On the definition and properties of p-harmonious functions

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Inspiration: Games Mathematicians Play

- Y. Peres, O. Schramm, S. Sheffield and D. Wilson; *Tug-of-war and the infinity Laplacian.* J. Amer. Math. Soc., 22, (2009), 167-210.
- Y. Peres, S. Sheffield; *Tug-of-war with noise: a game theoretic view of the p-Laplacian.* Duke Math. J. 145(1), (2008), 91–120.
- E. Le Gruyer; *On absolutely minimizing Lipschitz extensions and PDE* $\Delta_\infty(u) = 0$, 2007 NoDEA .

MPR1 *An asymptotic mean value property characterization of p-harmonic functions*, 2009 preprint.

MPR2 *On the definition and properties of p-harmonious functions*, 2009 preprint.

Example: Trees

A directed tree with regular 3-branching T consists of

- the empty set $\emptyset$,
- 3 sequences of length 1 with terms chosen from the set $\{0, 1, 2\}$,
- 9 sequences of length 2 with terms chosen from the set $\{0, 1, 2\}$,
- $\dots$
- 3^r sequences of length r with terms chosen from the et $\{0, 1, 2\}$

and so on. The elements of T are called *vertices*.

Calculus on Trees

Each vertex v al level r has three children (successors)

$$v_0, v_1, v_2.$$

Let $u\colon T \mapsto \mathbb{R}$ be a real valued function.

Gradient

The gradient of u at the vertex v is the vector in $\mathbb{R}^3$

$$\nabla u(v) = \{u(v_0) - u(v), u(v_1) - u(v), u(v_2) - u(v)\}.$$

Divergence

The averaging operator or *divergence* of a vector $X = (x, y, z) \in \mathbb{R}^3$ as

$$\mathrm{div}(X) = x + y + z.$$

Harmonic functions

A function u is harmonic if satisfies the Laplace equation

$$\mathrm{div}(\nabla u) = 0.$$

The Mean Value Property

A function u is harmonic if and only if it satisfies the mean value property

$$u(v) = \frac{u(v_0) + u(v_1) + u(v_2)}{3}.$$

Thus the values of harmonic function at level r determine its values at all levels smaller than r.

The boundary of the tree

Branches and boundary

A **branch** of T is an infinite sequence of vertices, each followed by on of its immediate successors (this corresponds to the level $r = \infty$.) The collection of all branches forms the boundary of the tree T is denoted by ∂T.

The mapping $g\colon \partial T \mapsto [0, 1]$ given by

$$g(b) = \sum_{r=1}^{\infty} \frac{b_r}{3^r} \quad \text{(also denoted by } b\text{)}$$

is a bijection (think of an expansion in base 3 of the numbers in [0,1]).

The Dirichlet problem

• We have a natural metric and natural measure in ∂T inherited from the interval $[0,1]$.

• The **classical Cantor set** C is the subset of ∂T formed by branches that don't go through any vertex labeled 1.

The Dirichlet problem

Given a (continuous) function $f\colon \partial T \mapsto \mathbb{R}$ find a harmonic function $u\colon\ T \mapsto \mathbb{R}$ such that

$$\lim_{r\to\infty} u(b_r) = f(b)$$

for every branch $b = (b_r) \in \partial T$.

Dirichlet problem, II

Given a vertex $v \in T$ consider the subset of ∂T consisting of all branches that start at v. This is always an interval that we denote by I_v.

Solution to the Dirichlet problem, $p = 2$

The we have

$$u(v) = \frac{1}{|I_v|} \int_{I_v} f(b)\, db.$$

Note that u is a *martingale*.

We see that we can in fact solve the Dirichlet problem for $f \in L^1([0,1])$.

Game interpretation

Random Walk

Start at the top $\emptyset$. Move downward by choosing successors at random with uniform probability. When you get at ∂T at the point b you get paid $f(b)$ dollars.

Two player random Tug-of-War game

A coin is tossed. The player who wins the coin toss chooses the successor vertex (heads for player I, tails for player II.) The game *ends* when we reach ∂T at a point b in which case player II pays $f(b)$ dollars to player I.

More on Random Walk Game interpretation

Every time we run the game we get a sequence of vertices

$$v_1, v_2, \ldots, v_k, ...$$

that determines a point on b the boundary ∂T.

If we are at vertex v_1 and run the game, player II pays $f(b)$ dollars to player I. Let us average out over all possible plays that start at v_1.

