A Fokker-Planck model for two interacting populations of neurons

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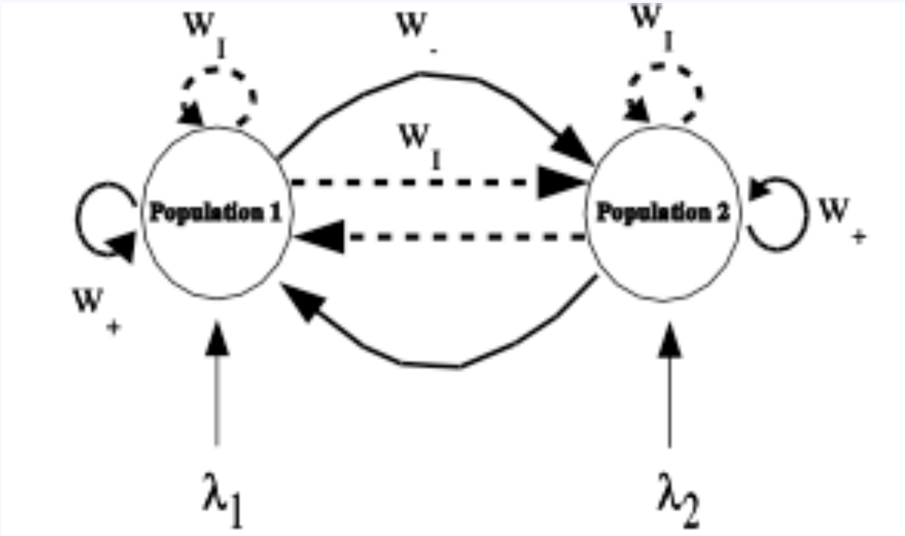
Outline

- Neuroscience model
- Fokker-Planck equation
- The stationnary problem
- The time dependent model
- Generalized relative entropy
- Numerical Results
- Slow-fast behaviour
- Conclusions and perspectives

The neuroscience model



Synaptic strength between population i and j:

$$w_{ij} = \begin{cases} w_+ - w_I & i = j \\ w_- - w_I & i \neq j \end{cases}$$

w_+ weight excitation between neurons of a same population

w_- weight excitation between neurons of different populations

w_I inhibitory weight coupling with all other neurons

λ_i sensory input to the population i

The ODS system

The firing rates $\nu_1 = \nu_1(t)$, $\nu_2 = \nu_2(t)$ of two interacting neuron families may be modeled as follows (Wilson-Cowan):

$$\begin{cases} \dot{\nu_1} = -\nu_1 + \phi\left(\lambda_1 + \sum_{j=1,2} w_{1j}\nu_j\right) + \xi \\ \dot{\nu_2} = -\nu_2 + \phi\left(\lambda_2 + \sum_{j=1,2} w_{2j}\nu_j\right) + \xi \end{cases}$$

where $\xi = \xi(t)$ is a white noise of standard deviation β^2 (fluctuations) the function $\phi(x)$ (transfer response) is a sigmoidal function given by:

$$\phi(x) = \frac{\nu_c}{1 + \exp(-\alpha(x/\nu_c - 1))},$$

with $\alpha, \nu_c \in \mathbb{R}$.

$$\lambda_2 = \lambda_1 + \Delta\lambda\,, \qquad \Delta\lambda = 0 \text{ or } 0.1$$

Remark: ϕ is strictly monotone and bounded.

[DM] G.Deco, D.Marti, *Biological Cybernetics* (2007)

Fokker-Planck equation

Let $f(t,x,y)$ be a distribution function for $t \geq 0$ and $\nu = (\nu_1, \nu_2) \in \Omega$, s.t.:

$$\partial_t f + \nabla \cdot (F\, f) - \frac{\beta^2}{2}\Delta f = 0\,, \qquad \left(Ff - \frac{\beta^2}{2}\nabla f\right) \cdot n = 0 \qquad (FP)$$

with the flux $Ff = (-\nu + \Phi(\Lambda + W \cdot \nu))\, f$ not deriving by a potential V:

$$z_1 = \lambda + w_{11}\nu_1 + w_{12}\nu_2 \neq \lambda + w_{12}\nu_1 + w_{11}\nu_2 = z_2 \implies w_{12}\phi'(z1) \neq w_{12}\phi'(z2)$$

