Index Theory for Symplectic Matrix Paths with Applications

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Abstract

In this lecture notes, I give an introduction on the Maslov-type index theory for symplectic matrix paths and its iteration theory with applications to existence, multiplicity, and stability of periodic solution orbit problems for nonlinear Hamiltonian systems and closed geodesic problems on manifolds, including a survey on recent progresses in these areas.

Since the pioneering work of P. Rabinowitz in 1978, topological and variational methods have been widely and deeply applied to the study of nonlinear Hamiltonian systems. On the other hand, as well known Morse theory is a very powerful tool in mathematics. For example, based upon the work [Bot1] of R. Bott in 1956 on iteration theory of Morse index, there have been many deep results obtained in the study of closed geodesics on Riemannian manifolds.

Therefore in the study of periodic solutions of nonlinear Hamiltonian systems, it is natural to consider applications of the Morse theory. But unfortunately, the functionals on loop spaces corresponding to Hamiltonian systems are indefinite, whose positive and negative Morse indices are always infinite and the usual Morse theory is not directly applicable. For this reason, further understanding and development of possible homotopy invariants for linear Hamiltonian systems as well as for paths in the symplectic matrix group starting from the identity become necessary again. Interests on such invariants started from the earlier works on the stability problems for

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linear Hamiltonian systems of M. Krein, I. Gelfand, V. Lidskii, J. Moser and others in 1950's (cf. [GeL1], [Mos1], [YaS1]). On the other hand, even in the study on the closed geodesic problem itself, a better understanding on the Morse indices of iterates of closed geodesics will give us more information. Here we note that closed geodesics can be viewed as special examples of solution orbits of Hamiltonian systems on manifolds. Therefore an index theory and its iteration theory for periodic solution orbits of Hamiltonian systems are required.

Since early 1980's, efforts on index theories for Hamiltonian systems have appeared in two different directions. One is the index theory established by I. Ekeland for convex Hamiltonian systems, including its iteration theory with successful applications to various problems on convex Hamiltonian systems (cf. [Eke3] and the reference therein). The other development is the so called Maslov-type index theory for general Hamiltonian systems without any convexity type assumptions, which was defined by C. Conley, E. Zehnder, Y. Long, and C. Viterbo in a sequence of papers [CoZ1], [LoZ1], [Lon1], [Lon3], and [Vit2].

Motivated by the studying of the existence, multiplicity, and stability problems of periodic solution orbits of nonlinear Hamiltonian systems, in 1990s we have systematically developed the iteration theory of the Maslov-type index for symplectic paths. This iteration theory unifies the above mentioned iteration theory of Bott and Ekeland, and give more precise information. It has turned out to be a powerful tool in the study of various problems on periodic solution orbits of Hamiltonian systems including closed geodesic problems.

In this lecture notes, I give an introduction to this Maslov-type index theory, its iteration theory, and applications to periodic solution orbit problems of nonlinear Hamiltonian systems and closed geodesic problems on manifolds, including a survey on recent progress in these areas.

This note includes the following parts.

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Chapter 1. An Index Theory for Symplectic Paths

Let $\mathbf{N}$, $\mathbf{Z}$, $\mathbf{R}$, and $\mathbf{C}$ be the sets of natural, integral, real, and complex numbers respectively. Let $\mathbf{U}$ be the unit circle in $\mathbf{C}$. As usual for any $n \in \mathbf{N}$, we define the symplectic groups on $\mathbf{R}^{2n}$ by

$$\mathrm{Sp}(2n) = \{M \in \mathcal{L}(\mathbf{R}^{2n}) \mid M^T J M = J\},$$

where $J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}$, I_n denotes the identity matrix on $\mathbf{R}^n$, the subscript n will be omitted when there is no confusion. $\mathcal{L}(\mathbf{R}^{2n})$ is the set of all $2n \times 2n$ real matrices, M^T denotes the transpose of M. The topology of $\mathrm{Sp}(2n)$ is induced from that of $\mathbf{R}^{n^2}$. For $\tau > 0$ and $H \in C^2(S_\tau \times \mathbf{R}^{2n}, \mathbf{R})$ with $S_\tau = \mathbf{R}/(\tau\mathbf{Z})$, we consider the τ-periodic boundary value problem of the following Hamiltonian systems:

$$\dot{x}(t) = JH'(t, x(t)), \tag{0.1}$$

where $H'(t,x)$ denotes the gradient of H with respect to the x variables. Suppose $x = x(t)$ is a τ-periodic solution of (1.1) for some $\tau > 0$. Denote by γ_x the fundamental solution of the linearized Hamiltonian system

$$\dot{y} = JB(t)y, \tag{0.2}$$

where $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$ is defined by $B(t) = H''(t, x(t))$, and $\mathcal{L}_s(\mathbf{R}^{2n})$ is the subset of symmetric matrices in $\mathcal{L}(\mathbf{R}^{2n})$. Then γ_x is a path in $\mathrm{Sp}(2n)$ starting from the idetity matrix I. Based upon the work [AmZ1] of H. Amann and E. Zehnder in 1980 on the index theory for linear Hamiltonian systems with constant coefficients, C. Conley and E. Zehnder in their celebrated paper [CoZ1] of 1984 defined their index theory for non-degenerate paths in the symplectic matrix group $\mathrm{Sp}(2n)$ started from the identity when $n \geq 2$. This index theory was extended to non-degenerate paths in $\mathrm{Sp}(2)$ by the author and E. Zehnder in [LoZ1] of 1990. Then C. Viterbo in [Vit2] and the author in [Lon1] of 1990 extended this index theory to degenerate symplectic paths which are fundamental solutions of linear Hamiltonian systems with continuous symmetric periodic coefficients. In the work [Lon3], the author further extended this index theory to all continuous degenerate paths in $\mathrm{Sp}(2n)$ for all $n \geq 1$ and gave an axiom characterization of this index theory. We call this index theory the Maslov-type index theory in this paper. The Maslov-type index theory assigns a pair of numerical invariants to the periodic solution x through the associated path γ_x in $\mathrm{Sp}(2n)$ and reflects important properties of the periodic solution x.

1 Definitions and basic properties

We start from some notations introduced in [CoZ1], [LoZ1], [Lon1], and [Lon3] (cf. [Lon8]). Define

$$D_1(M) = (-1)^{n-1}\det(M-I), \qquad \forall M \in \mathrm{Sp}(2n).$$

Let

$$\begin{aligned} \mathrm{Sp}(2n)^{\pm} &= \{M \in \mathrm{Sp}(2n) \mid \pm D_1(M) < 0\}, \\ \mathrm{Sp}(2n)^{*} &= \mathrm{Sp}(2n)^{+} \cup \mathrm{Sp}(2n)^{-}, \qquad \mathrm{Sp}(2n)^{0} = \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^{*}. \end{aligned}$$

For any two matrices of square block form:

$$M_1 = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix}_{2i\times 2i}, \quad M_2 = \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}_{2j\times 2j},$$

the $\diamond$-product of M_1 and M_2 is defined by the $2(i+j)\times 2(i+j)$ matrix $M_1\diamond M_2$:

$$M_1\diamond M_2 = \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$

Denote by $M^{\diamond k}$ the k-fold $\diamond$-product $M\diamond\cdots\diamond M$. Note that the $\diamond$-multiplication is associative, and the $\diamond$-product of any two symplectic matrices is symplectic.

We define $D(a) = \mathrm{diag}(a, a^{-1})$ for $a \in \mathbf{R}\setminus\{0\}$. For θ, λ, and $b \in \mathbf{R}$ we define

$$R(\theta) = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \quad N_1(\lambda, b) = \begin{pmatrix} \lambda & b \\ 0 & \lambda \end{pmatrix}.$$

Define two $2n\times 2n$ diagonal matrices

$$M_n^{+} = D(2)^{\diamond n}, \qquad M_n^{-} = D(-2)\diamond D(2)^{\diamond(n-1)}.$$

Lemma 1.1. (cf. [CoZ1], [LoZ1], and [SaZ1]) 1^o $\mathrm{Sp}(2n)^{*}$ *contains two path connected components* $\mathrm{Sp}(2n)^{+}$ *and* $\mathrm{Sp}(2n)^{-}$, *and there hold* $M_n^{\pm} \in \mathrm{Sp}(2n)^{\pm}$.

2^o *Both of* $\mathrm{Sp}(2n)^{+}$ *and* $\mathrm{Sp}(2n)^{-}$ *are simply connected in* $\mathrm{Sp}(2n)$.

Idea of the proof. Since $D_1(M_n^{+})D_1(M_n^{-}) < 0$, $\mathrm{Sp}(2n)^{*}$ contains at least two path-connected components.

For any given $M \in \mathrm{Sp}(2n)^{*}$, by a small perturbation we can connect M to a matrix M_1 with only simple eigenvalues within $\mathrm{Sp}(2n)^{*}$. Then there holds

$$PMP^{-1} = M_1\diamond\cdots\diamond M_p\diamond N_1\diamond\cdots\diamond N_q = N,$$

where $P \in \mathrm{Sp}(2n)$, $M_i \in \mathrm{Sp}(2)$ for $1 \le i \le p$ and $N_j \in \mathrm{Sp}(4)$ for $1 \le j \le q$, each M_I ahs the form $R(\theta)$ with $\theta \in (0,\pi) \cup (\pi, 2\pi)$ or $D(a)$ with $a \in \mathbf{R} \setminus \{0\}$, each N_j has four simple eigenvalues λ_j, $\overline{\lambda_j}$, λ_j^{-1}, and $\overline{\lambda}^{-1}$ out side $\mathbf{R} \cup \mathbf{U}$.

By connecting P to I in $\mathrm{Sp}(2n)$, we get that M can be connected to N within $\mathrm{Sp}(2n)^{2n}$. Then it can be proved that these M_i's and N_j's can be connected to $D(2)$, $D(-2)$, or their $\diamond$-products within $\mathrm{Sp}(2)^*$ or $\mathrm{Sp}(4)^*$. Note that $D(-2)\diamond D(-2)$ can be connected to $D(2)\diamond D(2)$. This proves that N can be connected to one of M_n^+ and M_n^- within $\mathrm{Sp}(2n)^*$. Then $\mathrm{Sp}(2n)^*$ contains at most two path connected components, and 1^o is proved.

We refer the readers to [SaZ1] for the proof of 2^o. ∎

Fix $\tau > 0$. Let

$$\begin{aligned} \mathcal{P}_\tau(2n) &= \{\gamma \in C([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I\}, \\ \mathcal{P}_\tau^*(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(\tau) \in \mathrm{Sp}(2n)^*\}, \\ \mathcal{P}_\tau^0(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}_\tau^*(2n). \end{aligned}$$

The topology of $\mathcal{P}_\tau(2n)$ is defined by the $C^0([0,\tau], \mathrm{Sp}(2n))$-topology induced from the topology of $\mathrm{Sp}(2n)$. Note that the following subset of $\mathcal{P}_\tau(2n)$ consists of all foundamental solutions of linear Hamiltonian systems (1.2) with symmetric continuous and τ-periodic coefficients:

$$\hat{\mathcal{P}}_\tau(2n) = \{\gamma \in C^1([0,\tau], \mathrm{Sp}(2n)) \,|\, \gamma(0) = I,\ \dot{\gamma}(1) = \dot{\gamma}(0)\gamma(1)\}.$$

The topology of $\hat{\mathcal{P}}_\tau(2n)$ is defined to be the $C^1([0,\tau], \mathrm{Sp}(2n))$-topology induced from the topology of $\mathrm{Sp}(2n)$.

Definition 1.2. (cf. [Lon1], [Lon8]) *For every $\gamma \in \mathcal{P}_\tau(2n)$, we define*

$$\nu_1(\gamma) = \dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) - I).$$

Definition 1.3. (cf. [Lon1], [Lon8]) *Given two paths γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2n)$, if there is a map $\delta \in C([0,1] \times [0,\tau], \mathrm{Sp}(2n))$ such that $\delta(0,\cdot) = \gamma_0(\cdot)$, $\delta(1,\cdot) = \gamma_1(\cdot)$, $\delta(s,0) = I$, and $\nu_\tau(\delta(s,\cdot))$ is constant for $0 \le s \le 1$, then γ_0 and γ_1 are* **homotopic on** $[0,\tau]$ **along** *$\delta(\cdot,\tau)$ and we write $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ along $\delta(\cdot,\tau)$. This homotopy possesses fixed end points if $\delta(s,\tau) = \gamma_0(\tau)$ for all $s \in [0,1]$.*

As well known, every $M \in \mathrm{Sp}(2n)$ has its unique polar decomposition $M = AU$, where $A = (MM^T)^{1/2}$ is symmetric positive definite and symplectic, U is orthogonal and symplectic. Therefore U has the form

$$U = \begin{pmatrix} u_1 & -u_2 \\ u_2 & u_1 \end{pmatrix},$$

where $u = u_1 + \sqrt{-1}u_2 \in \mathcal{L}(\mathbf{C}^n)$ is a unitary matrix. So for every path $\gamma \in \mathcal{P}_\tau(2n)$ we can associate a path $u(t)$ in the unitary group on $\mathbf{C}^n$ to it. If $\Delta(t)$ is any continuous real function satisfying

$\det u(t) = \exp(\sqrt{-1}\Delta(t))$, the difference $\Delta(\tau) - \Delta(0)$ depends only on γ but not on the choice of the function $\Delta(t)$. Therefore we may define the mean rotation number of γ on $[0,\tau]$ by

$$\Delta_\tau(\gamma) = \Delta(\tau) - \Delta(0).$$

Lemma 1.4. (cf. [LoZ1], [Lon8]) *If γ_0 and $\gamma_1 \in \mathcal{P}_\tau(2n)$ possess common end point $\gamma_0(\tau) = \gamma_1(\tau)$, then $\Delta_\tau(\gamma_0) = \Delta_\tau(\gamma_1)$ if and only if $\gamma_0 \sim \gamma_1$ on $[0,\tau]$ with fixed end points.* ∎

By Lemma 1.1, for every path $\gamma \in \mathcal{P}_\tau^*(2n)$ there exists a path $\beta : [0,\tau] \to \mathrm{Sp}(2n)^*$ such that $\beta(0) = \gamma(\tau)$ and $\beta(\tau) = M_n^+$ or M_n^-. Define the product path $\beta * \gamma$ by

$$\beta * \gamma(t) = \begin{cases} \gamma(2t), & 0 \le t \le \frac{\tau}{2}, \\ \beta(2t-\tau), & \frac{\tau}{2} < t \le \tau. \end{cases}$$

Then $k \equiv \Delta_\tau(\beta * \gamma)/\pi \in \mathbf{Z}$ and is independent of the choice of the path β by 2^o of Lemma 1.1. In this case we write $\gamma \in \mathcal{P}_{\tau,k}^*(2n)$.

Lemma 1.5. (cf. [LoZ1], [Lon8]) *These $\mathcal{P}_{\tau,k}^*(2n)$'s give a homotopy classification of $\mathcal{P}_\tau^*(2n)$.*

Definition 1.6. (cf. [CoZ1], [LoZ1], [Lon8]) *If $\gamma \in \mathcal{P}_{\tau,k}^*(2n)$, we define $i_1(\gamma) = k$.*

We define the standard non-degenerate symplectic paths by

$$\begin{aligned} \hat{\alpha}_{1,0,\tau}(t) &= D(1 + \frac{t}{\tau}), \qquad \text{for } 0 \le t \le \tau, \\ \hat{\alpha}_{1,k,\tau} &= (D(2)\phi_{k\pi,\tau}) * \hat{\alpha}_{1,0,\tau}, \quad \forall k \in \mathbf{Z} \setminus \{0\}, \end{aligned}$$

where $\phi_{\theta,\tau}(t) = R((\theta t/\tau)$. When $n \ge 2$, we define

$$\begin{aligned} \hat{\alpha}_{n,0,\tau} &= (\hat{\alpha}_{1,0,\tau})^{\diamond n}, \\ \hat{\alpha}_{n,k,\tau} &= ((D(2)\phi_{k\pi,\tau}) * \hat{\alpha}_{1,0,\tau}) \diamond (\hat{\alpha}_{1,0,\tau})^{\diamond(n-1)}, \quad \forall k \in \mathbf{Z} \setminus \{0\}, \end{aligned}$$

Then there hold

$$\hat{\alpha}_{n,k,\tau} \in \mathcal{P}_{\tau,k}^*(2n), \qquad \forall k \in \mathbf{Z}.$$

The following lemma is crucial in the study of degenerate symplectic paths.

