_c-\overline{K}_H)\right]+M_cB_c-D\frac{\left|\vec{V}_c-\overline{\vec{V}}\right|_H^2}{2}-E\frac{\left|\vec{V}_c-\overline{\vec{V}}\right|^2}{2}$$

$$-\sigma\left[\left(\vec{V}_c-\overline{\vec{V}}\right)\bullet\left(\nabla_H p'\right)_c+\overline{\rho}w_c\left(\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c\right] \qquad (3)$$

↑ parameterized (the $\left(\vec{V}_c-\overline{\vec{V}}\right)\bullet\left(\nabla_H p'\right)_c$ term)

Combine (2) +(3):

$$\frac{\partial\left(\overline{\rho}(\overline{h}+\overline{K}_H)\right)}{\partial t}+\nabla\bullet\left(\overline{\rho}\overline{\vec{V}}(\overline{h}+\overline{K}_H)\right)-\left(\frac{\partial\overline{p}}{\partial t}\right)=\overline{Q}_R-\frac{\partial\left(M_c\left[(h_c+K_c)-\left(\overline{h}+\overline{K}_H\right)\right]\right)}{\partial z}$$

$$+\left\{\overline{\mathrm{D}}-\left[(D+E)\frac{\left|\vec{V}_c-\overline{\vec{V}}\right|_H^2}{2}+E\frac{\left(w_c-\overline{w}\right)^2}{2}\right]\right\}$$

The R.H.S. should be in flux form. Q_R is the radiative flux divergence.
Dissipational heating should be positive. Suggests :

$$\overline{\mathrm{D}}=\left[(D+E)\frac{\left|\vec{V}_c-\overline{\vec{V}}\right|_H^2}{2}+E\frac{\left(w_c-\overline{w}\right)^2}{2}\right]=\mathrm{D}_c+\mathrm{D}_E$$

$$\Rightarrow\quad \frac{\partial\left(\overline{\rho}\overline{h}\right)}{\partial t}+\nabla\bullet\left(\overline{\rho}\left(\overline{\vec{V}}\right)\overline{h}\right)-\left(\frac{\partial\overline{p}}{\partial t}+\overline{\vec{V}}\bullet\nabla_H\overline{p}\right)\cong\overline{Q}-\frac{\partial\left(M_c\left(h_c-\overline{h}\right)\right)}{\partial z}+$$

$$\sigma\left(\vec{V}_c-\overline{\vec{V}}_H\right)\bullet\left(\nabla_H p'+\overline{\rho}\hat{k}\frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c-M_cB_c$$

In summary, assuming top-hat cumulus profiles

$$\frac{\partial M_c}{\partial z} + D - E = 0 \qquad \text{(a)}$$

$$\frac{\partial \left(M_c h_c\right)}{\partial z} + Dh_c - Eh_E = -M_c B_c + \sigma \vec{V}_c \bullet \left(\nabla_H p' + \overline{\rho} \frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c + E\frac{\left|\vec{V}_c - \vec{V}_E\right|^2}{2} \qquad \text{(b)} \qquad \left(h_E \cong \overline{h}\right)$$

$$\frac{\partial (M_c \vec{V}_c)}{\partial z} + D\vec{V}_c - E\vec{V}_E = -\sigma \left(\nabla_H p' + \overline{\rho}\hat{k} \frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c + \overline{\rho}\sigma B_c \hat{k} \qquad \text{(c)} \qquad (\vec{V}_E \cong \overline{\vec{V}})$$

$$\frac{\partial \left(\overline{\rho}\overline{h}\right)}{\partial t} + \nabla \bullet \left(\overline{\rho}\left(\overline{\vec{V}}\right)\overline{h}\right) - \left(\frac{\partial \overline{p}}{\partial t} + \overline{\vec{V}} \bullet \nabla_H \overline{p}\right) = \overline{Q}_R - \frac{\partial \left(M_c\left(h_c - \overline{h}\right)\right)}{\partial z} + \sigma\left(\vec{V}_c - \overline{\vec{V}}_H\right) \bullet \left(\nabla_H p' + \overline{\rho}\hat{k} \frac{\partial}{\partial z}\left(\frac{p'}{\overline{\rho}}\right)\right)_c$$

$$- M_c B_c + D\frac{\left|\vec{V}_c - \overline{\vec{V}}\right|_H^2}{2} + E\frac{\left|\vec{V}_c - \overline{\vec{V}}_H\right|^2}{2} \qquad \text{(d)}$$

$$\frac{\partial \left(\overline{\rho}\overline{\vec{V}}_H\right)}{\partial t} + \nabla \bullet \left(\overline{\rho}\overline{\vec{V}}\,\overline{\vec{V}}_H\right) + f\hat{k} \times \overline{\vec{V}} = -\nabla_H \overline{p} - \frac{\partial \left(M_c (\vec{V}_c - \overline{\vec{V}})_H\right)}{\partial z} \qquad \text{(e)}$$

The cumulus pgf term must be parameterized, e.g. Gregory et al, 1997 propose the following for the horizontal component associated with updrafts:

$$-\sigma\left(\nabla_H p'\right)_u = \alpha M_u \frac{\partial \vec{\bar{V}}}{\partial z} \qquad \alpha \cong .7$$

For the vertical component, the pgf is often assumed to partially offset the buoyancy and enhance the drag effect of entrainment.

Since $|w_E| << |w_c|$

$$\frac{\partial}{\partial z}\left(M_c w_c\right) + D w_c \cong -\bar{\rho}\left(\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)_c + \bar{\rho}\sigma B_c$$

$$\Rightarrow \bar{\rho}\sigma w_c \frac{\partial w_c}{\partial z} + E w_c = -\bar{\rho}\sigma\left(\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)_c + \bar{\rho}\sigma B_c$$

Let $$\bar{\rho}\sigma\left(\frac{\partial}{\partial z}\left(\frac{p'}{\bar{\rho}}\right)\right)_c = aEw_c + \bar{\rho}\sigma B_c \gamma / \left(1+\gamma\right)$$

Typical choice: $a = 1$ $\quad 0 \le \gamma \le 2$ (Siebesma et al, 2003)