Moreover F verifies:

$$\nabla \cdot F \leq 0 \qquad (H1)$$

$$F \cdot n \leq 0 \qquad (H2)$$

$$\int_\Omega f\, d\nu = 1 \qquad (H3)$$

[AC] A. Arnold, E.Carlen, EQUADIFF 99, Proc. Intern. Conf. Diff. Eqs. (2000)

Stationnary problem

Consider now the stationanry problem associated to (P):

$$\mathcal{A}f = -\frac{\beta^2}{2}\Delta f + \nabla\cdot(Ff) = 0\,, \qquad \left(Ff - \frac{\beta^2}{2}\nabla f\right)\cdot n = 0 \qquad (S)$$

Theorem: Assume $(H2)$ and $(H3)$,
then there exists a unique positive solution $f_\infty(\nu)$ to (S).

Proof: *Based on Krein-Rutman theorem :*

- $T : L^2(\Omega) \to L^2(\Omega)$, s.t. $\forall g \in L^2(\Omega)$, $Tg = f$, with f the unique solution of :

$$\mathcal{A}f + \rho f = g \quad \text{in } \Omega, \qquad (Ff - \frac{\beta^2}{2}\nabla f)\cdot n = 0 \quad \text{on } \partial\Omega$$

- $T : H^2 \to H^2$ is a compact operator, and $T : K \to K$ strong. pos., with $K = W^{2,2}_+(\Omega)$.
- KR th. $\Longrightarrow r(T) > 0$ and $\exists\, g > 0$ s.t. $Tg = r(T)g$. So that,

$$\mathcal{A}f + \rho f = \lambda f, \quad f = r(T)g > 0, \quad \lambda = \frac{1}{r(T)} \quad \text{and}$$

$$\mathcal{A}f = (\lambda - \rho)f \quad \Rightarrow (\lambda-\rho)\int_\Omega f\,dx = 0 \quad \Rightarrow \rho = \lambda \quad \Rightarrow \mathcal{A}f = 0.$$

Time depending problem

We consider the parabolic problem:

$$\partial_t f + \mathcal{A} f = 0 \,, \qquad \left(F f - \frac{\beta^2}{2} \nabla f \right) \cdot n = 0 \qquad (P)$$

and the initial condition: $f_0(\cdot) \in L^2(\Omega)$

Theorem: Assume that $(H1)$ holds,
then (P) has a unique solution $f(t,x,y)$.

Consider the bilinear form associated to $\mathcal{A}$:

$$a(t,f,g) = \int_\Omega \frac{\beta^2}{2} \nabla f \cdot \nabla g \, d\nu - \int_\Omega f F \cdot \nabla g \, d\nu \,, \quad \forall \, f, g \in H^1(\Omega) \,, \qquad (a)$$

- $a(t,f,g)$ is continuous,
- $a(t,f,g) + \rho < f, g >$ is coercive for $\rho \in \mathbb{R}$ large enough.

Remark : Maximum principle doesn't apply.

Generalised relative entropy

Theorem: Let $f_1, f_2 > 0$ **solutions of** (P), **and** $g > 0$ **a solution of :**

$$\begin{cases} \partial_t g = -F \cdot \nabla g - \frac{\beta^2}{2}\Delta g, & \textbf{in } \Omega \times [0,T], \\ \frac{\partial g}{\partial n} = 0 \quad \textbf{on } \partial\Omega \end{cases}$$

then we have:

$$\frac{d}{dt}\int_\Omega g f_1 H \, d\nu = -\frac{\beta^2}{2}\int_\Omega g f_1 H'' \left|\nabla(f_2/f_1)\right|^2 d\nu \leq 0 \,, \qquad \forall\, H \textbf{ convex}.$$

Proof : $\quad \frac{\partial}{\partial t}[g f_1 H] = -\nabla \cdot [F g f_1 H] + \frac{\beta^2}{2}\nabla \cdot \left[g^2 \nabla\left(\frac{f_1}{g}H\right)\right] - \frac{\beta^2}{2} g f_1 H'' \left|\nabla\left(\tilde{f}\right)\right|^2$

From this we can proove positivity of the solution f of (P) and its L^2 convergence to the stationnary solution f_∞ of (S).

[MMP] P. Michel, S.Mischler, B.Perthame, *J.Math. Pures Appl.* (2005).