Lemma 1.7. (cf. [Lon1], [Lon5], [Lon8]) *For any $\gamma \in \mathcal{P}_\tau^0(2n)$, there exists a one parameter family of symplectic paths γ_s with $s \in [-1,1]$ and a $t_0 \in (0,\tau)$ sufficiently close to τ such that*

$$\begin{aligned} \gamma_0 &= \gamma, \quad \gamma_s(t) = \gamma(t) \ \ \textit{for} \ \ 0 \le t \le t_0, && (1.1) \\ \gamma_s &\in \mathcal{P}_\tau^*(2n) \ \ \forall s \in [-1,1] \setminus \{0\}, && (1.2) \\ i_1(\gamma_s) &= i_1(\gamma_{s'}), \quad \textit{if} \ \ ss' > 0, && (1.3) \\ i_1(\gamma_1) &- i_1(\gamma_{-1}) = \nu_1(\gamma), && \\ \gamma_s &\to \gamma_0 = \gamma \quad \textit{in} \ \ \mathcal{P}_\tau(2n) \quad \textit{as} \ \ s \to 0. && (1.4) \end{aligned}$$

When $\gamma \in \hat{P}_\tau(2n)$, we also have

$$\begin{aligned} \gamma_s &\in \hat{P}_\tau(2n) \qquad \forall s \in [-1,1], && (1.5)\\ \gamma_s &\to \gamma \quad in \ \hat{\mathcal{P}}_\tau(2n) \quad as \ \ s \to 0. && (1.6)\end{aligned}$$

Idea of the proof. Among these properties of $\{\gamma_s\}$, the most important one is (1.3). The construction of $\{\gamma_s\}$ uses the results on normal forms of symplectic matrices proved in [LoD1] and [HaL1]. Here we briefly indicate how this family of paths $\{\gamma_s\}$ is constructed.

For every integer m, $1 \le m \le n$, and $\theta \in \mathbf{R}$, a $2n \times 2n$ rotation matrix $R_m(\theta) = (r_{i,j})$ is defined in [Lon1] and [Lon2] (cf. [Lon8]) by

$$\begin{cases} r_{m,m} = r_{n+m,n+m} & = \cos\theta, \\ r_{n+m,m} = -r_{m,n+m} & = \sin\theta, \\ r_{i,i} = 1, & \text{if } i \neq m \text{ and } n+m, \\ r_{i,j} = 0, & \text{otherwise.} \end{cases}$$

Fix $\gamma \in \mathcal{P}^0_\tau(2n)$. Then there exist an integer q, $1 \le q \le n$, a strictly increasing subsequence $\{m_1, \ldots, m_q\}$ of $\{1, \ldots, n\}$, $\theta_0 \in (0, \frac{\pi}{8n})$ small enough depending on $\gamma(\tau)$, and $P \in \mathrm{Sp}(2n)$ such that for $i = 1, \ldots, q$ the m_i is the least positive integer which satisfies for $0 < |\theta| \le \theta_0$:

$$\begin{aligned} &\dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) P R_{m_1}(\theta) \cdots R_{m_{i-1}}(\theta) P^{-1} - I) \\ &\qquad - \dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) P R_{m_1}(\theta) \cdots R_{m_{i-1}}(\theta) R_{m_i}(\theta) P^{-1} - I) \ge 1, \\ &\dim_{\mathbf{R}} \ker_{\mathbf{R}}(\gamma(\tau) P R_{m_1}(\theta) \cdots R_{m_q}(\theta) P^{-1} - I) = 0. \end{aligned}$$

Here we set $R_{m_0}(\theta) \equiv I$. Note that the integers q, m_1, …, m_q, and P are determined by the normal form of the matrix $\gamma(\tau)$.

For $t_0 \in (0, \tau)$, let $\rho \in C^2([0,\tau],[0,1])$ such that $\rho(t) = 0$ for $0 \le t \le t_0$, $\dot{\rho}(t) \ge 0$ for $0 \le t \le \tau$, $\rho(\tau) = 1$, and $\dot{\rho}(\tau) = 0$. For any $(s,t) \in [-1,1] \times [0,\tau]$, the path γ_s is defined by

$$\gamma_s(t) = \gamma(t) P R_{m_1}(s\rho(t)\theta_0) \cdots R_{m_q}(s\rho(t)\theta_0) P^{-1}. \qquad (1.7)$$

When $t_0 \in (0,\tau)$ is sufficiently close to τ, the properties (1.1) to (1.6) hold. ∎

With lemma 1.7, we can give

Definition 1.8. (cf. [Lon1], [Lon8]) *Define $i_1(\gamma) = i_1(\gamma_s)$ for $s \in [-1,0)$.*

Definition 1.9. *For every path $\gamma \in \mathcal{P}_\tau(2n)$, the definitions 1.2, 1.6 and 1.8 assign a pair of integers*

$$(i_1(\gamma), \nu_1(\gamma)) \in \mathbf{Z} \times \{0, \ldots, 2n\}$$

to it. This pair of integers is called the **Maslov-type index** *of* γ*. When* $\gamma=\gamma_x$ *for a solution* x *of (1.1), we also write*

$$(i_1(x),\nu_1(x))=(i_1(\gamma_x),\nu_1(\gamma_x)).$$

The following theorem shows that the Definition 1.8 of $i_\tau(\gamma)$ for $\gamma\in\mathcal{P}_\tau^0(2n)$ is independent from the way which is defined.

Theorem 1.10. (cf. [Lon1], [Lon3], [Lon8]) *For any* $\gamma\in\mathcal{P}_\tau^0(2n)$*, and every* $\beta\in\mathcal{P}_\tau^*(2n)$ *which is sufficiently close to* γ*, there holds*

$$i_1(\gamma)=i_1(\gamma_{-1})\le i_1(\beta)\le i_1(\gamma_1)=i_1(\gamma)+\nu_1(\gamma). \tag{1.8}$$

Specially we obtain

$$i_1(\gamma)=\inf\{i_1(\beta)\,|\,\beta\in\mathcal{P}_\tau^*(2n) \textit{ and } \beta \textit{ is sufficiently close to } \gamma \textit{ in } \mathcal{P}_\tau(2n)\}. \tag{1.9}$$

Idea of the proof of (1.8). Firstly we reduce the general case to the case of that all the paths in consideration are in $\hat{\mathcal{P}}_\tau(2n)$. Then the later case can be proved by using Theorem 2.1 below and a perturbation argument on the Morse index for finite dimensional symmetric matrices. ∎

The following theorem characterizes the Maslov-type index on any continuous symplectic paths in $\mathcal{P}_\tau(2n)$.

Theorem 1.11.(cf. [Lon3], [Lon8]) *The Maslov-type index* $i_1:\cup_{n\in\mathbf{N}}\mathcal{P}_\tau(2n)\to\mathbf{Z}$*, is uniquely determined by the following five axioms:*

1° (**Homotopy invariant**) *For* γ_0 *and* $\gamma_1\in\mathcal{P}_\tau(2n)$*, if* $\gamma_0\sim\gamma_1$ *on* $[0,\tau]$*, then*

$$i_1(\gamma_0)=i_1(\gamma_1). \tag{1.10}$$

2° (**Symplectic additivity**) *For any* $\gamma_i\in\mathcal{P}_\tau(2n_i)$ *with* $i=0$ *and* 1*, there holds*

$$i_1(\gamma_0\diamond\gamma_1)=i_1(\gamma_0)+i_1(\gamma_1). \tag{1.11}$$

3° (**Clockwise continuity**) *For any* $\gamma\in\mathcal{P}_\tau^0(2)$ *with* $\gamma(\tau)=N_1(1,b)$ *for* $b=\pm1$ *or* 0*, there exists a* $\theta_0>0$ *such that*

$$i_1([\gamma(\tau)\phi_{-\theta,\tau}]*\gamma)=i_1(\gamma),\qquad \forall\, 0<\theta\le\theta_0. \tag{1.12}$$

4° (**Counterclockwise jumping**) *For any* $\gamma\in\mathcal{P}_\tau^0(2)$ *with* $\gamma(\tau)=N_1(1,b)$ *for* $b=\pm1$*, there exists a* $\theta_0>0$ *such that*

$$i_1([\gamma(\tau)\phi_{\theta,\tau}]*\gamma)=i_1(\gamma)+1,\qquad \forall\, 0<\theta\le\theta_0. \tag{1.13}$$

5° (**Normality**) *For the standard path* $\hat{\alpha}_{1,0,\tau}$*, there holds*

$$i_1(\hat{\alpha}_{1,0,\tau}) = 0. \tag{1.14}$$

Idea of the proof. Using normal forms and perturbation techniques together with the properties 1^o and 2^o to reduce the uniqueness to the case of paths in $\mathcal{P}_\tau(2)$. Then it follows from the $\mathbf{R}^3$-cylindrical coordinate representation introduced in the section 3 below immediately. The proof for the sufficiency can be found in [Lon4]. ∎

The following theorem is very useful in the study of the iteration theory for the Maslov-type index.

Theorem 1.12. (**Inverse homotopy invariant**) (cf. [Lon3], [Lon8]) *For any two paths* γ_0 *and* $\gamma_1 \in \mathcal{P}_\tau(2n)$ *with* $i_1(\gamma_0) = i_1(\gamma_1)$*, suppose that there exists a continuous path* $h : [0,1] \to \mathrm{Sp}(2n)$ *such that* $h(0) = \gamma_0(\tau)$*,* $h(1) = \gamma_1(\tau)$*, and* $\dim\ker(h(s)-I) = \nu_\tau(\gamma_0)$ *for all* $s \in [0,1]$*. Then* $\gamma_0 \sim \gamma_1$ *on* $[0,\tau]$ *along* h*.*

Idea of the proof. Note that $\gamma_0 \sim (h * \gamma_0)$. Since $(h * \gamma_0) \sim \gamma_1$ and γ_1 have the same end points and index, they must be homotopic. This proves the theorem. ∎

2 An intuitive explanation of the index theory for symplectic paths in $\mathrm{Sp}(2)$

At the last part of this section, we give an intuitive interpretation of the index theory defined above in terms of the cylinderical coordinate representation in $\mathbf{R}^3$ of $\mathrm{Sp}(2)$ firstly introduced in [Lon2] of 1991 by the author as follows. As well known, $M \in \mathrm{Sp}(2)$ if and only if $\det M = 1$. Via the polar decomposition of each element M in $\mathrm{Sp}(2)$,

$$M = \begin{pmatrix} r & z \\ z & (1+z^2)/r \end{pmatrix} \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}, \tag{2.1}$$

we can define a map Φ from the element M in $\mathrm{Sp}(2)$ to $(r,\theta,z) \in \mathbf{R}^+ \times S_{2\pi} \times \mathbf{R}$, where $\mathbf{R}^+ = \{r \in \mathbf{R}\,|\, r > 0\}$. This map Φ is a C^∞-diffeomorphism. In the following, for simplicity, we identify elements in $\mathrm{Sp}(2)$ and their images in $\mathbf{R} \setminus \{z-\text{axis}\}$ under Φ.

Remark 2.1. Note that a different representation of $\mathrm{Sp}(2)$ was given by I. Gelfand and V. Lidsikii in [GeL1] of 1955 which was based on the hyperbolic functions and which maps $\mathrm{Sp}(2)$ into a solid torus.

By this $\mathbf{R}^3$-cylindrical coordinate representation of $\mathrm{Sp}(2)$, it is easy to see that $\mathrm{Sp}(2)$ is homeomorphic to $S^1 \times \mathbf{R}^2$. This can be generalized to general $\mathrm{Sp}(2n)$ which is homeomorphic to a product of S^1 and a simply connected space. Therefore any path $\gamma \in \mathcal{P}_\tau(2)$ rotates around the deleted z-axis

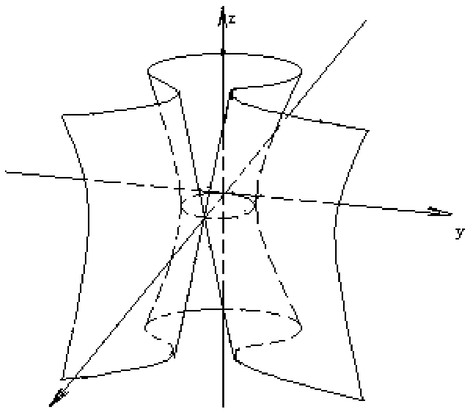

Figure 2.1: $\mathrm{Sp}(2)^0$ in cylindrical coordinates of $\mathbf{R}^3 \setminus \{z-\text{axis}\}$

in $\mathbf{R}^3$ in someway. There are infinitely many topologically meaningful ways to count the rotation number of γ. But the key point here is to find a natural way to count this rotation so that the rotation number reflects intrinsically analytical properties of the corresponding Hamiltonian system when $\gamma \in \hat{\mathcal{P}}_\tau(2)$.

Under this $\mathbf{R}^3$-cylindrical coordinate representation we have

$$\begin{aligned}
\mathrm{Sp}(2)^+ &= \{(r,\theta,z) \in \mathbf{R}^+ \times S^1 \times \mathbf{R} \,|\, (r^2+z^2+1)\cos\theta > 2r\},\\
\mathrm{Sp}(2)^0 &= \{(r,\theta,z) \in \mathbf{R}^+ \times S^1 \times \mathbf{R} \,|\, (r^2+z^2+1)\cos\theta = 2r\},\\
\mathrm{Sp}(2)^- &= \{(r,\theta,z) \in \mathbf{R}^+ \times S^1 \times \mathbf{R} \,|\, (r^2+z^2+1)\cos\theta < 2r\},\\
\mathrm{Sp}(2)^0_\pm &= \{(r,\theta,z) \in \mathrm{Sp}(2)^0 \,|\, \pm\sin\theta > 0\} = \left\{PN_1(1,\mp 1)P^{-1} \,|\, P \in \mathrm{Sp}(2)\right\},\\
\mathrm{Sp}(2)^* &= \mathrm{Sp}(2)^+ \cup \mathrm{Sp}(2)^-, \quad \mathrm{Sp}(2)^0 = \mathcal{M}_2^1 \cup \{I\}, \quad \mathcal{M}_2^1 = \mathrm{Sp}(2)^0_+ \cup \mathrm{Sp}(2)^0_-.
\end{aligned}$$

Note that $\mathrm{Sp}(2)^0$ is a codimension 1 hypersurface in $\mathrm{Sp}(2)$, $\mathcal{M}_2^1$ is its regular part. Note that $\mathcal{M}_2^1$ contains two path connected components $\mathrm{Sp}(2)^0_+$ and $\mathrm{Sp}(2)^0_-$, which are two smooth surfaces both diffeomorphic to $\mathbf{R}^2\setminus\{0\}$ as shown in the Figure 2.1. The following Figure 2.2 shows the picture of $\mathrm{Sp}(2)^0 \cap \{z=0\}$.

Note that for the case of $\mathrm{Sp}(2)$, Lemma 1.1 follows from these two pictures immediately.

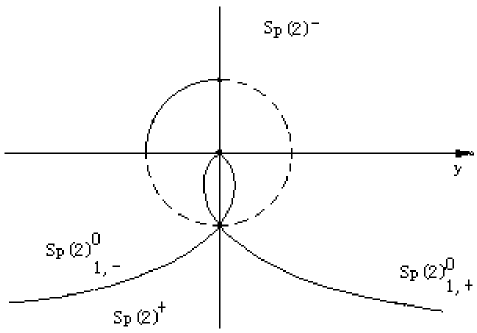

Figure 2.2: The intersection of $\mathrm{Sp}(2)^0$ and the plane $\{z = 0\}$

Now based upon the standard non-degenerate symplectic paths defined in the section 1, from Figures 2.1 and 2.2, it is obvious that for any given $\gamma \in \mathcal{P}_\tau^*(2)$, there exists one and only one $k \in \mathbf{Z}$ such that

$$\gamma \sim \hat{\alpha}_{1,k,\tau}.$$

This proves Lemma 1.5 for the case of $n = 1$, and then makes the Definition 1.6 become meaningful.