The value function is harmonic, $p = 2$.

$$\text{Expected pay-off} = \mathbb{E}^{v_1}[f(t)] = u(v_1) = \frac{1}{|I_{v_1}|}\int_{I_{v_1}} f(b)\,db.$$

Two player random Tug-of-War game, $p = \infty$

In this case, say that f is monotonically increasing. When player I moves he tries to move to the right. When player II moves he moves to the left.These are examples of *strategies*.

Definition of Value functions

$$u^I(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v[f(b)] \quad \textit{and} \quad u^{II}(v) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^v[f(b)]$$

DPP (Dynamic Programming Principle)

We have $u^I = u^{II}$. Moreover, if we denote the common function by u, it is the only function on the tree such that:

$$u = f \text{ on } \partial T, \quad u(v) = \frac{1}{2}\left[\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right].$$

Random Walk + Tug-of-War

Let us combine random choice of successor plus tug of war. Choose $\alpha \geq 0$, $\beta \geq 0$ such that $\alpha + \beta = 1$. Start at $\emptyset$. With probability α the players play Tug-of-War. With probability β move downward by choosing successors at random. When you get at ∂T at the point b player II pays $f(b)$ dollars to player I.

DPP for Tug-of-War with noise, DPP = MVP

The value function u verifies the equation

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0) + u(v_1) + u(v_2)}{3}\right).$$

Where are the PDEs?

Setting

$$\mathrm{div}_\infty(X) = \max\{x, y, z\} + \min\{x, y, z\}$$

the value function u of the tug-of-war game satisfies

$$\mathrm{div}_\infty(\nabla u) = 0$$

Setting

$$\mathrm{div}_p(X) = \frac{\alpha}{2}(\max\{x, y, z\} + \min\{x, y, z\}) + \beta\left(\frac{x + y + z}{3}\right)$$

the value function u of the tug-of-war game with noise satisfies

$$\mathrm{div}_p(\nabla u) = 0.$$

This operator is the homogeneous p-Laplacian.

The (homogeneous) p-Laplacian on trees

The equations

$$\operatorname{div}_2(\nabla u) = 0, \quad \operatorname{div}_p(\nabla u) = 0, \quad \operatorname{div}_\infty(\nabla u) = 0$$

DPP for Tug-of-War with noise

$$u(v) = \frac{\alpha}{2}\left(\max_i\{u(v_i)\} + \min_i\{u(v_i)\}\right) + \beta\left(\frac{u(v_0) + u(v_1) + u(v_2)}{3}\right).$$

1. The case $p = 2$ corresponds to $\alpha = 0$, $\beta = 1$.
2. The case $p = \infty$ corresponds to $\alpha = 1$, $\beta = 0$.
3. In general, there is no explicit solution formula for $p \neq 2$

Formulas for f monotone, $p = \infty$

Suppose that f is monotonically increasing. In this case the best strategy $S_I^\star$ for player I is always to move right and the best strategy $S_{II}^\star$ for player II always to move left. Starting at the vertex v at level k

$$v = 0.b_1 b_2 \dots b_k, \qquad b_j \in \{0, 1, 2\}$$

we always move either left (adding a 0) or right (adding a 1). In this case I_v is a Cantor-like set $I_v = \{0.b_1 b_2 \dots b_k d_1 d_2 \dots\}$, $d_j \in \{0, 2\}$

Formula for $p = \infty$

$$u(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v_{S_I, S_{II}}[f(b)] = E^v_{S_I^\star, S_{II}^\star}[f(b)] = \int_{I_v} f(b) d\mathcal{C}_v(b)$$

The best strategy $S_I^\star$ for player I is always to move right and the best strategy $S_{II}^\star$ for player II always to move left.

Formula for $2 \le p \le \infty$

$$u(v) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^v_{S_I,S_{II}}[f(b)] = E^v_{S_I^\star,S_{II}^\star}[f(b)]$$
$$= \alpha \fint_{I_v} f(b)\, d\mathcal{C}_v(b) + \beta \fint_{I_v} f(b)\, db$$

Plan of the rest of the talk:

1. Asymptotic Mean Value Properties for p-harmonic functions.
2. Definition, existence and uniqueness of p-harmonious functions.
3. Strong comparison principle for p-harmonious functions for $2 \leq p < \infty$.
4. Approximation of p-harmonic functions by p-harmonious functions.