Numerical approximation - FVM

Let $f^k(i,j) = f(k\Delta t, n_i, n_j)$ with $n_i = (i+\frac{1}{2})\Delta N_1,\ i = 0 \ldots N_1 - 1$ and $n_j = (j+\frac{1}{2})\Delta N_2,\ j = 0 \ldots N_2 - 1$. Then, the discretised Fokker-Planck equation is given by:

$$\begin{aligned} f^{k+1}(i,j) &= f^k(i,j) \\ &+ \Delta t \left(F^k(i+1/2,j) - F^k(i-1/2,j)\right)/\Delta N_1 \\ &+ \Delta t \left(G^k(i,j+1/2) - G^k(i,j-1/2)\right)/\Delta N_2, \end{aligned}$$

with: $F^k(i+\frac{1}{2},j)$, $G^k(i,j+\frac{1}{2})$ the fluxes at the interfaces:

$$\begin{aligned} F^k(i+1/2,j) &= \left(-n_{i+1/2} + \Phi(\lambda + w_{11} n_{i+1/2} + w_{12} n_j)\right) f^k(i+1/2,j) \\ &- \frac{\beta^2}{2\Delta N_1}\left(f^k(i+1,j) - f^k(i,j)\right), \\ G^k(i,j+1/2) &= \left(-n_{j+1/2} + \Phi(\lambda + w_{21} n_i + w_{22} n_{j+1/2})\right) f^k(i,j+1/2) \\ &- \frac{\beta^2}{2\Delta N_2}\left(f^k(i,j+1) - f^k(i,j)\right). \end{aligned}$$

and we choose the most simple interpolation at the interfaces for f:

$$f^k(i+1/2,j) = \frac{f^k(i+1,j) + f^k(i,j)}{2}, \qquad f^k(i,j+1/2) = \frac{f^k(i,j+1) + f^k(i,j)}{2}.$$

Remark adaptative Δt (gain factor 100) $\Rightarrow$ for i,j s.t. $f^k(i,j) \neq 0$ and $\mathcal{F}^k(i,j) \neq 0$:

$$\Delta t = \min_{i,j} \frac{f^k(i,j)}{2|\mathcal{F}^k(i,j)|}$$

Computed quantities

Marginals of $f(t,\nu_1,\nu_2)$ with respect to ν_2, and to ν_1 :

$$\mathcal{N}_1(t,\nu_1) = \int_0^{\nu_M} f(t,\nu_1,\nu_2)d\nu_2\,, \quad \mathcal{N}_2(t,\nu_2) = \int_0^{\nu_M} f(t,\nu_1,\nu_2)d\nu_1.$$

First order moments :

$$\mu_i(t) = \int\int_\Omega \nu_i f(\nu_1,\nu_2,t)d\nu_1 d\nu_2, \quad i = 1,2$$

Second order moments :