Now for $\gamma \in \mathcal{P}_\tau^0(2)$, from Figures 2.1 and 2.2, we immediately obtain the following results:

If $\gamma(\tau) \in \mathrm{Sp}(2)^0 \setminus \{I\}$, all paths $\beta \in \mathcal{P}_\tau^*(2)$ which are C^0-close to γ belong to two homotopy classes, one contains γ_{-1} and the other contains γ_1 defined by (1.10), and there holds

$$i_1(\gamma_{-1}) + 1 = i_1(\gamma_1).$$

If $\gamma(\tau) = I$, all paths $\beta \in \mathcal{P}_\tau^*(2)$ which are C^0-close to γ belong to three homotopy classes, one contains γ_{-1}, and another one contains γ_1 defined by (1.7). We pick up a path β in the third homotopy class. Then there holds

$$i_1(\gamma_{-1}) + 2 = i_1(\beta) + 1 = i_1(\gamma_1).$$

These results shows that the following definition (1.7) makes sense:

$$i_1(\gamma) = \inf\{i_1(\beta) \,|\, \beta \in \mathcal{P}_\tau^*(2) \textit{ and } \beta \textit{ is sufficiently close to } \gamma \textit{ in } \mathcal{P}_\tau(2)\}.$$

3 Relation with the Morse indices

Fix $\tau > 0$. Suppose $H \in C^2(S_\tau \times \mathbf{R}^{2n}, \mathbf{R})$ and $\|H\|_{C^2}$ is finite. Recall $S_\tau = \mathbf{R}/(\tau\mathbf{Z})$. The classical direct functional corresponding to the system (1.1) is

$$f(x) = \int_0^\tau (-\frac{1}{2}J\dot{x} \cdot x - H(t,x))dt, \tag{3.1}$$

for $x \in \mathrm{dom}(A) \subset L_\tau \equiv L^2(S_\tau, \mathbf{R}^{2n})$ with $A = -J\frac{d}{dt}$. It is well-known that critical points of f on L_τ and τ-periodic solutions of (0.1) are one-to-one correspondent. The Morse indices of f at its critical point x is defined by those of the following quadratic form on L_τ:

$$\phi(y) = \int_0^\tau (-J\dot{y} \cdot y - B(t)y \cdot y)dt, \tag{3.2}$$

where $B(t) = H''(t, x(t))$. Note that the positive and negative Morse indices of f at its critical point x, i.e. the total multiplicities of positive and negative eigenvalues of the quadratic form (3.2), are always infinite. Using the saddle point reduction method on the space L_τ (cf. [AmZ1] and [Cha1]), we obtain a finite dimensional subspace $Z \subset L_\tau$ consisting of finite Fourier polynomials with $2d = \dim Z$ being sufficiently large, an injective map $u : Z \to L_\tau$ and a functional $a : Z \to \mathbf{R}$, such that there holds

$$a(z) = f(u(z)), \qquad \forall z \in Z, \tag{3.3}$$

and that the critical points of a and f are one to one correspondent. Note that the following important result holds.

Theorem 3.1. (cf. [CoZ1], [LoZ1], [Lon1], [Lon8]) *Under the above conditions, let z be a critical point of a and $x = u(z)$ be the corresponding solution of the system (0.1). Denote the Morse indices of the functional a at z by $m^*(z)$ for $* = +, 0, -$. Then the Maslov-type index $(i_1(x), \nu_1(x))$ satisfy*

$$m^-(z) = d + i_1(x), \quad m^0(z) = \nu_1(x), \quad m^+(z) = d - i_1(x) - \nu_1(x). \tag{3.4}$$

Idea of the proof. 1^o For the non-degenerate case with $n \geq 2$ or $n = 1$ and $i_1(x) \in (2\mathbf{Z}+1) \cup \{0\}$ as in [CoZ1], it suffices to use the homotopy invariance of the Maslov-type index to reduce the computation of the indices to the case of liner Hamiltonian systems with constant coefficients.

2^o For the non-degenerate case with $n = 1$, we first couple the given linearized Hamiltonian system H_0 with a linear Hamiltonian system H_1 on $\mathbf{R}^2$ with constant coefficients and Maslov-type index 1 to get a new linear Hamiltonian system H_2 on $\mathbf{R}^4$. Then the index formula (3.4) for H_0 follows from that for H_1 subtract from that of H_2.

3^o For the degenerate case, use the paths γ_s and perturbation techniques to reduce the problem to the comparison of non-degenerate cases of γ_1 and γ_{-1}. ∎

Note that from (3.4), the Maslov-type indices can be viewed as a finite representation of the infinite Morse indices of the direct variational formulations. Note also that for general Hamiltonian H whose second derivative may not be bounded, results similar to Theorem 3.1 was proved via Galerkin approximations in [FeQ1] by G. Fei and Q. J. Qiu.

Next we consider the periodic problem of the calculus of variation, i.e. finding extremal loops of the following functional

$$F(x) = \int_0^\tau L(t, x, \dot{x})dt, \qquad \forall x \in W_\tau = W^{1,2}(S_\tau, \mathbf{R}^n). \tag{3.5}$$

Here we suppose $\tau > 0$ and $L \in C^2(S_\tau \times \mathbf{R}^n \times \mathbf{R}^n, \mathbf{R})$ such that $L_{p,p}(t, x, p)$ is symmetric and positive definite, and $L_{x,x}(t, x, p)$ is symmetric. An extremal loop x of F corresponds to a 1-periodic solution of the Lagrangian system

$$\frac{d}{dt} L_p(t, x, \dot{x}) - L_x(t, x, \dot{x}) = 0. \tag{3.6}$$

Fix such an extremal loop x, define

$$P(t) = L_{p,p}(t, x(t), \dot{x}(t)), \quad Q(t) = L_{x,p}(t, x(t), \dot{x}(t)), \quad R(t) = L_{x,x}(t, x(t), \dot{x}(t)). \tag{3.7}$$

The Hessian of F at x corresponds to a linear periodic Sturm system,

$$-(P\dot{y} + Qy)^{\cdot} + Q^T\dot{y} + Ry = 0. \tag{3.8}$$

It corresponds to the linear Hamiltonian system (0.2) with

$$B(t) \equiv B_x(t) = \begin{pmatrix} P^{-1}(t) & -P^{-1}(t)Q(t) \\ -Q(t)^T P^{-1}(t) & Q(t)^T P^{-1}(t)Q(t) - R(t) \end{pmatrix}. \tag{3.9}$$

Denote by γ_x the fundamental solution of this linearized Hamiltonian system (0.2). The Morse index and nullity of the functional F at an extremal loop x in W_τ are always finite. We denote them by $m^-(x)$ and $m^0(x)$ respectively.

Theorem 3.2. (cf. [Vit2], [LoA1], [AnL1]) *Under the above conditions, there hold*

$$m^-(x) = i_1(\gamma_x), \qquad m^0(x) = \nu_1(\gamma_x). \tag{3.10}$$

Idea of the proof. We apply the index theory of [Dui1]. Using the homotopy invariance properties of this index theory and the Maslov-type index theory to simple standard cases, then (3.10) is proved by concrete computations on these simple cases. ∎

Remark 3.3. (i) Note that in the sense of Theorems 3.1 and 3.2, our Definition 1.9 of the Maslov-type index is natural.

(ii) Similar to Theorem 3.2, one can prove that for every closed geodesic c with the linearized Poincaré map P_c ($\in \mathrm{Sp}(2n-2)$) on a Finsler (as well as Riemannian) manifold (M, F) with $n = \dim M$, there exists a symplectic path $\gamma \in \mathcal{P}_1(2n-2)$ such that there hold $\gamma(1) = P_c$ and

$$i_1(\gamma) = i(c), \quad \nu_1(\gamma) = \nu(c). \tag{3.11}$$

(iii) For the Ekeland index $(i^E(x), \nu^E(x))$ defined for every periodic solution x of a convex Hamiltonian system (0.1) on $\mathbf{R}^{2n}$, similar to Theorem 3.2, one can prove

$$i_1(x) = i^E(x) + n, \qquad \nu_1(x) = \nu^E(x). \tag{3.12}$$

Chapter 2. The Index Iteration Theory for Symplectic Paths

For $\tau > 0$ and any $\gamma \in \mathcal{P}_\tau(2n)$, the iteration path $\tilde{\gamma} \in C([0,+\infty), \mathrm{Sp}(2n))$ of γ is defined by

$$\tilde{\gamma}(t) = \gamma(t - j\tau)\gamma(\tau)^j, \qquad \text{for } j\tau \le t \le (j+1)\tau \text{ and } j \in \{0\} \cup \mathbf{N},$$

and $\gamma^m = \tilde{\gamma}|_{[0,m\tau]}$ for all $m \in \mathbf{N}$. Then we can associate to γ through γ^m a sequence of Maslov-type indices

$$\{(i_1(\gamma^m), \nu_1(\gamma^m))\}_{m\in\mathbf{N}}.$$

When $\gamma : [0,+\infty) \to \mathrm{Sp}(2n)$ is the fundamental solution of (0.2) with $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$, where $\mathcal{L}_s(\mathbf{R}^{2n})$ is the set of symmetric $2n \times 2n$ real matrices, there holds $(\gamma|_{[0,\tau]})^\sim = \gamma$. When x is a τ-periodic solution of (1.1), we define the iterations of x by

$$x^m(t) = x(t - j), \qquad \forall j \le t \le j+1, j = 0, 1, \ldots, m-1.$$

Denotes by

$$(i_1(x^m), \nu_1(x^m)) = (i_1(\gamma_x^m), \nu_1(\gamma_x^m)).$$

Thus the corresponding index sequence with $\gamma = \gamma_x$ reflects important properties of the τ-periodic solution x of the Hamiltonian system (1.1).

In the celebrated work [Bot1] of R. Bott in 1956 as well as [BTZ1], the iteration theory of Morse index for closed geodesics is established. In the works of I. Ekeland (cf. [Eke1]-[Eke3]) the iteration theory of his index for convex Hamiltonian systems is established. In [Vit1] of C. Viterbo, the iteration theory for an index theory of nondegenerate star-shaped Hamiltonian systems is partially established. But for our purpose in the study of existence, multiplicity, and stability problems of periodic orbits of nonlinear Hamiltonian systems with no any convexity assumptions, all these results are not applicable. The only paper we know which studied certain iteration properties of certain Maslov index in such a generality is [CuD1] of R. Cushman and J. Duistermaat in 1977. But their result is not good enough for our purposes and contains some flaws in certain cases.

The basic question in an index iteration theory is for any given positive integer m to determine precisely the index pair $(i_1(x^m), \nu_1(x^m))$ of the m-th iteration x^m of a τ-periodic solution x of a given Hamiltonian system in terms of its initial index $(i_1(x), \nu_1(x))$ and information from the linearized Poincaré map $\gamma_x(\tau)$. In a sequence of papers in 1990s up to 2002, we have established such an index iteration theory for any symplectic paths including the Bott-type iteration formulae, precise iteration formulae, various iteration inequalities, and the common index jump theorem. In this chapter, I shall give an introduction to this index iteration theory.

4 The ω-index theory, splitting numbers and basic normal form decompositions

As we have mentioned in the Section 1, the Maslov-type index theory is defined via the singular hypersurface $\mathrm{Sp}(2n)^0$ in $\mathrm{Sp}(2n)$. This hypersurface is formed by all matrices in $\mathrm{Sp}(2n)$ which possesses 1 as its eigenvalues. In the study of the iteration properties of the Malsov-type index theory, as in [Lon4] for any $\omega \in \mathbf{U}$ it is natural to consider the generalization $D \in C^\infty(\mathbf{U} \times \mathrm{Sp}(2n), \mathbf{R})$ of the determinant function defined by

$$D_\omega(M) = (-1)^{n-1}\omega^{-n}\det(M - \omega I), \quad \forall \omega \in \mathbf{U}, M \in \mathrm{Sp}(2n), \tag{4.1}$$

and the hypersurface

$$\mathrm{Sp}(2n)^0_\omega = \{M \in \mathrm{Sp}(2n) \,|\, D_\omega(M) = 0\}, \tag{4.2}$$

which contains all symplectic matrices having ω as an eigenvalue. Similarly for any $\omega \in \mathbf{U}$ we define

$$\begin{aligned}
\mathrm{Sp}(2n)^\pm_\omega &= \{M \in \mathrm{Sp}(2n) \,|\, \pm D_\omega(M) < 0\}, \\
\mathrm{Sp}(2n)^*_\omega &= \mathrm{Sp}(2n)^+_\omega \cup \mathrm{Sp}(2n)^-_\omega = \mathrm{Sp}(2n) \setminus \mathrm{Sp}(2n)^0_\omega, \\
\mathcal{P}^*_{\tau,\omega}(2n) &= \{\gamma \in \mathcal{P}_\tau(2n) \,|\, \gamma(\tau) \in \mathrm{Sp}(2n)^*_\omega\}, \\
\mathcal{P}^0_{\tau,\omega}(2n) &= \mathcal{P}_\tau(2n) \setminus \mathcal{P}^*_{\tau,\omega}(2n).
\end{aligned}$$

In [Lon5], for $\omega \in \mathbf{U}$, the ω-nullity of any symplectic path is defined by

$$\nu_\omega(\gamma) = \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I), \qquad \forall \gamma \in \mathcal{P}_\tau(2n). \tag{4.3}$$

In [Lon4], the author proved the following result similar to Lemma 1.1.

Lemma 4.1. (cf. [Lon4]) *For any $\omega \in \mathbf{U}$, $\mathrm{Sp}(2n)^*_\omega$ contains two path connected components $\mathrm{Sp}(2n)^+_\omega$ and $\mathrm{Sp}(2n)^-_\omega$, and $M_n^\pm \in \mathrm{Sp}(2n)^\pm_\omega$. Both of these two sets are simply connected in $\mathrm{Sp}(2n)$.*

Based upon this result, the index $i_\omega(\gamma)$ is defined in [Lon5] for any $\gamma \in \mathcal{P}^*_{\tau,\omega}(2n)$ in a similar way to that used in the Definition 1.6.

Then based upon the results obtained in [Lon4] on the properties of and near $\mathrm{Sp}(2n)^0_\omega$ in $\mathrm{Sp}(2n)$, for any $\omega \in \mathbf{U}$ and $\gamma \in \mathcal{P}^0_{\tau,\omega}(2n)$ it is defined in [Lo11] that

$$i_\omega(\gamma) = \inf\{i_\omega(\beta) \,|\, \beta \in \mathcal{P}^*_{\tau,\omega}(2n) \textit{ and } \beta \textit{ is sufficiently close to } \gamma \textit{ in } \mathcal{P}_\tau(2n)\}.$$

In such a way, the ω-index theory assigns a pair of integers to each $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega \in \mathbf{U}$:

$$(i_\omega(\gamma), \nu_\omega(\gamma)) \in \mathbf{Z} \times \{0, \ldots, 2n\}. \tag{4.4}$$

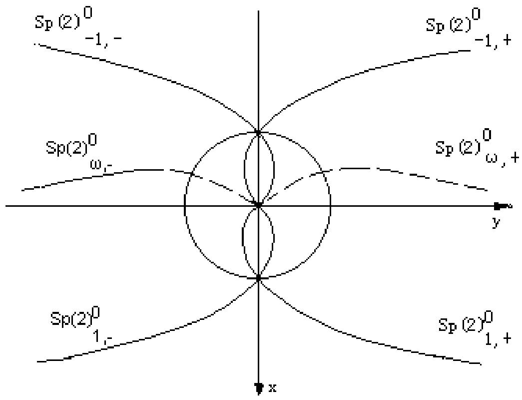

Figure 4.1: Oriented $\mathrm{Sp}(2)^0_\omega$ for $\omega = \pm 1$ and $\omega \in \mathbf{U} \setminus \mathbf{R}$

When $\omega = 1$, the ω-index theory coincides with the Maslov-type index theory. Similarly to Theorem 1.11, an axiom characterization of the ω-index theory can be given as in [Lon5].

Now let us fix a path $\gamma \in \mathcal{P}_\tau(2n)$, and move ω on $\mathbf{U}$ from 1 to -1, and study the properties of the ω-index of γ as functions of ω. In [Lon5], the following result is proved.

Lemma 4.2. (cf. [Lon5], [Lon8]) *For fixed $\gamma \in \mathcal{P}_\tau(2n)$, $i_\omega(\gamma)$ as a function of ω is constant on each connected component of $\mathbf{U} \setminus \sigma(\gamma(\tau))$. There holds*

$$\nu_\omega(\gamma) = 0, \qquad \forall \omega \in \mathbf{U} \setminus \sigma(\gamma(\tau)). \tag{4.5}$$

Idea of the proof. It follows from that the index functions are locally constant. ∎

By this lemma, in order to understand the properties of the ω-index as a function of $\omega \in \mathbf{U}$, it is important to study the possible jumps of $i_\omega(\gamma)$ at $\omega \in \mathbf{U} \setminus \sigma(\gamma(\tau))$. These jumps are usually called **splitting numbers**, which play a crucial role in iteration theory of the Maslov-type index theory for symplectic paths. The precise definition of the splitting number is contained in the following result.