1. Asymptotic mean-value properties for p-harmonic functions.

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. Consider the Taylor expansion:

$$u(x+h) = u(x) + \langle \nabla u(x), h \rangle + \frac{1}{2} \langle D^2 u(x) h, h \rangle + o(|h|^2), \text{ as } h \to 0.$$

Averaging on a ball $B_\epsilon(x) \subset \Omega$ we get:

$$\fint_{B_\epsilon(0)} u(x+h)\, dh = u(x) + \frac{1}{2(N+2)} \epsilon^2 \Delta(u)(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$

Lemma

$u \in C^2(\Omega)$ is harmonic in Ω if and only if for all $x \in \Omega$

$$\fint_{B_\epsilon(0)} u(x+h)\, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0.$$

Since viscosity harmonic functions are harmonic in the classical sense, we indeed have:

Lemma

$u \in C(\Omega)$ is harmonic in Ω if and only if for all $x \in \Omega$

$$\fint_{B_\epsilon(0)} u(x+h)\, dh = u(x) + o(\epsilon^2), \text{ as } \epsilon \to 0$$

The case $p = \infty$, $\nabla u(x) \neq 0$

Let $u \in C^2(\Omega)$, $\Omega \subset \mathbb{R}^N$. In the Taylor expansion, use

$$h = \epsilon \frac{\nabla u(x)}{|\nabla u(x)|} \quad \text{and} \quad h = -\epsilon \frac{\nabla u(x)}{|\nabla u(x)|},$$

add, and compute to get:

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) = u(x) + \epsilon^2 \Delta_\infty u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

where

$$\Delta_\infty u(x) = \frac{1}{|\nabla u(x)|^2} \langle D^2 u(x) \nabla u(x), \nabla u(x) \rangle$$

is the *homogeneous* ∞-Laplacian.

The case $p = \infty$, $\nabla u(x) \neq 0$

Lemma

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is ∞-harmonic in Ω if and only if for all $x \in \Omega$

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0.$$

Lemma

Let $u \in C(\Omega)$ be just continuous. Suppose that for all $x \in \Omega$ we have

$$\frac{1}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) = u(x) + o(\epsilon^2) \text{ as } \epsilon \to 0,$$

then u is ∞-harmonic in Ω.

The converse to the previous lemma does not hold.

Example: Aronsson's function near $(x, y) = (1, 0)$

$$u(x, y) = |x|^{4/3} - |y|^{4/3}$$

Aronsson's function is ∞-harmonic in the viscosity sense but it is not of class C^2. A calculation shows that

$$\lim_{\varepsilon \to 0+} \frac{\frac{1}{2}\left\{ \max\limits_{B_\varepsilon(1,0)} u + \min\limits_{B_\varepsilon(1,0)} u \right\} - u(1,0)}{\varepsilon^2} = \frac{1}{18}.$$

But if an asymptotic expansion held in the classical sense, this limit would have to be zero.

Let $u \in C^2(\Omega)$ and α, β non-negative such that $\alpha + \beta = 1$.

$$\begin{aligned} \frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \fint_{B_\epsilon(x)} u &= u(x) \\ &+ \alpha \Delta_\infty u(x) + \beta \tfrac{1}{(N+2)} \Delta u(x) \\ &+ o(\epsilon^2), \qquad \text{as } \epsilon \to 0, \end{aligned}$$

Let us rewrite the second order operator

$$\alpha \Delta_\infty u(x) + \beta \frac{1}{(N+2)} \Delta u(x) = \beta \frac{1}{(N+2)} \left(\Delta u(x) + \frac{\alpha}{\beta \frac{1}{(N+2)}} \Delta_\infty u(x) \right).$$

The case $1 < p < \infty$, $\nabla u(x) \neq 0$

Next, choose $2 < p < \infty$ such that

$$p - 2 = \frac{\alpha}{\beta \frac{1}{(N+2)}}.$$

We then have

$$\Delta u(x) + \frac{\alpha}{\beta \frac{1}{(N+2)}} \Delta_\infty u(x) = |\nabla u(x)|^{2-p} \mathrm{div} \left(|\nabla u(x)|^{p-2} \nabla u(x) \right).$$

Lemma

$u \in C^2(\Omega)$, $\nabla u(x) \neq 0$, is p-harmonic in Ω if and only if for all $x \in \Omega$

$$\frac{\alpha}{2} \left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u \right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

The case $1 < p < \infty$

Lemma

Let be $u \in C(\Omega)$. Suppose that for all $x \in \Omega$ we have

$$\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

where $\alpha \geq 0$, $\beta \geq 0$, and $\alpha + \beta = 1$ and

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta},$$

then u is p-harmonic in Ω

Question: Can we modify these lemmas so that they **characterize** p-harmonic functions?