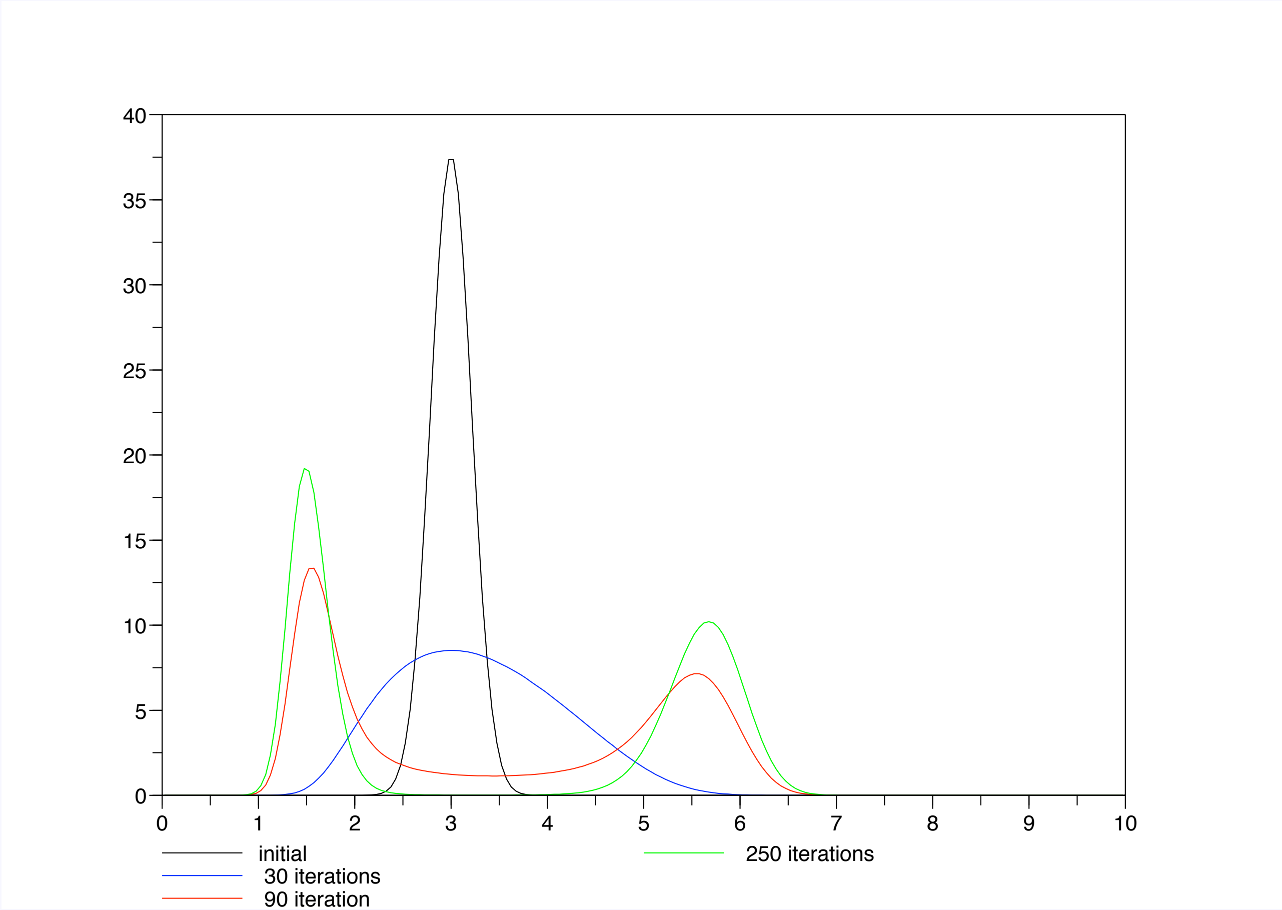
$$\gamma_{ij}(t) = \int\int_\Omega \nu_i\nu_j f(\nu_1,\nu_2,t)d\nu_1 d\nu_2, \quad i,j = 1,2.$$

Distributions $\rho_i(t)$ with respect to the domains, Ω_i, with $i = 1,2,3$:

$$\rho_i(t) = \int\int_{\Omega_i} f(\nu_1,\nu_2,t)d\nu_1 d\nu_2.$$

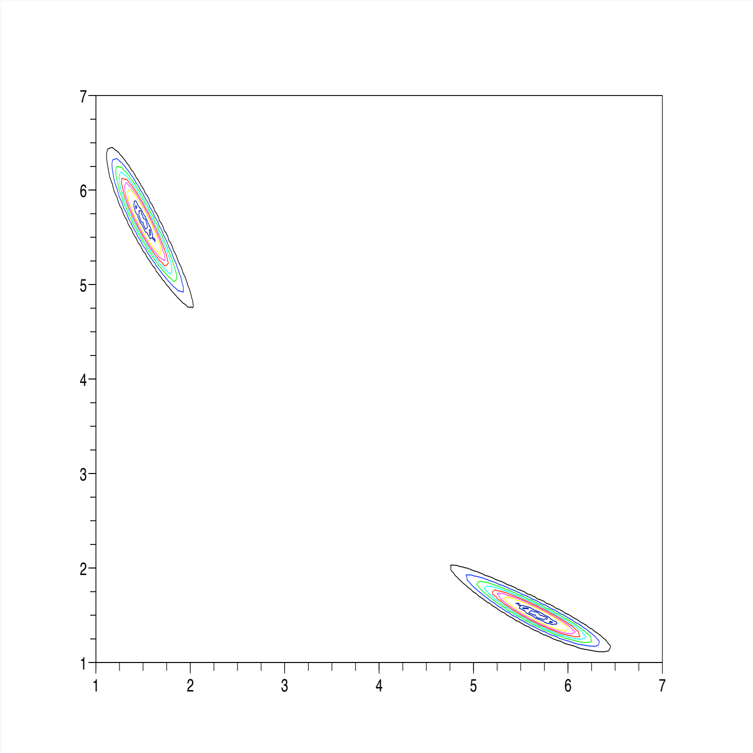
We choose $N_1 = N_2 = 200$ points of discretisation, and compute the solution up to a precision of order 10^{-10}, with the same values used in [1]. ($\beta = 0.3$, $\alpha = 4$, $\nu_c = 20$, $\lambda = 15$.)