Theorem 4.3. (cf. [Lon5], [Lon8]) *For any $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U}$, choose $\tau > 0$ and*

$\gamma \in \mathcal{P}_\tau(2n)$ *with* $\gamma(\tau) = M$*, and define*

$$S_M^{\pm}(\omega) = \lim_{\epsilon\to 0^+} i_{\exp(\pm\epsilon\sqrt{-1})\omega}(\gamma) - i_\omega(\gamma). \tag{4.6}$$

Then these two integers are independent of the choice of the path γ*. They are called the* **splitting numbers** *of* M *at* ω*.*

In order to further understand the splitting number, new concepts of the homotopy component of $M \in \mathrm{Sp}(2n)$ and the **ultimate type** of $\omega \in \mathbf{U}$ for $M \in \mathrm{Sp}(2n)$ is introduced by the author in [Lon5] as follows.

Definition 4.4. (cf. [Lon5], [Lon8]) *For any* $M \in \mathrm{Sp}(2n)$*, define the* **homotopy set** *of* M *in* $\mathrm{Sp}(2n)$ *by*

$$\begin{aligned}\Omega(M) = \{N \in \mathrm{Sp}(2n) \quad | \quad & \sigma(N)\cap\mathbf{U} = \sigma(M)\cap\mathbf{U}, \text{ and} \\ & \dim_{\mathbf{C}}\ker_{\mathbf{C}}(N-\lambda I) = \dim_{\mathbf{C}}\ker_{\mathbf{C}}(M-\lambda I),\ \forall\,\lambda\in\sigma(M)\cap\mathbf{U}\}.\end{aligned}$$

We denote by $\Omega^0(M)$ *the path connected component of* $\Omega(M)$ *which contains* M*, and call it the* **homotopy component** *of* M *in* $\mathrm{Sp}(2n)$*.*

For any $M \in \mathrm{Sp}(2n)$, define its conjugate set by

$$[M] = \{N \in \mathrm{Sp}(2n)\,|\,N = P^{-1}MP \text{ for some } P \in \mathrm{Sp}(2n)\}.$$

Then $[M] \subset \Omega^0(M)$ for all $\omega \in \mathbf{U}$.

Definition 4.5.(cf. [Lon5], [Lon8]) *The following matrices in* $Sp(2n)$ *are called* **basic normal forms** *for eigenvalues on* $\mathbf{U}$*:*

$$\begin{aligned}
N_1(\lambda,b) & \qquad \text{with } \lambda=\pm1,\ \ b=\pm1, \text{ or } 0, \\
R(\theta) & \qquad \text{with } \omega = e^{\theta\sqrt{-1}} \in \mathbf{U}\setminus\mathbf{R}, \\
N_2(\omega,b) & = \begin{pmatrix} R(\theta) & b \\ 0 & R(\theta)\end{pmatrix}, \quad \text{with } b = \begin{pmatrix} b_1 & b_2 \\ b_3 & b_4\end{pmatrix} \in \mathcal{L}(\mathbf{R}^2), \\
& \qquad b_2-b_3 \neq 0, \quad \text{and } \omega = e^{\theta\sqrt{-1}} \in \mathbf{U}\setminus\mathbf{R}.
\end{aligned}$$

A basic normal form matrix M *is* **trivial***, if for sufficiently small* $a>0$*,* $MR((t-1)a)^{\diamond n}$ *possesses no eigenvalue on* $\mathbf{U}$ *for* $t\in[0,1)$*, and is* **nontrivial** *otherwise.*

Note that by direct computations, $N_1(1,-1)$, $N_1(-1,1)$, $N_2(\omega,b)$, and $N_2(\overline{\omega},b)$ with $\dim_{\mathbf{C}}\ker_{\mathbf{C}}(M-\omega I) = 1$, $\omega = \exp(\theta\sqrt{-1}) \in \mathbf{U}\setminus\mathbf{R}$ and $(b_2-b_3)\sin\theta > 0$ are trivial, and any other basic normal form matrix is nontrivial.

Theorem 4.6.(cf. [Lon5], [Lon8]) *For any* $M \in \mathrm{Sp}(2n)$*, there is a path* $f \in C^\infty([0,1],\Omega^0(M))$ *such that* $f(0) = M$ *and*

$$f(1) = M_1\diamond\cdots\diamond M_k\diamond M_0, \tag{4.7}$$

where the integer $p \in [0,n]$, each M_i is a basic normal form of eigenvalues on $\mathbf{U}$ for $1 \le i \le k$, and the symplectic matrix M_0 satisfies $\sigma(M_0) \cap \mathbf{U} = \emptyset$.

Idea of the proof. Firstly we connect M within $\Omega^0(M)$ to a product of normal forms via the results of [LoD1] and [HaL1]. Then by carefully chosen perturbations and connecting paths, we connect all these normal forms to basic normal forms within $\Omega^0(M)$. ∎

Recall that (cf. Section I.2 of [Eke3] or [YaS1]) for $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U} \cap \sigma(M)$ being an m-fold eigenvalue, the Hermitian form $\langle \sqrt{-1}J\cdot, \cdot\rangle$, which is called the **Krein form**, is always nondegenerate on the invariant root vector space $E_\omega(M) = \ker_{\mathbf{C}}(M - \omega I)^m$, where $\langle\cdot,\cdot\rangle$ denotes the inner product in $\mathbf{C}^{2n}$. Then ω is of **Krein type** (p,q) with $p+q=m$ if the restriction of the Krein form on $E_\omega(M)$ has signature (p,q). ω is **Krein positive** if it has Krein type $(p,0)$, is **Krein negative** if it has Krein type $(0,q)$. If $\omega \in \mathbf{U} \setminus \sigma(M)$, we define the Krein type of ω by $(0,0)$.

Definition 4.7.(cf. [Lon5], [Lon8]) *For any basic normal form $M \in \mathrm{Sp}(2n)$ and $\omega \in \mathbf{U} \cap \sigma(M)$, we define the* **ultimate type** *(p,q) of ω for M to be its usual Krein type if M is nontrivial, and to be $(0,0)$ if M is trivial. For any $M \in \mathrm{Sp}(2n)$, we define the ultimate type of ω for M to be $(0,0)$ if $\omega \in \mathbf{U} \setminus \sigma(M)$. For any $M \in \mathrm{Sp}(2n)$, by Theorem 4.6 there exists a $\diamond$-product expansion (4.7) in the homotopy component $\Omega^0(M)$ of M where each M_i is a basic normal form for $1 \le i \le k$ and $\sigma(M_0) \cap \mathbf{U} = \emptyset$. Denote the ultimate type of ω for M_i by (p_i, q_i) for $0 \le i \le k$. Let $p = \sum_{i=0}^k p_i$ and $q = \sum_{i=0}^k q_i$. We define the* **ultimate type** *of ω for M by (p,q).*

It is proved in [Lon5] that the ultimate type of $\omega \in \mathbf{U}$ for M is uniquely determined by ω and M, therefore is well defined. It is constant on $\Omega^0(M)$ for fixed $\omega \in \mathbf{U}$.

Lemma 4.8. (cf. [Lon5], [Lon8]) *For $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, denote the Krein type and the ultimate type of ω for M by (P,Q) and (p,q). Then there holds*

$$P - p = Q - q \ge 0. \tag{4.8}$$

The following theorem completely characterizes the splitting numbers.

Theorem 4.9. (cf. [Lon5], [Lon8]) *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, there hold*

$$S_M^+(\omega) = p \quad \text{and} \quad S_M^-(\omega) = q, \tag{4.9}$$

where (p,q) is the ultimate type of ω for M.

Idea of the proof. Use Theorem 4.6 to reduce the proof to the case of basic normal forms. Then carry out the direct computation for each basic normal form. The difficulty part is the computation for $N_2(\omega, b)$'s. We refer to [Lon5] (cf. [Lon8]) for details. ∎

Corollary 4.10. *If $\omega \in \mathbf{U} \cap \sigma(\gamma(\tau))$ is of Krein type (p, q), there holds*

$$\lim_{\epsilon \to 0^+} \left(i_{e^{\epsilon\sqrt{-1}}\omega}(\gamma) - i_{e^{-\epsilon\sqrt{-1}}\omega}(\gamma) \right) = p - q. \tag{4.10}$$

Corollary 4.11. *For any $\omega \in \mathbf{U}$ and $M \in \mathrm{Sp}(2n)$, there holds*

$$0 \leq S_M^{\pm}(\omega) \leq \dim_{\mathbf{C}} \ker_{\mathbf{C}}(M - \omega I). \tag{4.11}$$

Remark 4.12. Theorem 4.9 and Corollary 4.10 generalize Theorem IV on p.180 of [Bot1] and Proposition 9 on p.44 of [Eke3]. Note that there is a sign difference between our J and that in [Eke3]. Note also that the conclusion of our Theorem 4.9 coincides with the Example II on p.181 of [Bot1].

5 Bott-type iteration formulae and the mean index

Based upon our preparations in the above subsection, next we establish the Bott-type formulae for the Maslov-type index theory.

Fix $\tau > 0$ and $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$. Let $\gamma \in \hat{\mathcal{P}}_\tau(2n)$, i.e. γ is the fundamental solution of (0.2) for some $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$. Fix $k \in \mathbf{N}$. The bilinear form corresponding to the system (0.2) is given by

$$\phi_{k\tau}(x, y) = \frac{1}{2}\langle (A - B)x, y \rangle_{L_\tau}, \qquad \forall x, y \in E_{k\tau} = W^{1,2}(S_{k\tau}, \mathbf{R}^{2n})) \subset L_{k\tau}. \tag{5.1}$$

For $\omega \in \mathbf{U}$ define

$$E_{k\tau}(\tau, \omega) = \{ y \in E_{k\tau} \,|\, y(t + \tau) = \omega y(t),\ \forall t \}.$$

For simplicity we identify $E_{k\tau}(\tau, \omega)$ with $E_\tau(\tau, \omega)$. Define $\omega_p = \exp(2p\pi/k\sqrt{-1})$ for $0 \leq p \leq k$. Then $\omega_p^k = 1$. By direct computation we obtain that $E_{k\tau}(\tau, \omega_p)$ and $E_{k\tau}(\tau, \omega_q)$ is $\phi_{k\tau}$-orthogonal for $0 \leq p \neq q \leq k$, and there holds

$$E_{k\tau} = \oplus_{\omega^k = 1} E_{k\tau}(\tau, \omega). \tag{5.2}$$

Thus we obtain

$$\phi_{k\tau}|_{E_{k\tau}} = \sum_{i=0}^{k-1} \phi_{k\tau}|_{E_{k\tau}(\tau, \omega_i)} = k \sum_{i=0}^{k-1} \phi_\tau|_{E_\tau(\tau, \omega_i)}. \tag{5.3}$$

Now we carry out the saddle point reduction for $\phi_{k\tau}$ on $E_{k\tau}$, and obtain the functional $a_{k\tau} = \phi_{k\tau} \circ u_{k\tau}$ defined on $Z_{k\tau}$. Simultaneously this induces saddle point reductions for ϕ_τ on $E_\tau(\tau, \omega_i)$ for $0 \leq i \leq k-1$, and yields the functional $a_{\tau,\omega_i} = \phi_\tau \circ u_{\tau,\omega_i}$ defined on $Z_{\tau,\omega}$. By the orthogonality claim (5.3), the Morse index of $\phi_{k\tau}$ on the left hand side of (5.3) splits into the sum of the Morse indices of the functional on the right hand side of (5.3). Note that the dimensions of spaces appeared in (5.3) satisfy

$$d_{k\tau} = \sum_{\omega^k=1} d_{\tau,\omega}. \tag{5.4}$$

Thus by Theorem 2.1, we obtain the following Bott-type formula for $\gamma \in \hat{\mathcal{P}}_\tau(2n)$.

Theorem 5.1. (cf. [Lon5], [Lon8]) *For any $\tau > 0$, $\gamma \in \mathcal{P}_\tau(2n)$, and $m \in \mathbf{N}$, there hold*

$$i_1(\gamma^m) = \sum_{\omega^m=1} i_\omega(\gamma), \tag{5.5}$$

$$\nu_1(\gamma^m) = \sum_{\omega^m=1} \nu_\omega(\gamma). \tag{5.6}$$

Idea of the proof. For the general case of $\gamma \in \mathcal{P}_\tau(2n)$. Choose $\beta \in \hat{P}_\tau(2n)$ such that $\beta(\tau) = \gamma(\tau)$ and $\beta \sim \gamma$. We obtain $i_\omega(\beta) = i_\omega(\gamma)$ for all $\omega \in \mathbf{U}$. From $\beta \sim \gamma$ with fixed end points, this homotopy can be extended to $[0,1] \times [0, k\tau]$. By the inverse homotopy Theorem 1.12, we then obtain $\beta^k \sim \gamma^k$. Thus $i_1(\beta^k) = i_1(\gamma^k)$ holds. Then the Bott-type formulae (5.5) and (5.6) for β imply those for γ. This completes the proof of Theorem 5.1. ∎

As a direct consequence of Theorem 5.1, we obtain

$$\frac{i_1(\gamma^k)}{k} = \frac{1}{2\pi} \sum_{\omega^k=1} i_\omega(\gamma) \frac{2\pi}{k},$$

$$\frac{\nu_1(\gamma^k)}{k} = \frac{1}{2\pi} \sum_{\omega^k=1} \nu_\omega(\gamma) \frac{2\pi}{k}.$$

By Lemma 4.2, as functions of ω, the function $i_\omega(\gamma)$ is locally constant and $\nu_\tau(\omega)$ is locally zero on $\mathbf{U}$ except at finitely many points. Therefore the right hand sides of above equalities are Riemannian sums, and converge to the corresponding integrals as $k \to \infty$. This proves the following result.

Theorem 5.2. (cf. [Lon5], [Lon8]) *For any $\tau > 0$ and $\gamma \in \mathcal{P}_\tau(2n)$ there hold*

$$\hat{i}(\gamma) \equiv \lim_{k\to+\infty} \frac{i_1(\gamma^k)}{k} = \frac{1}{2\pi} \int_{\mathbf{U}} i_\omega(\gamma) d\omega, \tag{5.7}$$

$$\hat{\nu}(\gamma) \equiv \lim_{k\to+\infty} \frac{\nu(\gamma^k)}{k} = \frac{1}{2\pi} \int_{\mathbf{U}} \nu_\omega(\gamma) d\omega = 0. \tag{5.8}$$

Specially, $\hat{i}(\gamma)$ is always a finite real number, and is called the **mean Maslov-type index** *per τ for γ.*

As a direct consequence of Theorem 5.2, for any $\gamma \in \mathcal{P}_\tau(2n)$ we obtain

$$\hat{i}(\gamma^k) = k\hat{i}(\gamma), \qquad \forall k \in \mathbf{N}. \tag{5.9}$$

Then through the fundamental solution γ_x of (0.2) with $B(t) = H''(t, x(t))$, the mean index per period τ of a τ-periodic solution x of the nonlinear system (0.1) can be defined by

$$\hat{i}(x) = \hat{i}(\gamma_x). \tag{5.10}$$

This yields a new invariant to each periodic solution of the system (0.1).

Remark 5.3. As proved in [Lon5] (cf. [Lon8]), for a fixed Sturm system (3.8) and the corresponding path $\gamma \in \mathcal{P}_\tau(2n)$ as the fundamental solution of the system (0.2) with coefficient B defined by (3.9), our ω-index pair $(i_\omega(\gamma), \nu_\omega(\gamma))$ and the index functions $\Lambda(\omega)$ and $N(\omega)$ of R. Bott defined in [Bot1] satisfy

$$i_\omega(\gamma) = \Lambda(\omega), \quad \nu_\omega(\gamma) = N(\omega), \quad \forall \omega \in \mathbf{U}. \tag{5.11}$$

Note that in [Eke3] the standard symplectic matrix is given by $-J$. For the fundamental solution γ of a fixed linear Hamiltonian system (0.2) with negative definite coefficient $B \in C(S_\tau, \mathcal{L}_s(\mathbf{R}^{2n}))$, our ω-index and the index functions $j_\tau(\omega)$ and $n_\tau(\omega)$ of I. Ekeland defined in the section I.5 of [Eke3] satisfy

$$\begin{aligned} \nu_\omega(\gamma) &= n_\tau(\omega), \qquad \forall\, \omega \in \mathbf{U}, & (5.12)\\ i_1(\gamma) + \nu_1(\gamma) &= -j_\tau(1) - n, & (5.13)\\ i_\omega(\gamma) + \nu_\omega(\gamma) &= -j_\tau(\omega), \qquad \forall\, \omega \in \mathbf{U} \setminus \{1\}. & (5.14) \end{aligned}$$

By (5.11) and (5.12)-(5.14), our above theorems generalize the well known Bott formulae (Theorem A of [Bo] with periodic boundary condition) for Morse indices of closed geodesics, and the Bott-type formulae of Ekeland indices (Corollary I.4 of [Eke3]) for convex Hamiltonian systems, and corresponding result of C. Viterbo in [Vit1] for non-degenerate star-shaped Hamiltonian systems.