The case $1 < p \leq \infty$

Theorem

$u \in C(\Omega)$ is p-harmonic in Ω if and only if for all $x \in \Omega$ we have that the asymptotic expansion

$$\frac{\alpha}{2}\left(\sup_{B_\epsilon(x)} u + \inf_{B_\epsilon(x)} u\right) + \beta \fint_{B_\epsilon(x)} u = u(x) + o(\epsilon^2), \qquad \text{as } \epsilon \to 0,$$

holds in the **VISCOSITY SENSE**, where $\alpha \geq 0$, $\beta \geq 0$, $\alpha + \beta = 1$ and

$$\frac{p-2}{N+2} = \frac{\alpha}{\beta}.$$

Similar results hold for p-subharmonic and p-superharmonic functions.

Asymptotic Mean Value Expansions

Definition

A continuous function u verifies

$$u(x) = \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} u + \min_{B_\varepsilon(x)} u \right\} + \beta \fint_{B_\varepsilon(x)} u(y)\, dy + o(\varepsilon^2)$$

as $\varepsilon \to 0$ *in the viscosity sense* if
(i) for every $\phi \in C^2$ that touches u from below at x ($u - \phi$ has a strict minimum at the point $x \in \overline{\Omega}$ and $u(x) = \phi(x)$) we have

$$\phi(x) \geq \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy + o(\varepsilon^2).$$

Asymptotic Mean Value Expansions

Definition (continued)

(ii) for every $\phi \in C^2$ that touches u from above at x ($u - \phi$ has a strict maximum at the point $x \in \overline{\Omega}$ and $u(x) = \phi(x)$) we have

$$\phi(x) \leq \frac{\alpha}{2} \left\{ \max_{B_\varepsilon(x)} \phi + \min_{B_\varepsilon(x)} \phi \right\} + \beta \fint_{B_\varepsilon(x)} \phi(y)\, dy + o(\varepsilon^2).$$

Sketch of the proof

u p-harmonic $\iff$ u p-harmonic in the viscosity sense $\iff$
Use Taylor theorem applied to the test function ϕ.
(We can safely avoid points x for which $\nabla u(x) = 0$)

2. Definition, $2 \le p < \infty$ ($p = \infty$ Le Gruyer)

Let Ω be a (bounded) domain in $\mathbb{R}^N$ and consider

$$\Gamma_\epsilon = \{x \in \mathbb{R}^N \setminus \Omega \,:\, \text{dist}(x, \partial\Omega) \le \epsilon\}, \qquad \Omega_\varepsilon = \Omega \cup \Gamma_\varepsilon$$

The function u_ε is p-harmonious in Ω with continuous boundary values $F : \Gamma_\varepsilon \to \mathbb{R}$ if $u_\varepsilon(x) = F(x), \ \ x \in \Gamma_\varepsilon$ and

$$u_\varepsilon(x) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x)} u_\varepsilon\right\} + \beta \fint_{B_\varepsilon(x)} u_\varepsilon\, dy \quad \text{for every } x \in \Omega,$$

where

$$\alpha = \frac{p-2}{p+N}, \quad \text{and} \quad \beta = \frac{2+N}{p+N}.$$

WARNING! Solutions to this equation may be discontinuous as 1-d examples show.

Tug-of-War Games with Noise $2 \leq p < \infty$

Fix $1 > \alpha \geq 0$, $\beta > 0$ such that $\alpha + \beta = 1$.
Fix $\varepsilon > 0$ and place a token at starting point $x_0 \in \Omega$. Move the token to the next state x_1 as follows:

- With probability α play tug-of war: a fair coin is tossed and the winner of the toss moves the token to any $x_1 \in \overline{B}_\varepsilon(x_0)$.
- With probability β the token moves according to a uniform probability density to a random point in the ball $\overline{B}_\varepsilon(x_0)$.

This procedure yields an infinite sequence of game states $x_0, x_1, \ldots$ where every x_k, except x_0, is a random variable.