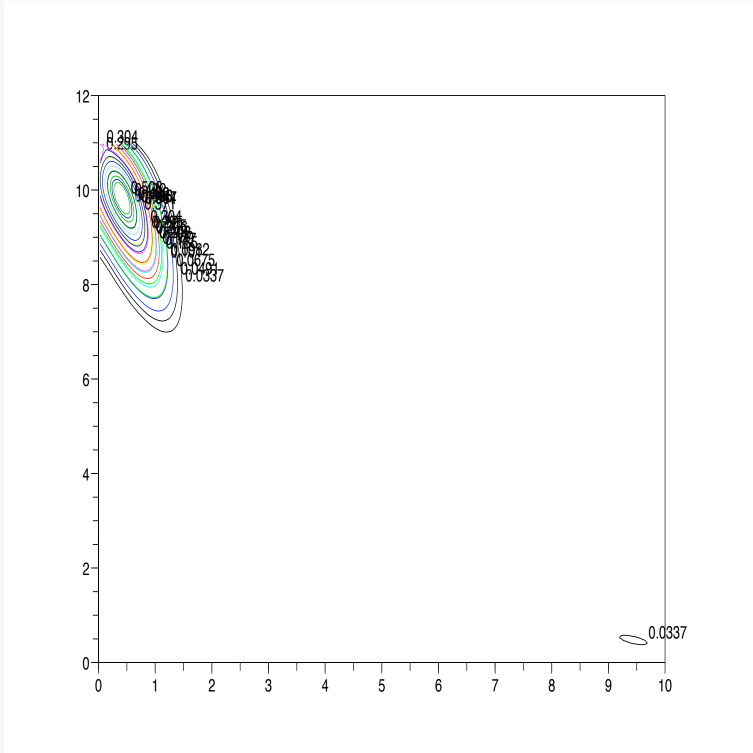
Time evolution for the marginals



Evolution from one initial gaussian distribution centered in $S = (3.3, 3.3)$ - near the unstable point $S_0 = (3.19, 3.19)$ - to a double picked distribution centerd on the two stable points S_1 and S_2.