6 Precise iteration formulae

The basic question of the index iteration theory is compute the index of the m-th iteration γ^m for any given $m \in \mathbf{N}$ and $\gamma \in \mathcal{P}_\tau(2n)$ in terms of the initial information $i_1(\gamma)$, $\nu_1(\gamma)$ and $\gamma(\tau)$. The Bott-type formula Theorem 5.1 is a powerful tool for this purpose. Such a formula is established based on the usual symplectic coordinate changes. Thus in many cases, each normal form in the decomposition of the end point matrix $\gamma_x(\tau) \in \mathrm{Sp}(2n)$ of the fundamental solution γ_x of a periodic solution x of a Hamiltonian system may still possesses very high order which makes the computation and estimates of the indices become very difficulty. On the other hand, in the Bott-type formula, the index of the m-th iteration is computed in terms of the ω-indices for all m-th roots of unit ω, which is also not easy to compute neither.

To answer for this basic question was given by [Lon6] of the author in 2000, where a different method of computing and estimating the Maslov-type indices for such iterated paths was developed. The main idea is to reduce the computation of the index of a given path in $\mathrm{Sp}(2n)$ to those of paths in $\mathrm{Sp}(2)$ and some special paths in $\mathrm{Sp}(4)$ ending at the basic normal forms by a sequence of homotopies in the sense of Definition 1.3. But in terms of the cylinderical coordinate representation of $\mathrm{Sp}(2)$ in $\mathbf{R}^3$, the computation of the Maslov-type index of any path in $\mathrm{Sp}(2)$ starting from I is almost obvious. By certain careful studies, the cases in $\mathrm{Sp}(4)$ can also be reduced to the case of $\mathrm{Sp}(2)$. This method yields rather precise information on the Maslov-type indices for iterations with very simple paths. Using this method, a different proof of the Bott-type formulae (5.5) and (5.6) can be given by computing both sides of them on paths in $\mathcal{P}_\tau(2) \cup \mathcal{P}_\tau(4)$ ending at basic normal forms in Definition 4.5.

For any $a \in \mathbf{R}$, we define

$$[a] = \max\{k \in \mathbf{Z} \mid k \le a\}, \qquad E(a) = \min\{k \in \mathbf{Z} \mid k \ge a\},$$
$$\varphi(a) = E(a) - [a], \qquad \{a\} = a - [a].$$

The following is the main result in [Lon6].

Theorem 6.1. ([Lon6], [Lon8]) *Let $\gamma \in \mathcal{P}_\tau(2n)$. There is a path $f \in C([0,1], \Omega^0(\gamma(\tau))$ by Theorem 1.8.10 such that $f(0) = \gamma(1)$ and*

$$\begin{aligned} f(1) \;\; = \;\; & N_1(1,1)^{\diamond p_-} \diamond I_{2p_0} \diamond N_1(1,-1)^{\diamond p_+} \\ & \diamond N_1(-1,1)^{\diamond q_-} \diamond (-I_{2q_0}) \diamond N_1(-1,-1)^{\diamond q_+} \\ & \diamond R(\theta_1) \diamond \cdots \diamond R(\theta_r) \\ & \diamond N_2(\omega_1, u_1) \diamond \cdots \diamond N_2(\omega_{r_*}, u_{r_*}) \\ & \diamond N_2(\lambda_1, v_1) \diamond \cdots \diamond N_2(\lambda_{r_0}, v_{r_0}) \\ & \diamond M_0, \end{aligned} \tag{6.1}$$

where $N_2(\omega_j, u_j)$'s are nontrivial and $N_2(\lambda_j, v_j)$'s are trivial basic normal forms; $\sigma(M_0) \cap \mathbf{U} = \emptyset$; p_-, p_0, p_+, q_-, q_0, q_+, r, r_*, and r_0 are nonnegative integers; $\omega_j = e^{\sqrt{-1}\alpha_j}$, $\lambda_j = e^{\sqrt{-1}\beta_j}$; θ_j, α_j, $\beta_j \in (0,\pi) \cup (\pi, 2\pi)$; these integers and real numbers are uniquely determined by $\gamma(\tau)$. Then there hold

$$i(\gamma, m) \;\; = \;\; m(i(\gamma,1) + p_- + p_0 - r) + 2\sum_{j=1}^{r} E(\frac{m\theta_j}{2\pi}) - r$$

$$
\begin{aligned}
& -p_- - p_0 - \frac{1+(-1)^m}{2}(q_0+q_+) \\
& +2(\sum_{j=1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi}) - r_*), \qquad (6.2) \\
\nu(\gamma,m) \;=\; & \nu(\gamma,1) + \frac{1+(-1)^m}{2}(q_- + 2q_0 + q_+) + 2\varsigma(m,\gamma(\tau)), \qquad (6.3)
\end{aligned}
$$

where we denote by

$$\varsigma(m,\gamma(\tau)) = r - \sum_{j=1}^{r}\varphi(\frac{m\theta_j}{2\pi}) + r_* - \sum_{j=1}^{r_*}\varphi(\frac{m\alpha_j}{2\pi}) + r_0 - \sum_{j=1}^{r_0}\varphi(\frac{m\beta_j}{2\pi}). \qquad (6.4)$$

Remark 6.2. Note that using Theorem 6.1, results in the Section 5 can also be proved.

Based upon teh basic normal form decomposition (6.1), the proofs of (6.2) and (6.3) in Theorem 6.2 are reduced to paths in $\mathcal{P}_\tau(2)$ and $\mathcal{P}_\tau(4)$ with end matrices listed in (6.1). To illustrate the computations of Maslov-type indices for iterated paths, next we give a pictorial proof of the iteration formulae for the case of $\gamma(\tau) \in \mathrm{Sp}(2)^0_-$, and explain several most important cases which we shall need in the later sections of our applications. For more details about this computation, we refer to [Lon6] as well as [Lon8].

Let $\mathrm{Sp}(2)^0_\pm = \Omega^0(N_1(1,\mp 1))$.

Case 1. $\gamma \in \mathcal{P}^0_\tau(2)$ *and* $\gamma(\tau) \in \mathrm{Sp}(2)^0_-$.

In this case we must have $k \equiv i_1(\gamma)$ being odd and $\nu_1(\gamma) = 1$. From the fact $(\mathrm{Sp}(2)^0_-)^m \subset \mathrm{Sp}(2)^0_-$, we obtain $\nu_1(\gamma^m) = 1$ for all $m \in \mathbf{N}$. From the Figure 6.1 we obtain $i_1(\gamma) = (k+1)-1$, and $i_1(\gamma^m) = m(k+1)-1$ for all $m \in \mathbf{N}$. Thus in this case we obtain

$$i_1(\gamma^m) = m(i_1(\gamma)+1)-1, \quad \nu_1(\gamma^m) = 1, \quad \forall m \in \mathbf{N}. \qquad (6.5)$$

Note that this formula can also be obtained from the Bott-type formula (5.7).

Case 2. $\gamma \in \mathcal{P}_\tau(2n)$ *and* $\gamma(\tau) = I$.

Similar to the case 1, we must have $i_1(\gamma)$ being odd and $\nu_1(\gamma) = 2$. In this case we obtain

$$i_1(\gamma^m) = m(i_1(\gamma)+1)-1, \quad \nu_1(\gamma^m) = 2, \quad \forall m \in \mathbf{N}. \qquad (6.6)$$

Case 3. $\gamma \in \mathcal{P}_\tau(2n)$ *and* $\gamma(\tau) \in \mathrm{Sp}(2)^0_+$.

In this case, we must have $i_1(\gamma)$ being even and $\nu_1(\gamma) = 2$. Similar to the case 1 we obtain

$$i_1(\gamma^m) = mi_\tau(\gamma), \quad \nu_1(\gamma^m) = 1, \quad \forall m \in \mathbf{N}. \qquad (6.7)$$

Case 4. $\gamma \in \mathcal{P}_\tau(2n)$ *and* $\sigma(\gamma(\tau)) = \{a, a^{-1}\}$ *with* $a \in \mathbf{R} \setminus \{0, \pm 1\}$.

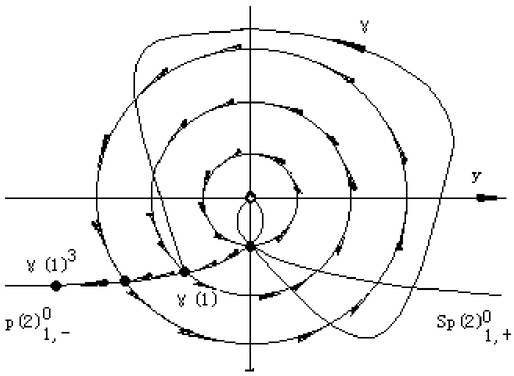

Figure 6.1: Computation of indices for iterations of the path γ in the Case 1

In this case we have that $i_1(\gamma)$ is odd if $a < 0$ and $i_1(\gamma)$ is even if $a > 0$. Similar to the case 1 we obtain

$$i_1(\gamma^m) = mi_1(\gamma), \quad \nu_1(\gamma^m) = 0, \quad \forall m \in \mathbf{N}. \tag{6.8}$$

Case 5. *$\gamma \in \mathcal{P}_\tau(2n)$ and $\sigma(\gamma(\tau)) \subset \mathbf{U} \setminus \mathbf{R}$, i.e., $R(\theta) \in \Omega^0(\gamma(\tau))$ for some $\theta \in [0, 2\pi]$.*

In this case, we must have $i_1(\gamma)$ being odd and $\nu_1(\gamma) = 0$. Similarly to the case 1 we obtain

$$i_1(\gamma^m) = m(i_1(\gamma) - 1) + 2E(\frac{m\theta}{2\pi}) - 1, \quad \nu_1(\gamma^m) = 2 - 2\varphi(\frac{m\theta}{2\pi}), \quad \forall m \in \mathbf{N}. \tag{6.9}$$

7 Iteration inequalities

In many of our applications, we need sharp increasing estimates on the iterated Maslov-type index $i_1(\gamma^m)$ for $\gamma \in \mathcal{P}_\tau(2n)$. These results are proved in [LiL1] based on results in [DoL1] and in [LiL2] based on results in [Lon5].

Theorem 7.1.(cf. [LiL1], [LiL2], [Lon8]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, there holds*

$$m\hat{i}(\gamma) - n \le i_1(\gamma^m) \le m\hat{i}(\gamma) + n - \nu_1(\gamma^m). \tag{7.1}$$

Theorem 7.2.(cf. [LiL1], [LiL2], [Lon8]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $m \in \mathbf{N}$, there holds*

$$m(i_1(\gamma) \quad + \quad \nu_1(\gamma) - n) + n - \nu_1(\gamma) \leq i_1(\gamma^m) \tag{7.2}$$

$$\leq m(i_1(\gamma) + n) - n - (\nu_1(\gamma^m) - \nu_1(\gamma)). \tag{7.3}$$

Remark 7.3. Necessary and sufficient conditions so that equality holds on each side of (7.1) or (7.3) are found in [LiL2], cf. [Lon8] for details.

Our proof of these theorems is based on the following result proved in [LL3]. In particular, this proof uses the properties of the ω-index theory, splitting numbers on homotopy components of symplectic matrices, and mean indices are very crucial in the proofs.

Proposition 7.4. (cf. [LiL2], [Lon8]) *For any $\gamma \in \mathcal{P}_\tau(2n)$ and $\omega \in \mathbf{U} \setminus \{1\}$, there always holds*

$$i_1(\gamma) + \nu_1(\gamma) - n \leq i_\omega(\gamma) \leq i_1(\gamma) + n - \nu_\omega(\gamma). \tag{7.4}$$

Idea of the proof. The proof of (6.6) is based on the estimate of the difference between $i_1(\gamma) = i_1(\gamma)$ and $i_1\omega(\gamma)$, which is expressed by a sum of splitting numbers when the parameter runs from 1 to ω on $\mathbf{U}$:

$$i_\omega(\gamma) = i_1(\gamma) + S^+_{\gamma(\tau)}(1) - \sum_{j=1}^{k}[S^-_{\gamma(\tau)}(\omega_j) - S^+_{\gamma(\tau)}(\omega_j)] - S^-_{\gamma(\tau)}(\omega).$$

Then apply properties of the splitting numbers to get (7.4). For proofs of other parts of the theorem, we refer to [LiL2], [Lon8] for details. ∎

Now based on the Proposition 7.4, we can give the proofs of Theorems 7.1 and 7.2 below.

Proof of Theorem 7.1. By Theorem 5.2, integrating (7.4) on $\mathbf{U}$ we obtain

$$i_1(\gamma) + \nu_1(\gamma) - n \leq \hat{i}(\gamma) \leq i_1(\gamma) + n. \tag{7.5}$$

Replacing τ by $m\tau$ in (7.5), by (5.9) we obtain (7.1). ∎

Proof of Theorem 7.2. By Theorem 5.1, summing (7.4) up over all m-th roots of unit, we obtain

$$\begin{aligned}(m-1)(i_1(\gamma) \quad &+ \quad \nu_1(\gamma) - n) + i_1(\gamma) \leq i_1(\gamma^m) \\ &\leq (m-1)(i_1(\gamma) + n) + i_1(\gamma) - (\nu_1(\gamma^m) - \nu_1(\gamma)).\end{aligned}$$

This yields (7.3). ∎

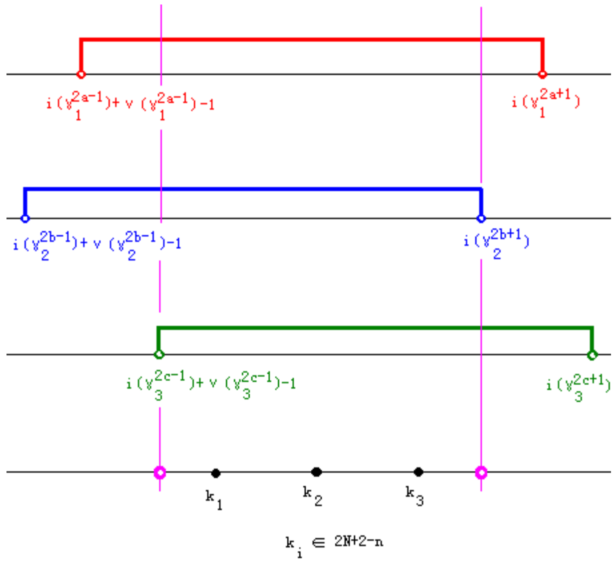

Figure 8.1: Common index jump of 3 paths

8 The common index jump theorem

An abstract precise index iteration theorem was obtained in [LZh1] of Y. Long and C. Zhu in 2002:

Theorem 8.1. ([LZh1]) *For $\gamma \in \mathcal{P}_\tau(2n)$ and any $m \geq 1$, there holds:*

$$\begin{aligned} i_1(\gamma^m) &= m(i_1(\gamma) + S_M^+(1) - C(M)) \\ &\quad + 2\sum_{\theta\in(0,2\pi)} E(\frac{m\theta}{2\pi})S_M^-(e^{\sqrt{-1}\theta}) - (S_M^+(1) + C(M)), \end{aligned}$$

where $M = \gamma(\tau)$, and $C(M) = \sum_{0<\theta<2\pi} S_M^-(e^{\sqrt{-1}\theta})$.

Besides other results in the index iteration theory described in [Lon5], the following common index jump theorem for finitely many symplectic paths was proved in [LZh1] by C. Zhu and the author. For any $\gamma \in \mathcal{P}_\tau(2n)$, its m-th index jump $\mathcal{G}_m(\gamma)$ is defined to be the open interval

$$\mathcal{G}_m(\gamma) = (i_1(\gamma^m) + \nu_1(\gamma^m) - 1, i_1(\gamma^{m+2})).$$

Note that under the assumption (8.1) below, the interval $\mathcal{G}_m(\gamma)$ is meaningful and non-empty.

Theorem 8.2. (cf. [LZh1], [Lon8]) *Let $\gamma_j \in \mathcal{P}_{\tau_j}(2n)$ with $1 \leq j \leq q$ satisfy*

$$\hat{i}(\gamma_j) > 0, \quad i_1(\gamma_j) \geq n, \qquad 1 \leq j \leq q. \tag{8.1}$$

Then there exist infinitely many positive integer tuples $(N, m_1, \ldots, m_q) \in \mathbf{N}^{q+1}$ *such that*

$$\emptyset \neq [2N - \kappa_1, 2N + \kappa_2] \subset \bigcap_{j=1}^{q} \mathcal{G}_{2m_j-1}(\gamma_j), \tag{8.2}$$

where $\kappa_1 = \min_{1\le j\le q}(i_1(\gamma_j) + 2S^+_{\gamma_j(\tau_j)}(1) - \nu_1(\gamma_j))$ *and* $\kappa_2 = \min_{1\le j\le q} i_1(\gamma_j) - 1$.