Tug-of-War Games with Noise

- A run of the game is $\mathbf{x} = (x_0, x_1, \ldots, x_k, \ldots)$, where $\mathbf{x}(k) = x_k$.
- The game stops the first time it hits Γ_ε. Write

$$\tau(\mathbf{x}) = \min\{k \colon x_k \in \Gamma_\varepsilon\}.$$

  The random variable τ is a STOPPING TIME. We write

$$\mathbf{x}(\tau(\mathbf{x})) = x_\tau.$$

- $F : \Gamma_\varepsilon \to \mathbb{R}$ is a given (Lipschitz, bounded) *payoff function*. The game payoff is $F(\mathbf{x}) = F(x_\tau)$.
- Player I earns \$ $F(x_\tau)$ while Player II earns \$ $-F(x_\tau)$.

Tug-of-War Games with Noise

- Fix strategies S_I and S_{II} for players I and II respectively.
- Start the game at x_0.
- The probability measure $\mathbb{P}^{x_0}_{S_I,S_{II}}$ is defined on the set of all game histories $H \subset \Omega_\varepsilon^\infty$ by the transition probabilities

$$\pi_{S_I,S_{II}}(x_0,\dots,x_k;A) = \frac{\alpha}{2}\left(\delta_{S_I(x_0,\dots,x_k)}(A) + \delta_{S_{II}(x_0,\dots,x_k)}(A)\right) + \beta\frac{\left|A\cap \overline{B}_\varepsilon(x_k)\right|}{\left|\overline{B}_\varepsilon(x_k)\right|}$$

and Kolmogorov's extension theorem.

Tug-of-War Games with Noise, $2 \le p < \infty$

Games end almost surely

$\mathbb{P}^x_{S_I,S_{II}}(H) = 1$ because $\beta > 0$.

Value of the game for player I

$$u_I^\varepsilon(x) = \sup_{S_I} \inf_{S_{II}} \mathbb{E}^x_{S_I,S_{II}}[F(x_\tau)]$$

Value of the game for player II

$$u_{II}^\varepsilon(x) = \inf_{S_{II}} \sup_{S_I} \mathbb{E}^x_{S_I,S_{II}}[F(x_\tau)]$$

Comparison Principle

$$u_I^\varepsilon(x) \le u_{II}^\varepsilon(x)$$

THEOREM

The value functions u_I^ε and u_{II}^ε are p-harmonious. They satisfy the equation

$$u(x) = \frac{\alpha}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\} + \beta \fint_{B_\varepsilon(x)} u(y)\,dy, \qquad x \in \Omega,$$

$$u(x) = F(x), \qquad x \in \Gamma_\varepsilon.$$

(In the case $p = \infty$ Le Gruyer showed that the mapping

$$T(u) = \frac{1}{2}\left\{\sup_{\overline{B}_\varepsilon(x)} u + \inf_{\overline{B}_\varepsilon(x)} u\right\}$$

has a fixed point.)

Comparison I

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set.

- If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v(y) \geq u_I^\varepsilon(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \geq u_I^\varepsilon(x)$ for $x \in \Omega_\varepsilon$.
- If v_ε is a p-harmonious function with boundary values F_v in Γ_ε such that $F_v(y) \leq u_I^\varepsilon(y)$ for $y \in \Gamma_\varepsilon$, then $v_\varepsilon(x) \leq u_{II}^\varepsilon(x)$ for $x \in \Omega_\varepsilon$.

That is u_I^ε is the smallest p-harmonious function with given boundary values and u_{II}^ε is the largest p-harmonious function with given boundary values.

Comparison I, Proof

Player I arbitrary strategy S_I, player II strategy S_{II}^0 that almost minimizes v_ε, $\quad v_\varepsilon(x_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v_\varepsilon(y) + \eta 2^{-k}$

Key Point

$$M_k = v_\varepsilon(x_k) + \eta 2^{-k}$$

is a supermartingale for any $\eta > 0$.

$$\begin{aligned}\mathbb{E}^{x_0}_{S_I, S^0_{II}}[M_k \mid x_0, \ldots, x_{k-1}] &= \mathbb{E}^{x_0}_{S_I, S^0_{II}}[v^\varepsilon(x_k) + \eta 2^{-k} \mid x_0, \ldots, x_{k-1}] \\ &\leq \frac{\alpha}{2}\left\{ \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} v^\varepsilon(y) + \sup_{y \in \overline{B}_\varepsilon(x_{k-1})} v^\varepsilon(y) + \eta 2^{-k} \right\} \\ &+ \beta \fint_{B_\varepsilon(x_{k-1})} v^\varepsilon \, dy + \eta 2^{-k} \leq v^\varepsilon(x_{k-1}) + \eta 2^{-(k-1)} = M_{k-1}\end{aligned}$$