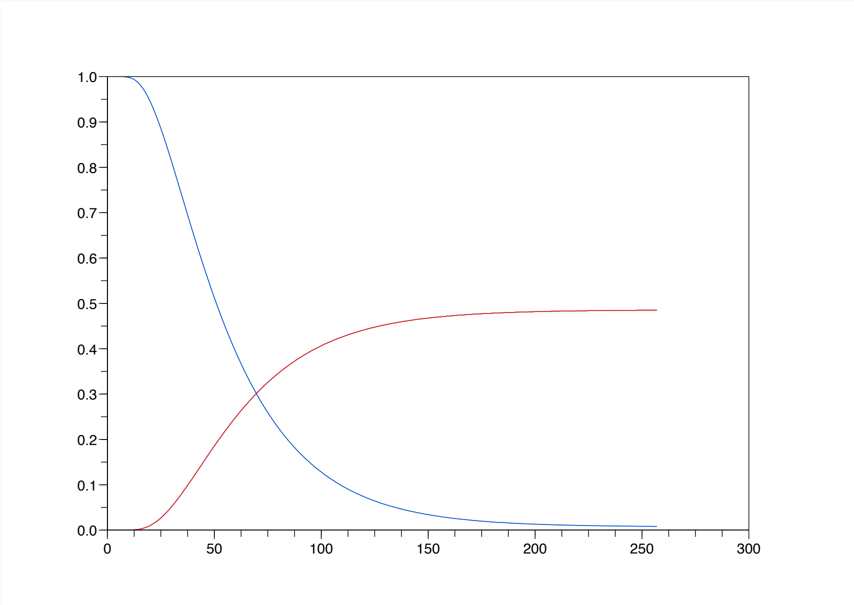
Equilibrium state

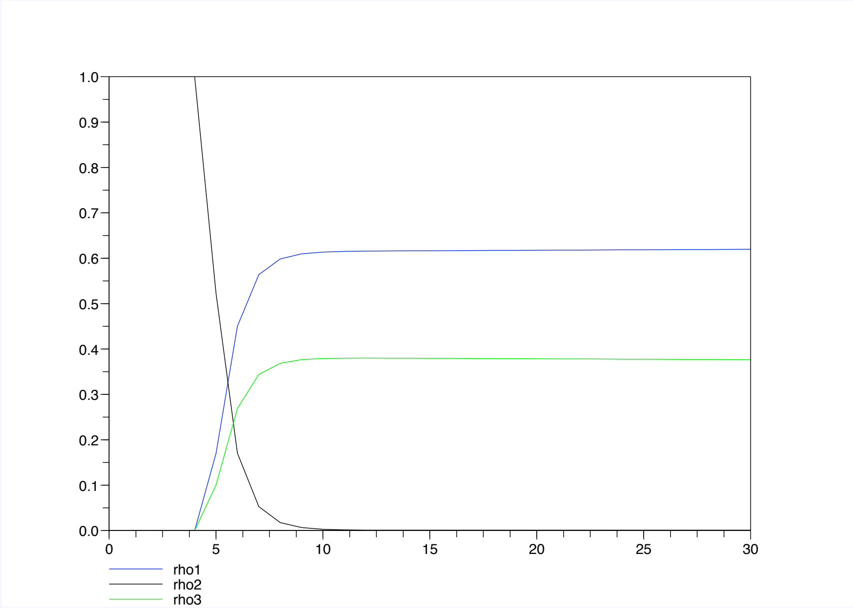




Contour levels of the density $f(\nu_1, \nu_2)$ at equilibrium. We note that there are two points of mass concentration around $S_1 = (1.32, 5.97)$ and $S_2 = (5.97, 1.32)$ which are the stable equilibrium points of the ODS.

Densities distributions





Probablity densities $\rho_i(t)$, $i = 1, 2, 3$, computed on three different domains $\Omega_1 = [5, 10] \times [0, 2]$, $\Omega_2 = [2, 5] \times [2, 5]$, $\Omega_3 = [2, 5] \times [5, 10]$.

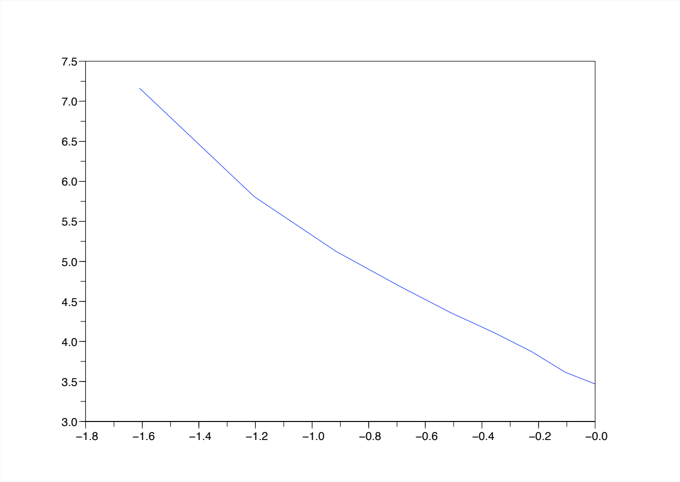
Each domain contains one of the three equilibrium points.

The initial condition $\Rightarrow$ $\rho_1(0) = \rho_3(0) = 0$ and $\rho_2(0) = 1$.