Idea of the proof. In order to prove this theorem, we need to use Theorem 8.1 and to make each index jump to be as big as possible, and to make their largest jumps happen simultaneously to guarantee the existence of a non-empty largest common intersection interval among them. This problem is reduced to a dynamical system problem on a torus, and is solved by properties of closed additive subgroups of tori. Intuitively speaking, we tried to find a certain common multiple of the iteration times so that the indices of each path reaches a similar value. ∎

The index iteration theory reviewed in this chapter has been applied to many problems including the prescribed minimal period problem for periodic solutions of the systems (0.1), periodic points of Poincaré maps of Lagrangian systems on tori, closed characteristics on convex hypersurfaces in R^{2n}, closed geodesic problems, etc. We refer readers to the book [Lon8] as well as references therein for such applications. In the following chapters we give some examples on such applications.

Chapter 3. Closed Characteristics on Convex Hypersurfaces in $\mathbf{R}^{2n}$

9 Existence, multiplicity and stability problems of closed characteristics

Let $\mathcal{H}(2n)$ denote the set of all compact strictly convex C^2-hypersurfaces in $\mathbf{R}^{2n}$. For $\Sigma \in \mathcal{H}(2n)$ and $x \in \Sigma$, let $N_\Sigma(x)$ be the unit outward normal vector at x of Σ. We consider the problem of finding $\tau > 0$ and a curve $x \in C^1([0,\tau], \mathbf{R}^{2n})$ such that

$$\begin{cases} \dot{x}(t) &= JN_\Sigma(x(t)), \quad x(t) \in \Sigma, \quad \forall t \in \mathbf{R}, \\ x(\tau) &= x(0). \end{cases} \tag{9.1}$$

A solution (τ, x) of the problem (9.1) is called a *closed characteristic* on Σ. Two closed characteristics (τ, x) and (σ, y) are *geometrically distinct*, if $x(\mathbf{R}) \neq y(\mathbf{R})$. We denote by $\mathcal{T}(\Sigma)$ the set of all closed characteristics (τ, x) on Σ with τ being the minimal period of x, and by $\tilde{\mathcal{T}}(\Sigma)$ the set of all geometrically distinct closed characteristics (τ, x) on Σ with τ being the minimal period of x respectively.

Note that the problem (9.1) can be described in a Hamiltonian system version and solved by variational methods as mentioned in Section 1. A closed characteristic (τ, x) is *non-degenerate*, if 1 is a Floquet multiplier of x, i.e., an eigenvalue of $\gamma_x(\tau)$, of precisely algebraic multiplicity 2, and is *elliptic*, if all the Floquet multipliers of x are on $\mathbf{U}$.

The problem on closed characteristics has been studied for more than 100 years. Two long standing conjectures on the multiplicity and the stability of closed characteristics are the following:

Multiplicity conjecture. ${}^\#\tilde{\mathcal{T}}(\Sigma) \geq n$ *holds for every* $\Sigma \in \mathcal{H}(2n)$.

Stability conjecture. *There exists at least one elliptic closed characteristic on every* $\Sigma \in \mathcal{H}(2n)$.

9.1 On the multiplicity conjecture

The most famous known results on the local multiplicity conjecture can be traced back to A. Liapunov's [Lia1] in 1892, which was improved by [Wei1] of A. Weinstein in 1973 to the following result

$${}^\#\tilde{\mathcal{T}}(H^{-1}(\epsilon)) \geq n, \quad \text{if } H \text{ is } C^2 \text{ near } 0,\ H''(0) > 0 \text{ and } \epsilon > 0 \text{ is sufficiently small.}$$

Then this local theorem was further generalized by J. Moser in [Mos1] of 1976 and T. Bartsch in [Bar1] of 1997.

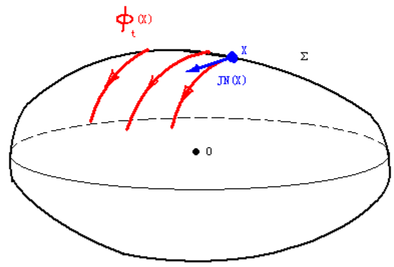

Figure 9.1: x on Σ and $JN(x)$

The first break through on the multiplicity conjecture in the global sense was made by P. Rabinowitz (for star-shaped hypersufaces) and A. Weinstein in 1978. They proved

Theorem 9.1. ([Rab1], [Wei1], 1978) ${}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq 1, \quad \forall \Sigma \in \mathcal{H}(2n)$.

Besides many results under pinching conditions (cf. for example, [ELy1], [AmM1], [Hof1], [BLRM], [DyL1]), in 1987-1988, I. Ekeland-L. Lassoued, I. Ekeland-H. Hofer, and A, Szulkin proved

Theorem 9.2. ([ELa1], [EkH1], [Szu1], 1987) ${}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq 2, \quad \forall \Sigma \in \mathcal{H}(2n)$ *with* $n \geq 2$.

In 1998, H. Hofer, K. Wysocki, and E. Zehnder proved the following remarkable result:

Theorem 9.3. ([HWZ1], 1998) ${}^{\#}\tilde{\mathcal{T}}(\Sigma) = 2$ *or* $+\infty, \ \forall \Sigma \in \mathcal{H}(4)$.

The proof of Theorem 9.3 depending on their construction of an open book structure to parametrize the hypersurface S^3 and a theorem of J. Franks in [Fra1] and [Fra2] on periodic points of area preserving homeomorphisms on annulus. Because the theorem of J. Franks is for 2-dimensional case, so far it is not clear whether the method of [HWZ1] can be generalized to higher dimensional cases.

On the other hand, using the index iteration theory for symplectic paths, the author and C. Zhu gave the following answer to the multiplicity conjecture in 2002:

Theorem 9.4. ([LZh1], 2002) *There holds*

$$ {}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq [\frac{n}{2}] + 1 \qquad \forall \Sigma \in \mathcal{H}(2n). $$

Moreover, if all the closed characteristics on Σ *are non-degenerate, then* ${}^{\#}\mathcal{T}(\Sigma) \geq n$.

Then C. Liu, C. Zhu and the author proved the following result in 2002:

Theorem 9.5. ([LLZ1], 2002) *For any* $\Sigma \in \mathcal{H}(2n)$*, if* Σ *is symmetric with respect to the origin, i.e.,* $x \in \Sigma$ *implies* $-x \in \Sigma$*, then* ${}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq n$.

Recently, W. Wang, X. Hu and the author further proved the following result:

Theorem 9.6. ([WHL1], 2007) *There holds*

$$^{\#}\tilde{\mathcal{T}}(\Sigma) \geq 3 \qquad \forall \Sigma \in \mathcal{H}(6).$$

9.2 On the stability conjecture

Concerning the stability conjecture, because the existence of closed characteristics on $\Sigma \in \mathcal{H}(2n)$ is usually proved by variational methods, very little is known on their stability. Up to the author's knowledge, the existence of one elliptic closed characteristic on $\Sigma \in \mathcal{H}(2n)$ was proved by I. Ekeland in 1990 when Σ is $\sqrt{2}$-pinched by two spheres. The following beautiful stability theorem was proved by G.-F. Dell'Antonio, B. D'Onofrio, and I. Ekeland in 1992.

Theorem 9.7. ([DDE1]) *For any $\Sigma \in \mathcal{H}(2n)$, if Σ is symmetric with respect to the origin, then there exists at least one elliptic closed characteristic on Σ.*

Idea of the Proof. Try to find a global minimal point x of the dual action functional on the 1/2-antisymmetric function space. Then this solution must satisfy $i_{-1}(x) = 0$. On the other hand, we always have $i_1(x) \geq n$ because Σ is convex. Therefore the total multiplicity of ω on the semi unit circle from $\omega = 1$ to $\omega = -1$ on which a change of $i_\omega(x)$ happens must be n. This proves that all the Floquet multipliers of x must locate on $\mathbf{U}$, i.e. x is elliptic. ∎

Note that based on the conclusion of the above Theorem 9.3 of [HWZ1], using the precise index iteration formulae established in [Lon6], the author proved

Theorem 9.8. ([Lon6]) *For $\Sigma \in \mathcal{H}(4)$ with $^{\#}\tilde{\mathcal{T}}(\Sigma) = 2$, both closed characteristics must be elliptic.*

Idea of the Proof. The idea of the proof of Theorem 4.7 is that all the integers in the set $2\mathbf{N} - 2 + n$ should be covered by the union $\mathcal{L}(\Sigma)$ of all the index intervals $[i(x^m), i(x^m) + \nu(x^m) - 1]$ of all iterations of all closed characteristics. Then using the above mentioned precise index iteration formulae, we can classify the two closed characteristics into 9 different classes, and then get $\mathcal{L}(\Sigma)$ precisely for each case. We prove first that $2\mathbf{N} - 2 + n \not\subset \mathcal{L}(\Sigma)$ if both of them are not elliptic, and then if at least one of the two closed characteristics is not elliptic. These contradictions prove the theorem. ∎

Using an index iteration estimate on the elliptic height and the index jump theorem, the following result was further proved by C. Zhu and the author.

Theorem 9.9. ([LZh1]) *For $\Sigma \in \mathcal{H}(2n)$ with $n \geq 2$, suppose $^{\#}\tilde{\mathcal{T}}(\Sigma) < +\infty$. Then (i) there exists at least one elliptic closed characteristic on Σ.*

(ii) at least one closed characteristic on Σ satisfying $\hat{i}(x) \in \mathbf{R} \setminus \mathbf{Q}$.

(iii) If $n \geq 2$ and $^{\#}\tilde{\mathcal{T}}(\Sigma) \leq 2[n/2]$. Then there exist at least two elliptic elements in $\tilde{\mathcal{T}}(\Sigma)$.

Other results on the multiplicity and stability problems can be found in the reference below.

In the next two sections, we shall give sketches on the proofs of Theorems 9.4 and 9.6 to show how the index iteration theory play a crucial role in the studies.

10 The multiplicity theorem of Long and Zhu

In [LZh1], C. Zhu and the author proved the following result:

Theorem 9.4. ([LZh1]) *There holds*

$$^{\#}\tilde{\mathcal{T}}(\Sigma) \geq [\frac{n}{2}] + 1 \qquad \forall \Sigma \in \mathcal{H}(2n).$$

Moreover, if all the closed characteristics on Σ *are non-degenerate, then* $^{\#}\tilde{\mathcal{T}}(\Sigma) \geq n$.

The main ingredient in the proof of Theorems 9.4 is our index iteration theory mentioned above. To illustrate this method, we briefly describe below the main ideas in this proof.

Ideas of the Proof of Theorem 9.4. The proof is carried out in 6 steps.

Step 1. *The Hamiltonian formulation.*

To cast the problem (9.1) of closed characteristics on compact convex hypersurfaces in $\mathbf{R}^{2n}$ into a Hamiltonian version, following I. Ekeland's book [Eke3], fix a $\Sigma \in \mathcal{H}(2n)$ bounding a convex set C. Then the origin is in the interior of C. Let $j_\Sigma : \mathbf{R}^{2n} \to [0, +\infty)$ be the gauge function of Σ defined by

$$\left\{ \begin{array}{rcl} j_\Sigma(x) & = & \inf\{\lambda > 0 \,|\, \frac{x}{\lambda} \in C\}, \quad \text{for} \quad x \neq 0, \\ j_\Sigma(0) & = & 0. \end{array} \right. \tag{10.1}$$

Fix a constant α satisfying $1 < \alpha < 2$ in this chapter. As usual we define the Hamiltonian function $H_\alpha : \mathbf{R}^{2n} \to [0, +\infty)$ by

$$H_\alpha(x) = j_\Sigma(x)^\alpha, \qquad \forall x \in \mathbf{R}^{2n}. \tag{10.2}$$

Then $H_\alpha \in C^1(\mathbf{R}^{2n}, \mathbf{R}) \cap C^2(\mathbf{R}^{2n} \setminus \{0\}, \mathbf{R})$ is convex and $\Sigma = H_\alpha^{-1}(1)$. The problem (9.1) is equivalent to the following given energy problem of the Hamiltonian system

$$\dot{x}(t) = JH'_\alpha(x(t)), \quad \forall t \in \mathbf{R}, \tag{10.3}$$

$$x(\tau) = x(0), \tag{10.4}$$

$$H_\alpha(x(t)) = 1, \quad \forall t \in \mathbf{R}. \tag{10.5}$$

Denote by $\mathcal{T}(\Sigma, \alpha)$ the set of all solutions (τ, x) of the problem (10.3)-(10.5) where τ is the minimal period of x. Note that elements in $\mathcal{T}(\Sigma)$ and $\mathcal{T}(\Sigma, \alpha)$ are one to one correspondent to each other.

Step 2. The given period problem and the dual action principle.

Consider the following given period problem:

$$\begin{aligned} \dot{z}(t) &= JH_\alpha'(z(t)), \quad \forall t\in \mathbf{R}, && (10.6)\\ z(1) &= z(0). && (10.7)\end{aligned}$$

Let

$$E_\alpha = \{u \in L^{\alpha/(\alpha-1)}(\mathbf{R}/\mathbf{Z}, \mathbf{R}^{2n}) \mid \int_0^1 u dt = 0\},$$

with the usual $L^{(\alpha-1)/\alpha}$ norm. The Clarke-Ekeland dual action functional $f_\alpha : E_\alpha \to \mathbf{R}$ is defined by

$$f_\alpha(u) = \int_0^1 \{\frac{1}{2}Ju\cdot \Pi u + H_\alpha^*(-Ju)\}dt,$$

where Πu is defined by $\frac{d}{dt}\Pi u = u$ and $\int_0^1 \Pi u dt = 0$. The Fenchel conjugate $H_\alpha^* : \mathbf{R}^{2n} \to \mathbf{R}$ is defined by

$$H_\alpha^*(x) = \sup_{y\in\mathbf{R}^{2n}} \{x\cdot y - H_\alpha(y)\}.$$

Then $f_\alpha \in C^2(E_\alpha, \mathbf{R})$. Suppose $u \in E_\alpha \setminus \{0\}$ is a critical point of f_α. By [Eke3], there exists $\xi_u \in \mathbf{R}^{2n}$ such that $z_u(t) = \Pi u(t) + \xi_u$ is a 1-periodic solution of the problem (10.6).

Let $h = H_\alpha(z_u(t))$ and $1/m$ be the minimal period of z_u for some $m \in \mathbf{N}$. Define

$$x_u(t) = h^{-1/\alpha} z_u(h^{(2-\alpha)/\alpha}t) \quad \text{and} \quad \tau = \frac{1}{m}h^{(\alpha-2)/\alpha}. \tag{10.8}$$

Then there hold $x_u(t) \in \Sigma$ for all $t \in \mathbf{R}$ and $(\tau, x_u) \in \mathcal{T}(\Sigma, \alpha)$. Note that the period 1 of z_u corresponds to the period $m\tau$ of the solution $(m\tau, x_u^m)$ of (10.3)-(10.5) with minimal period τ.

Note that by iteration, every solution $(\tau, x) \in \mathcal{T}(\Sigma, \alpha)$ gives rise to a sequence $\{z_m^x\}_{m\in\mathbf{N}}$ of solutions of the problem (10.3)-(10.5), and a sequence $\{u_m^x\}_{m\in\mathbf{N}}$ of critical points of f_α defined by

$$\begin{aligned} z_m^x(t) &= (m\tau)^{-1/(2-\alpha)}x(m\tau t), && (10.9)\\ u_m^x(t) &= (m\tau)^{(\alpha-1)/(2-\alpha)}\dot{x}(m\tau t). && (10.10)\end{aligned}$$

For every $m \in \mathbf{N}$ there holds

$$f_\alpha(u_m^x) = -(1-\frac{\alpha}{2})(\frac{2m}{\alpha}A(\tau, x))^{-\alpha/(2-\alpha)}, \tag{10.11}$$

where

$$A(\tau, x) = \frac{1}{2}\int_0^\tau (-J\dot{x}\cdot x)dt.$$

Step 3. Ekeland index, Maslov-type index, and estimates.

In 1980s, I. Ekeland proved that the Hessian of f_α at its critical point u possesses finite Morse index and nullity, which are the so called Ekeland index and nullity, and denoted by $i^E(u)$ and $\nu^E(u)$ respectively.

The following propositions relate Ekeland index to the Maslov-type index given in Chapter 1.