Comparison I, Proof

By optimal stopping

$$\begin{aligned}
u_I^\varepsilon(x_0) &= \sup_{S_I} \inf_{S_{II}} \mathbb{E}^{x}_{S_I,S_{II}}[F(x_\tau)] \\
&\leq \sup_{S_I} \mathbb{E}^{x}_{S_I,S^0_{II}}[F(x_\tau)] \\
&\leq \sup_{S_I} \mathbb{E}^{x}_{S_I,S^0_{II}}[v_\varepsilon(x_\tau)] \\
&\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[v_\varepsilon(x_\tau) + \eta 2^{-\tau}] \\
&\leq \sup_{S_I} \mathbb{E}^{x_0}_{S_I,S^0_{II}}[M_\tau] \\
&\leq \sup_{S_I} M_0 = v^\varepsilon(x_0) + \eta
\end{aligned}$$

The game has a value

Theorem

$M_k = u_I^\varepsilon(x_k) + \eta 2^{-k}$ is a supermartingale.
We have $u_I^\varepsilon = u_{II}^\varepsilon$

The proof is a variant of the proof of comparison.
Player II follows a strategy S_{II}^0 such that at $x_{k-1} \in \Omega_\varepsilon$, he always chooses to step to a point that almost minimizes u_I^ε; that is, to a point x_k such that

$$u_I^\varepsilon(x_k) \leq \inf_{y \in \overline{B}_\varepsilon(x_{k-1})} u_I^\varepsilon(y) + \eta 2^{-k}$$

3. Maximum and Comparison Principles

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. If u_ε is p-harmonious in Ω with a boundary data F, then $\sup_{\Gamma_\varepsilon} F \geq \sup_\Omega u_\varepsilon$. Moreover, if there is a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = \sup_{\Gamma_\varepsilon} F$, then u_ε is constant in Ω.

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. and let u_ε and v_ε be p-harmonious with boundary data $F_u \geq F_v$ in Γ_ε. Then if there exists a point $x_0 \in \Omega$ such that $u_\varepsilon(x_0) = v_\varepsilon(x_0)$, it follows that $u_\varepsilon = v_\varepsilon \quad \text{in} \quad \Omega$, and, moreover, the boundary values satisfy $F_u = F_v \quad \text{in} \quad \Gamma_\varepsilon$.

Proof of Strong Comparison

The proof uses the fact that $p < \infty$. The strong comparison principle **does not hold** for $p = \infty$.

$$F_u \geq F_v \implies u_\varepsilon \geq v_\varepsilon .$$

We have

$$u_\varepsilon(x_0) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy$$

and

$$v_\varepsilon(x_0) = \frac{\alpha}{2}\left\{ \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon + \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \right\} + \beta \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy .$$

Next we compare the right hand sides. Because $u_\varepsilon \geq v_\varepsilon$, it follows that

Proof of Strong Comparison, II

$$\sup_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \sup_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \geq 0,$$

$$\inf_{\overline{B}_\varepsilon(x_0)} u_\varepsilon - \inf_{\overline{B}_\varepsilon(x_0)} v_\varepsilon \geq 0, \quad \text{and}$$

$$\fint_{B_\varepsilon(x_0)} u_\varepsilon \, dy - \fint_{B_\varepsilon(x_0)} v_\varepsilon \, dy \geq 0$$

But since

$$u_\varepsilon(x_0) = v_\varepsilon(x_0),$$

and $\beta > 0$ must have $u_\varepsilon = v_\varepsilon$ almost everywhere in $B_\varepsilon(x_0)$. In particular,

$$F_u = F_v \quad \textbf{everywhere in} \quad \Gamma_\varepsilon$$

since F_u and F_v are continuous. By uniqueness $u_\varepsilon = v_\varepsilon$ everywhere in Ω.

4. Approximation of p-harmonic functions

Boundary Regularity Assumption

Ω bounded domain in $\mathbb{R}^n$ satisfying an exterior sphere condition: For each $y \in \partial\Omega$, there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$. $R > 0$ is chosen so that we always have $\Omega \subset B_{R/2}(z)$.