Escaping time

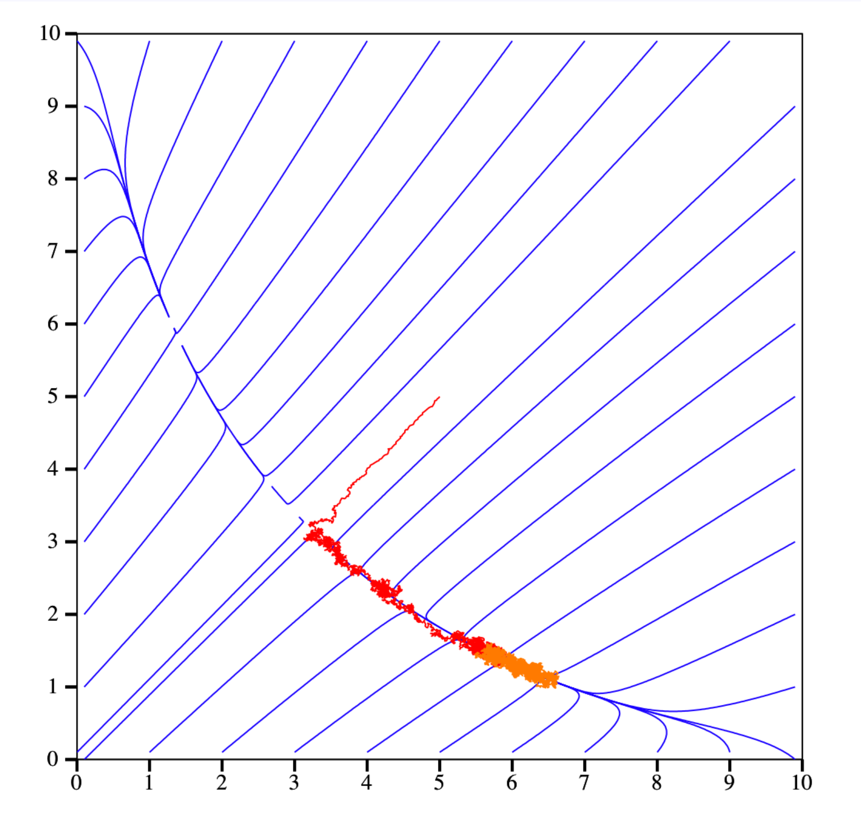
Let $f(0, \nu_1, \nu 2)$ be a gaussian distribution centered in S_1 and $\beta = 0.2, ..., 1$. Let T be the escaping time (ie. the time needed for half of the mass to pass from the neighborhood of S_1 to the neighborhood of S_2) : $\rho_1(T) < 2\rho_3(T)$. Then T has an exponential behaviour :

β	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
T	1290.3	332.7	168.5	109.3	78.2	60.9	48.0	37.1	32.1



Escaping time T with respect to the diffusion coefficient β in log scale

Slow-fast behaviour



One realisation of a trajectory for the ODS starting in $(5,5)$

The blue lines are numerical approximations of the solution of the deterministic system

$$\begin{cases} \dot{\nu_1} = -\nu_1 + \phi\left(\lambda + \sum_{j=1,2} w_{1j}\nu_j\right) \\ \dot{\nu_2} = -\nu_2 + \phi\left(\lambda + \sum_{j=1,2} w_{2j}\nu_j\right) \end{cases}$$

and highlight the slow manifold to which belongs the stable and unstable solutions of this system

$\Rightarrow$ fast to the manifold, slow on the manifold

[BG] N.Berglund, B.Gentz, *Noise-Induced Phenomena in Slow-Fast Dynamical Systems. A Sample-Paths Approach.* Springer, Probability and its Applications (2005)

Towards a one-dimensional problem

The change of variables: $x = \nu_1 + \nu_2$, $y = \nu_1 - \nu_2$ leads to the system :

$$\begin{cases} \varepsilon \dot{x} = h(x, y) \\ \dot{y} = g(x, y) \end{cases} , \qquad \varepsilon = \frac{\partial_y g}{\partial_x h}|_{S_0}$$

It is possible to find a function $x^*(y)$ and to reduce the system to a one-dimensional equation defined on the slow manifold :

$$\dot{y} = g(x^*(y), y)$$

Taking into account the white noise, the original FP model, reduces to a FP equation, for which the unknown distribution function depends on time t and the y variable, and the equilibrium solution is given by an exponential function:

$$\exp\left(\frac{-G}{\beta^2}\right), \qquad G = \partial_y g(x^*(y), y)$$

Conclusions and Perspectives

- We present theoretical results concerning the existence, uniqueness, positivity of the solution for the model (non-potential frame), and its convergence towards the solution of the stationnary associated problem.
- We propose a kinetic model for the evolution of two interacting populations (decision making), based on neurodynamical systems.
- Our numerical results agree with those find by G.Deco et al. applying moments methods on the ODS.

- Investigate the slow-fast behavior of the ODS and derive a one-dimensional pde for the distribution function defined along the slow-manifold.
- Study the system in the biased case ($\lambda_1 \neq \lambda_2$), or including some *adaptation in rivalry* term in the sigmoidal function.

- Stochastic Models in Neuroscience,
  18-22 January 2010,
  CIRM, Marseille (France).
  http://www.fdpoisson.org/colloques/neurostoch/