Proposition 10.1. ([Bro1] and [Lon5]) *For u and x_u defined above, there hold*

$$i_1(x_u) = i^E(u) + n, \qquad \text{and} \quad \nu_1(x_u) = \nu^E(u). \tag{10.12}$$

Proposition 10.2. ([Lon5], [Lon8]) *For x_u and z_u defined above, there hold*

$$i_1(x_u^m) = i_1(z_u), \qquad \text{and} \quad \nu_1(x_u^m) = \nu_1(z_u). \tag{10.13}$$

We need the following estimates on the iterated indices:

Proposition 10.3. ([LZh1], [Lon8]) *Fix $\Sigma \in \mathcal{H}(2n)$ and $\alpha \in (1,2)$. For any $(\tau, x) \in \mathcal{T}(\Sigma, \alpha)$ and $m \in \mathbf{N}$, there hold*

$$\begin{aligned}
\hat{i}(x) &> 2, && (10.14)\\
i_1(x) &\geq n, && (10.15)\\
\nu_1(x^m) &\geq \nu_1(x) \geq 1, && (10.16)\\
i_1(x^{m+1}) - i_1(x^m) - \nu_1(x^m) &\geq i_1(x) - \frac{e(\gamma_x(\tau))}{2} + 1 &&\\
&\geq i_1(x) - n + 1, &&\\
&\geq 1, && (10.17)\\
i(x, m+1) + \nu(x, m+1) - 1 &\geq i(x, m+1) &&\\
&> i(x, m) + \nu(x, m) - 1. && (10.18)
\end{aligned}$$

Step 4. Faddel-Rabinowitz index method (Liusternik-Schinirelmann method).

I. Ekeland proved the following result by using the Faddel-Rabinowitz cohomological index method:

Theorem 10.4. (cf. [Eke3]) *For*

$$[f_\alpha]_c \equiv \{u \in E_\alpha \,|\, f_\alpha(u) \leq c\},$$

define

$$c_k = \inf\{c < 0 \,|\, \mathrm{ind}([f_\alpha]_c) \geq k\}, \qquad \forall k \in \mathbf{N},$$

where ind *denotes the Fadell-Rabinowitz cohomological index. Then $\{c_k\}$ are critical values of f.*

They satisfy

$$-\infty < c_1 = \inf_{u\in E_\alpha} f_\alpha(u) \le c_2 \le \cdots \le c_k \le c_{k+1} \le \cdots < 0, \tag{10.19}$$

$$c_k \to +\infty, \text{as} \quad k \to +\infty, \tag{10.20}$$

$$^{\#}\tilde{\mathcal{J}}(\Sigma) = +\infty, \quad \text{if} \;\; c_k = c_{k+1} \;\; \text{for some} \;\; k \in \mathbf{N}. \tag{10.21}$$

For any given $k \in \mathbf{N}$, there exists $(\tau, x) \in \mathcal{J}(\Sigma, \alpha)$ and $m \in \mathbf{N}$ such that for u_m^x defined by (10.10), there hold

$$f'_\alpha(u_m^x) \;\; = \;\; 0, \quad f_\alpha(u_m^x) = c_k, \tag{10.22}$$

$$i_1(x^m) \;\; \le \;\; 2k - 2 + n \le i_1(x^m) + \nu_1(x^m) - 1. \tag{10.23}$$

Step 5. *Application of the common index jump theorem.*

Next we recall the common index jump theorem of Y. Long and C. Zhu. For any $\gamma \in \mathcal{P}_\tau(2n)$, its m-th index jump $\mathcal{G}_m(\gamma)$ is defined to be the open interval

$$\mathcal{G}_m(\gamma) = (i_1(\gamma^m) + \nu_1(\gamma^m) - 1, i_1(\gamma^{m+2})).$$

Note that under the assumption (10.24) below, the interval $\mathcal{G}_m(\gamma)$ is meaningful and non-empty.

Theorem 8.2. (cf. [LZh1], [Lon9]) *Let $\gamma_j \in \mathcal{P}_{\tau_j}(2n)$ with $1 \le j \le q$ satisfy*

$$\hat{i}(\gamma_j) > 0, \quad i_1(\gamma_j) \ge n, \qquad 1 \le j \le q. \tag{10.24}$$

Then there exist infinitely many positive integer tuples $(N, m_1, \ldots, m_q) \in \mathbf{N}^{q+1}$ such that

$$\emptyset \ne [2N - \kappa_1, 2N + \kappa_2] \subset \bigcap_{j=1}^{q} \mathcal{G}_{2m_j-1}(\gamma_j), \tag{10.25}$$

where $\kappa_1 = \min_{1\le j\le q}(i_1(\gamma_j) + 2S^+_{\gamma_j(\tau_j)}(1) - \nu_1(\gamma_j))$ and $\kappa_2 = \min_{1\le j\le q} i_1(\gamma_j) - 1$.

Let

$$q \equiv \, ^{\#}\tilde{\mathcal{T}}(\Sigma) < +\infty.$$

Then we obtain

$$q \ge \, ^{\#}\left((2\mathbf{N} - 2 + n) \cap \cap_{j=1}^{q} \mathcal{G}_{2m_j-1}(\gamma_{x_j})\right), \tag{10.26}$$

where a new version of the Liusternik-Schnirelmann theoretical argument at the iterated index level, which distinguishes solution orbits geometrically instead of critical points only as usual methods do. Here the estimate (10.17) and Liusternik-Schnirelmann type Theorem 10.4 are crucial, which guarantees that if one integer in $(2\mathbf{N} - 2 + n) \cap \cap_{j=1}^{q} \mathcal{G}_{2m_j-1}(\gamma_{x_j})$ corresponds to two orbits, then at least two corresponding critical values must equal and then yields infinitely many closed characteristics.

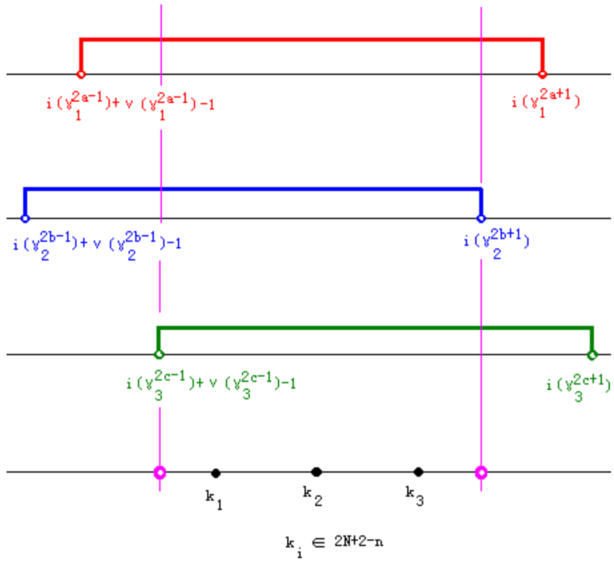

Figure 10.1: Common index jump of 3 paths

Next by the common index jump theorem 8.2 we have

$$\begin{aligned}
&\#\left((2\mathbf{N}-2+n)\cap\cap_{j=1}^{q}\mathcal{G}_{2m_j-1}(\gamma_{x_j})\right)\\
&\qquad\geq\ ^{\#}\left((2\mathbf{N}-2+n)\cap[2N-\kappa_1,2N+\kappa_2]\right) &(10.27)\\
&\qquad\geq\ \min\left\{\left[\frac{i_1(x)+2S^+_{\gamma_x(\tau)}(1)-\nu_1(x)+n}{2}\right]\ \Big|\ [(\tau,x)]\in\tilde{\mathcal{T}}(\Sigma,\alpha)\right\}\equiv\varrho_n(\Sigma), &(10.28)
\end{aligned}$$

where to get the last inequality we have used the definitions of κ_1 and κ_2.

We write $M\approx N$ if $M\in\Omega^0(N)$.

Now because the system is autonomous, we have the basic normal form decomposition of $\gamma_x(\tau)$ for each $(\tau,x)\in\mathcal{T}(\Sigma,\alpha)$:

$$\gamma_x(\tau)\approx N_1(1,1)\diamond M, \tag{10.29}$$

for some $M\in\mathrm{Sp}(2n-2)$. We have

$$2S^+(x)-\nu_1(x)=2S^+_{N_1(1,1)}(1)-\nu_1(N_1(1,1))+2S^+_M(1)-\nu_1(M)). \tag{10.30}$$

By Chapter 2, we obtain

$$S^+_{N_1(1,a)}(1)=\left\{\begin{array}{ll}1, & \text{if }\ a\geq 0,\\ 0, & \text{if }\ a<0.\end{array}\right. \tag{10.31}$$

Thus there holds

$$2S^+_{N_1(1,a)}(1) - \nu_1(N_1(1,a)) = a, \quad \text{for } a = \pm 1, 0. \tag{10.32}$$

By Chapter 2 again, we obtain

$$N_1(1,1)^{\diamond p_-} \diamond I_2^{\diamond p_0} \diamond N_1(1,-1)^{\diamond p_+} \diamond G \in \Omega^0(M), \tag{10.33}$$

for some nonnegative integers p_-, p_0, and p_+, and some symplectic matrix G satisfying $1 \notin \sigma(G)$. By (10.32) and (10.33), we then obtain

$$2S^+_M(1) - \nu_1(M) \geq p_- - p_+ \geq -p_+ \geq 1 - n. \tag{10.34}$$

Therefore we obtain

$$\begin{aligned} i_1(x) + 2S^+(x) - \nu_1(x) + n &\geq n + 1 + (1-n) + n \\ &= n + 2. \end{aligned} \tag{10.35}$$

Then it yields

$$\begin{aligned} \varrho_n(\Sigma) &= \min\left\{ \left[\frac{i_1(x) + 2S^+_{\gamma_x(\tau)}(1) - \nu_1(x) + n}{2} \right] \;\Big|\; (\tau, x) \in \tilde{\mathcal{T}}(\Sigma, \alpha) \right\} \\ &\geq [\frac{n+2}{2}] \\ &= [\frac{n}{2}] + 1. \end{aligned} \tag{10.36}$$

Therefore we obtain

$$q \equiv {}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq [\frac{n}{2}] + 1. \tag{10.37}$$

Step 6. *The non-degenerate case.*

When all closed characteristics are non-degenerate, i.e.,

$$\nu_1(x) = 0, \qquad \forall\, (\tau, x) \in \mathcal{T}(\Sigma, \alpha), \tag{10.38}$$

for the matrix M in the basic normal form decomposition (10.29) of $\gamma_x(\tau)$ we obtain $1 \notin \sigma(M)$, and then in stead of (10.34) we have

$$2S^+_M(1) - \nu_1(M) = 0. \tag{10.39}$$

Therefore we obtain

$$\begin{aligned} i_1(x) + 2S^+(x) - \nu_1(x) + n &\geq n + 1 + 0 + n \\ &= 2n + 1. \end{aligned} \tag{10.40}$$

Then it yields

$$\begin{aligned}\varrho_n(\Sigma) &= \min\left\{\left[\frac{i_1(x)+2S^+_{\gamma_x(\tau)}(1)-\nu_1(x)+n}{2}\right] \;\Big|\; (\tau,x)\in\tilde{\mathcal{T}}(\Sigma,\alpha)\right\} \\ &\geq [\frac{2n+1}{2}] \\ &= n. \end{aligned} \tag{10.41}$$

Therefore if all the closed characteristics are non-degenerate, we obtain

$$q \equiv {}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq n. \tag{10.42}$$

Here the common index jump Theorem 8.2 was discovered in fact when we tried to understand precisely the behavior of the iterated index sequences of two closed characteristics on any $\Sigma \in \mathcal{H}(4)$ and three closed characteristics on any $\Sigma \in \mathcal{H}(6)$ using the precise index iteration formulae of [Lon6]. Then it was proved by using the abstract precise index iteration Theorem 6.1.

11 The multiplicity theorem of Wang, Hu and Long

In [WHL1], W. Wang, X. Hu and the author proved the following result:

Theorem 9.6. ([WHL1], 2007) *There holds*

$${}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq 3 \qquad \forall \Sigma \in \mathcal{H}(6).$$

The main ingredients in the proof of Theorems 9.6 include a new resonance identity and our index iteration theory mentioned above. To illustrate this method, we briefly describe below the main ideas in this proof based on the description on the proof of Theorem 9.4.

Ideas of the Proof of Theorem 9.6. The proof is carried out in 6 steps.

Step 1. *A new resonance identity on closed characteristics.*

In [Eke1] of 1984, I. Ekeland claimed that there exists a resonance condition on closed characteristics whenever the total number of them is finite on the given Σ. In [Vit1] of 1989, C. Viterbo established such an identity for star-shaped $\Sigma \subset \mathbf{R}^{2n}$ with finitely many closed characteristics $\tilde{\mathcal{T}}(\Sigma) = \{(\tau_1, x_1), \ldots, (\tau_q, x_q)\}$:

$$\frac{(-1)^{i(x_1)}}{\hat{i}(x_1)} + \cdots + \frac{(-1)^{i(x_q)}}{\hat{i}(x_q)} = \frac{1}{2},$$

provided all $(m\tau_j, x_j^m) \in \mathcal{T}(\Sigma)$ are non-degenerate for all $m \in \mathbf{N}$ and $1 \leq j \leq q$. Note that in [Rad1] of 1989 and [Rad2] of 1992, H.-B. Rademacher established a mean index identity for closed geodesics on compact Finsler manifolds.

Motivated by these results, in [WHL1] we have established a new resonance identity for convex hypersurfaces with finitely many closed characteristics. To describe this result, we use the concepts and notations introduced in the above proof of Theorem 9.4 in Section 10 too. Specially we use the Steps 1 and 2 there and make corresponding modifications.

Theorem 11.1. ([WHL1]) *Suppose $\Sigma \in \mathcal{H}(2n)$ satisfies ${}^{\#}\tilde{\mathcal{T}}(\Sigma) < +\infty$. Denote all the geometrically distinct closed characteristics by $\{(\tau_j,\ y_j)\}_{1\le j\le k}$ for $k = {}^{\#}\tilde{\mathcal{T}}(\Sigma)$. Then the following identity holds*

$$\sum_{1\le j\le k}\frac{\hat{\chi}(y_j)}{\hat{i}(y_j)} = \frac{1}{2}, \tag{11.1}$$

where for each closed characteristic (τ, y), its the mean index is defined by

$$\hat{i}(y) \equiv \lim_{m\to+\infty} i_1(y^m)/m \in \mathbf{R},$$

$\hat{\chi}(y) \in \mathbf{Q}$ is the average Euler characteristic defined by

$$\hat{\chi}(y) = \frac{1}{K(y)}\sum_{\substack{1\le m\le K(y)\\ 0\le l\le 2n-2}}(-1)^{i(y^m)+l}k_l(y^m). \tag{11.2}$$

Here $K(y) \in \mathbf{N}$ is the minimal period of critical modules of iterations of y, $i(y^m)$ is the Morse index of a corresponding dual-action functional Ψ_a for some sufficiently large $a > 0$ at the m-th iteration y^m of y, $k_l(y^m)$ is the critical type numbers of y^m given by

$$k_l(y^m) = \dim\left(H_l(W(y^m)\cap\Lambda_a(y^m), (W(y^m)\setminus\{y^m\})\cap\Lambda_a(y^m))\right)^{\beta(y^m)\mathbf{Z}_m},$$

where $\beta(y^m) = (-1)^{i(y^m)-i(y)}$, $W(y^m)$ is the characteristic manifold of Gromoll-Meyer given by Ψ_a, and $\Lambda_a(y^m)$ is the subset of the free loop space not greater than the functional value $\Psi_a(y^m)$.

Idea of the proof. We apply the Morse theory for closed characteristics on Σ as follows:

$\langle 1\rangle$ Construction of the Hamiltonian function:

Let

$$\begin{aligned} j(\lambda x) &= \lambda, \qquad \forall x\in\Sigma,\ \ \lambda\ge 0.\\ \varphi_a(t) &= \begin{cases}\text{quadratic}, & \text{for } 0\le t\le t_0 << 1,\\ ct^{\alpha}, & \text{for } t_0 < t < T_0,\ \text{ where }\ \alpha\in(1,2),\\ \text{quadratic}, & \text{for } 1 << T_0 < t.\end{cases}\end{aligned}$$

Require $\varphi_a(t)$ to be convex for $t \ge 0$. Define

$$H_a(x) = a\varphi_a(j(x)), \qquad \forall\, x\in\mathbf{R}^{2n}.$$

$\langle 2\rangle$ Dual action principle.