THEOREM

F is Lipschitz in Γ_ε for small $0 < \varepsilon < \varepsilon_0$. Let u be the unique viscosity solution to

$$\begin{cases} \mathrm{div}(|\nabla u|^{p-2}\nabla u)(x) = 0, & x \in \Omega \\ u(x) = F(x), & x \in \partial\Omega, \end{cases}$$

and let u_ε be the unique p-harmonious function with boundary data F in Γ_ε, then $u_\varepsilon \to u$ uniformly in Ω as $\varepsilon \to 0$.

Approximation of p-harmonic functions, Proof I

Ascoli-Arzelá type theorem

Let $\{u_\varepsilon \,:\, u_\varepsilon : \overline{\Omega} \to \mathbb{R},\ \varepsilon > 0\}$ be a set of functions such that

1. there exists $C > 0$ so that $|u_\varepsilon(x)| < C$ for every $\varepsilon > 0$ and every $x \in \overline{\Omega}$,
2. given $\eta > 0$ there are constants r_0 and ε_0 such that for every $\varepsilon < \varepsilon_0$ and any $x, x' \in \overline{\Omega}$ with $|x - x'| < r_0$ it holds

$$|u_\varepsilon(x) - u_\varepsilon(x')| < \eta.$$

Then, there exists a sequence $\varepsilon_j \to 0$ and a uniformly continuous function $u : \overline{\Omega} \to \mathbb{R}$ such that

$$u_{\varepsilon_j} \to u$$

uniformly in $\overline{\Omega}$.

Condition 1 is clear:

$$\min_{y\in\Gamma_\varepsilon} F(y) \le F(x_\tau) \le \max_{y\in\Gamma_\varepsilon} F(y) \implies \min_{y\in\Gamma_\varepsilon} F(y) \le u_\varepsilon(x) \le \max_{y\in\Gamma_\varepsilon} F(y).$$

Condition 2, OSCILLATION ESTIMATE

The p-harmonious function u_ε with the boundary data F satisfies

$$|u_\varepsilon(x) - u_\varepsilon(y)| \le \mathrm{Lip}(F)\delta + C(R/\delta)(|x-y| + o(1)),$$

for every small enough $\delta > 0$ and for every two points $x, y \in \Omega \cup \Gamma_\varepsilon$. Here $C(R/\delta) \to \infty$ as $R/\delta \to \infty$. Furthermore the constant in $o(1)$ is uniform in x and y.

Ingredients in the Proof of the Oscillation Estimate

Exterior sphere condition $\Longrightarrow$ there exists there exists $B_\delta(z) \subset \mathbb{R}^n \setminus \Omega$ such that $y \in \partial B_\delta(z)$.
When Player I chooses the strategy of pulling towards z, denoted by S_I^z, Player II an arbitrary strategy.

$$M_k = |x_k - z| - C\varepsilon^2 k$$

is a supermartingale for a constant C large enough independent of ε.

By the optional stopping theorem

$$\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_\tau - z| - C\varepsilon^2\tau] \leq |x_0 - z|$$

$$\mathbb{E}^{x_0}_{S_I^z, S_{II}}[|x_\tau - z|] \leq |x_0 - z| + C\varepsilon^2 \mathbb{E}^{x_0}_{S_I^z, S_{II}}[\tau]$$

Sketch of the Proof of the Oscillation Estimate

Random Walk Exit Time Estimates

Consider a random walk on $B_R(z) \setminus \overline{B}_\delta(z)$ such that when at x_{k-1}, the next point x_k is chosen uniformly distributed in $B_\varepsilon(x_{k-1}) \cap B_R(z)$. For $\tau^\star = \inf\{k \,:\, x_k \in \overline{B}_\delta(z)\}$. we have

$$\mathbb{E}^{x_0}(\tau^\star) \leq \frac{C(R/\delta)\,\mathrm{dist}(\partial B_\delta(z), x_0) + o(1)}{\varepsilon^2},$$

for $x_0 \in B_R(y) \setminus \overline{B}_\delta(y)$. Here $C(R/\delta) \to \infty$ as $R/\delta \to \infty$.

This is surely known by experts in probability. We proved it by showing that $g(x) = \mathbb{E}^x(\tau^\star)$ can be estimated by the solution of a mixed Dirichlet-Newman problem in the ring $B_R(z) \setminus \overline{B}_\delta(z)$