Now we apply the dual action principle. Let

$$L_0^2(S^1,\mathbf{R}^{2n}) = \left\{ u \in L^2([0,1],\mathbf{R}^{2n}) \,\Big|\, \int_0^1 u(t)dt = 0 \right\}. \tag{11.3}$$

Define the anti-derivative linear operator $M : L_0^2(S^1,\mathbf{R}^{2n}) \to L_0^2(S^1,\mathbf{R}^{2n})$ by

$$\frac{d}{dt}Mu(t) = u(t), \qquad \int_0^1 Mu(t)dt = 0.$$

The dual action functional on $L_0^2(S^1,\mathbf{R}^{2n})$ is defined by

$$\Psi_a(u) = \int_0^1 \left(\frac{1}{2} Ju \cdot Mu + H_a^*(-Ju) \right) dt. \tag{11.4}$$

⟨3⟩ Computation on S^1-invariant relative homological groups at every critical orbits.

⟨4⟩ Vanishing of S^1-invariant relative homological group near 0.

⟨5⟩ Morse inequality argument.

Let X be an S^1-space such that the Betti numbers $b_i(X) = \dim H_{S^1,\,i}(X;\ \mathbf{Q})$ are finite for all $i \in \mathbf{Z}$. As usual the S^1-equivariant *Poincaré series* of X is defined by the formal power series $P(X)(t) = \sum_{i=0}^{\infty} b_i(X)t^i$. Note that the functional Ψ_a is bounded from below on $L_0^2(S^1,\ \mathbf{R}^{2n})$. Hence the S^1-equivariant *Morse series* $M(t)$ of the functional Ψ_a on the space $\Lambda_a^{-\varepsilon}$ is defined as usual by

$$M(t) = \sum_{q\geq 0,\ 1\leq j\leq p} \dim C_{S^1,\,q}(\Psi_a,\ S^1\cdot v_j)t^q,$$

where we denote by $\{S^1\cdot v_1,\ldots,S^1\cdot v_p\}$ the critical orbits of Ψ_a with critical values less than $-\varepsilon$. Then the Morse inequality in the equivariant sense yields a formal power series $Q(t) = \sum_{i=0}^{\infty} q_i t^i$ with nonnegative integer coefficients q_i such that

$$M(t) = P(t) + (1+t)Q(t), \tag{11.5}$$

where $P(t) \equiv P(\Lambda_a^{-\varepsilon})(t)$. For a formal power series $R(t) = \sum_{i=0}^{\infty} r_i t^i$, we denote by $R^k(t) = \sum_{i=0}^{k} r_i t^i$ for $k \in \mathbf{N}$ the corresponding truncated polynomial. Using this notation, (11.5) becomes

$$(-1)^p q_p = M^p(-1) - P^p(-1) = M^p(-1) - ([p/2]+1), \quad \forall\, p \in \mathbf{N}. \tag{11.6}$$

Then one can prove

Claim 1. *The coefficients w_h of $M(t) = \sum_{h=0}^{\infty} w_h t^h$ are bounded by some constant C independent of a. Consequently $\{q_p\}$ is bounded too.*

Thus we obtain:

$$\lim_{p\to+\infty} \frac{M^p(-1)}{p} = \lim_{p\to+\infty} \frac{P^p(-1)}{p} = \frac{1}{2},$$

On the other hand, one can prove the following claim:

Claim 2. *There is a real constant $C'>0$ independent of a such that*

$$\left| M^p(-1) - \sum_{\substack{1\le j\le k,\ 0\le l\le 2n-2\\ 1\le m\le K_j}} (-1)^{i(u_j^m)+l} k_l(u_j^m)\frac{p}{K_j\hat{i}(y_j)}\right| \le C', \tag{11.7}$$

where the sum in the left hand side of (11.7) equals to $p\sum_{1\le j\le k}\frac{\hat{\chi}(y_j)}{\hat{i}(y_j)}$.

By Claims 1 and 2 we get the identity (11.1). ∎

Step 2. *Long-Zhu's estimate.*

Now for a compact convex smooth hypersurface $\Sigma\subset\mathbf{R}^6$. Assume ${}^{\#}\tilde{\mathcal{T}}(\Sigma)\le 2$.

By Theorem 9.4, we have ${}^{\#}\tilde{\mathcal{T}}(\Sigma)=2$. Write

$$\tilde{\mathcal{T}}(\Sigma)=\{(\tau_1,x_1),(\tau_2,x_2)\}.$$

Recall that $\Sigma\subset\mathbf{R}^{2n}$ convex compact hypersurface implies $\hat{i}(x_j)>2$ for $j=1,2$.

Then Theorem 11.1 yields

$$\frac{\hat{\chi}(x_1)}{\hat{i}(x_1)}+\frac{\hat{\chi}(x_2)}{\hat{i}(x_2)}=\frac{1}{2}, \tag{11.8}$$

where both $\hat{\chi}(x_1)$ and $\hat{\chi}(x_1)$ are rational.

By (ii) of the stability Theorem 9.9 of Long-Zhu, at least one of x_1 and x_2 possesses irrational mean index, say $\hat{i}(x_1)\in\mathbf{R}\setminus\mathbf{Q}$.

Now we consider two cases: $\quad\hat{\chi}(x_1)\neq 0\quad$ or $\quad\hat{\chi}(x_1)=0$.

Step 3. *Case* $\hat{\chi}(x_1)\neq 0$

In this case, the identity (11.8) yields $\quad\hat{i}(x_2)\ \in\ \mathbf{R}\setminus\mathbf{Q}$.

Then by the common index jump theorem, as in the Step 5 of the proof of Theorem 9.4, we obtain

$$\begin{aligned}
2 \;&=\; q\equiv {}^{\#}\tilde{\mathcal{T}}(\Sigma)\\
&\ge\; {}^{\#}\left((2\mathbf{N}-2+n)\bigcap\bigcap_{j=1}^{q}\left(i(\gamma_j^{2m_j-1})+\nu(\gamma_j^{2m_j-1})-1,\, i(\gamma_j^{2m_j+1})\right)\right)\\
&\qquad\qquad (\text{by L}-\text{S theory})\\
&\ge\; {}^{\#}\left((2\mathbf{N}-2+n)\cap[2N-\kappa_1,2N+\kappa_2]\right)\quad (\text{by CIJ theorem})\\
&\ge\; \varrho_n(\Sigma)\qquad\qquad (\text{by index estimates})\\
&\equiv\; \min\left\{\left[\frac{i(x)+2S^+_{\gamma_x(\tau)}(1)-\nu(x)+n}{2}\right]\;\middle|\;(\tau,x)\in\tilde{\mathcal{T}}(\Sigma)\right\}.
\end{aligned} \tag{11.9}$$

Here $q = 2$, $n = 3$.

Next we need to prove

$$i(x_j) + 2S^+_{\gamma_j(\tau_j)}(1) - \nu(x_j) + 3 \geq 6, \qquad \text{for } \ j = 1, 2. \tag{11.10}$$

where $\gamma_j \equiv \gamma_{x_j} : [0, \tau_j] \to \mathrm{Sp}(6)$ for $j = 1$ and 2.

Here we have $i(x_j) \geq 3$ for $j = 1, 2$ always holds.

By the proof in Step 5 for Theorem 9.4, the worst case happens if $\gamma_j(\tau_j)$ is connected within $\Omega^0(\gamma_j(\tau_j))$ to:

$$\gamma_j(\tau_j) \ \approx \ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

In this case, we then obtain

$$i(x_j) + 2S^+_{\gamma_j(\tau_j)}(1) - \nu(x_j) + 3 \ = \ 3 + (2-1) + (0-1) + (0-1) + 3 \ = \ 5.$$

Then (11.9) becomes

$$2 = q \geq \varrho_n(\Sigma) \geq \left[\frac{5}{2}\right] = 2,$$

and yields no contradiction !

But here in this case, both x_1 and x_2 have irrational mean indices $\hat{i}(x_j) \in \mathbf{R} \setminus \mathbf{Q}$ for $j = 1, 2$. Thus $\gamma_j(\tau_j)$ can be connected within $\Omega^0(\gamma_j(\tau_j))$ to:

$$\gamma_j(\tau_j) \ \approx \ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} \cos\theta_j & -\sin\theta_j \\ \sin\theta_j & \cos\theta_j \end{pmatrix} \diamond M_j \ \equiv \ N_1(1,1) \diamond R(\theta_j) \diamond M_j,$$

where $M_j \in \mathrm{Sp}(2)$, $\ \theta_j/\pi \in \mathbf{R} \setminus \mathbf{Q}$. Then we have

$$\begin{aligned} & 2S^+_{N_1(1,1)}(1) - \nu(N_1(1,1)) = 1, \qquad 2S^+_{R(\theta_j)}(1) - \nu(R(\theta_j)) = 0, \\ & 2S^+_{M_j}(1) - \nu_1(M_j) \geq -1. \end{aligned}$$

where the worst case is given by $M_j = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$.

Therefore for $j = 1, 2$ we obtain

$$i(x_j) + 2S^+_{\gamma_j(\tau_j)}(1) - \nu(x_j) + 3 \geq 3 + (2-1) + (0-0) + (0-1) + 3 = 6. \tag{11.11}$$

which yields

$$2 \geq {}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq \varrho_3(\Sigma) \geq 3,$$

contradiction !

Step 4. *Case* $\hat{\chi}(x_1) = 0$

In this case, by Theorem 11.1, we obtain

$$\frac{\hat{\chi}(x_2)}{\hat{i}(x_2)} = \frac{1}{2}. \tag{11.12}$$

For x_1, as in Step 3, we still have

$$i(x_1) + 2S^+_{\gamma_1(\tau_1)}(1) - \nu(x_1) + 3 \geq 3 + (2-1) + (0-0) + (0-1) + 3 = 6. \tag{11.13}$$

If for x_2 we have $i(x_2)+2S^+_{\gamma_2(\tau_2)}(1)-\nu(x_2)+3 \geq 6$, then by (11.13) and (11.11) for x_1 we obtain

$$2 \geq {}^{\#}\tilde{\mathcal{T}}(\Sigma) \geq \varrho_3(\Sigma) \geq 3, \tag{11.14}$$

contradiction !

Thus for x_2, we have $i(x_2) + 2S^+_{\gamma_2(\tau_2)}(1) - \nu(x_2) + 3 \leq 5$.

This then implies $i(x_2) = 3$ and in $\Omega^0(\gamma_2(\tau_2))$ we have

$$\gamma_2(\tau_2) \quad \approx \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \diamond \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}.$$

Now by the precise index iteration Theorem 6.1 we obtain

$$i(x_2^m) = m(i(x_2)+1) - 1 = 4m - 1, \qquad \nu(x_2^m) = 3, \qquad \forall\, m \in \mathbf{N}. \tag{11.15}$$

Therefore we get $\hat{i}(x_2) = 4$, $K(x_2) = 1$, $i^E(x_2^m) = i(x_2, m) - 3 = 0$, $i^E(x_2^1) = i(x_2) - 3 = 0$, and

$$\hat{\chi}(x_2) \equiv \frac{1}{K(x_2)} \sum_{\substack{1 \leq m \leq K(x_2) \\ 0 \leq l \leq 2}} (-1)^{i^E(x_2^m)+l} k_l(x_2^m) = k_0(x_2) - k_1(x_2) + k_2(x_2) \leq 1.$$

Here note that both of $k_0(x_2)$ and $k_2(x_2)$ take values in $\{0,1\}$ and at most one of them is positive. Therefore we then obtain

$$\frac{1}{4} \geq \frac{\hat{\chi}(x_2)}{\hat{i}(x_2)} = \frac{1}{2}.$$

Contradiction !

Chapter 4. Closed geodesics on Spheres.

The content of this Chapter is based on my survey published in *J. of European Math. Soc.* 8 (2006) 341-353 and recent results after 2006 on this topic. In Section 15 we propose some questions and conjectures based on old and recent new results and considerations on closed geodesics as well as closed characteristics.

It is well known that the geodesic, i.e., the shortest curve, connecting two prescribed points in the Euclidean plane is the line segment connecting them. But the geodesic, especially the closed geodesic, problem on the earth is very difficult. In fact, the closed geodesic problem is a very important subject in both dynamical systems and differential geometry, and has stimulated many creative ideas and new developments in mathematics. For closed geodesics on spheres with Riemannian structures or Finsler structures, modern mathematical studies can be traced back at least to the work of J. Hadamard, H. Poincaré, G. D. Birkhoff, M. Morse, L. Lyusternik, L. Schnirlmann, and many other famous mathematicians. In this short survey, I can only introduce some of the vast literature which is related to closed geodesics on 2-dimensional and 3-dimensional spheres and to our current interests. This chapter is organized as follows: §12 A partial and certainly not complete history of the studies of closed geodesics mainly on spheres. §13 Recent results obtained by V. Bangert, H. Duan, W. Wang, H.-B. Rademacher and the author on the multiplicity and stability of closed geodesics on Finsler and Riemannian spheres. §14 Main ideas in the proof of the multiplicity theorem of V. Bangert and the author. §15 Open problems.

12 A partial history of closed geodesics on spheres

First we introduce the concept of Finsler and Riemannian metrics on manifolds.

Definition 12.1. (cf. [BCS1], [She1]) *Let M be a finite dimensional manifold. A function $F : TM \to [0, +\infty)$ is a* Finsler metric *if it satisfies*

(F1) F is C^∞ on $TM \setminus \{0\}$,

(F2) $F(x, \lambda y) = \lambda F(x, y)$ for all $y \in T_xM$, $x \in M$, and $\lambda > 0$,

(F3) For every $y \in T_xM \setminus \{0\}$, the quadratic form

$$g_{x,y}(u, v) \equiv \frac{1}{2}\frac{\partial^2}{\partial s \partial t} F^2(x, y + su + tv)|_{t=s=0}, \qquad \forall u, v \in T_xM,$$

is positive definite.

In this case, (M, F) is called a Finsler manifold*.*

F is reversible *if $F(x,-y) = F(x,y)$ holds for all $y \in T_xM$ and $x \in M$. F is* Riemannian *if $F(x,y)^2 = \frac{1}{2}G(x)y \cdot y$ for some symmetric positive definite matrix function $G(x) \in GL(T_xM)$ depending on $x \in M$ smoothly. We denote by $\mathcal{F}(M)$ and $\mathcal{R}(M)$ the set of all Finsler and Riemannian metrics on M respectively.*

Note that one of the major differences between Riemannian and Finsler metrics is the irreversibility in the condition (F2). For a closed geodesic c in a Finsler manifold (M,F), its inverse curve c^{-1} defined by $c^{-1}(t) = c(1-t)$ may not be a geodesic. If it is, it is usually viewed to be a closed geodesic different from c.

For any closed curve $f : S^1 \to M$ on a Finsler manifold (M,F) or a Riemannian manifold (M,g), the group $G = S^1$ or $G = O(2)$ acts on f by $\theta \cdot f(t) = f(t+\theta)$ for every $\theta \in G$. For a closed geodesic c, its m-th iterate is defined by $c^m(t) = c(mt)$. A closed geodesic is *prime*, if it is not any m-th iterate of any other closed geodesics with $m \geq 2$. Two prime closed geodesics c_1 and c_2 on an irreversible (or reversible) Finsler manifold (M,F) are *distinct* (or *geometrically distinct*), if they do not differ by an S^1-action (or $O(2)$-action). We denote the set of all distinct prime closed geodesics on an irreversible Finsler manifold by $\mathrm{CG}(M,F)$, and similarly by $\mathrm{CG}(M,F)$ for a reversible Finlser manifold.

It is a longstanding conjecture that there exist infinitely many distinct prime closed geodesics on every compact Riemannian manifold (cf. Problem 81 in [Yau1]). J. Hadamard in 1898 and H. Poincaré in 1905 studied closed geodesics on convex surfaces (cf. [Had1] and [Poi1]). Then G. D. Birkhoff proved the following remarkable result:

Theorem 12.2. (G. D. Birkhoff, [Bir1], 1917 and [Bir2], 1927)

$$^{\#}\mathrm{CG}(S^n, g) \geq 1, \qquad \forall g \in \mathcal{R}(S^n).$$

In 1951, L. Lyusternik and A. Fet proved the following important theorem:

Theorem 12.3. (L. Lyusternik and A. Fet, [LyF1], 1951) *For every compact manifold M, there holds*

$$^{\#}\mathrm{CG}(M, g) \geq 1, \qquad \forall g \in \mathcal{R}(M).$$

Note that this theorem holds also for Finsler metrics, because the proof of Theorem 12.3 (cf. [Kli1] and [Kli2]) is variational and does not really depend on the special properties of Riemannian metrics.

Denote by ΛM the free loop space of a Riemannian manifold (M,g) and by $\Lambda^0 M$ the single point loops on M. For the Finsler case, we choose a Riemannian metric on M, and define ΛM similarly. In 1969, D. Gromoll and W. Meyer proved the following important result:

Theorem 12.4. (D. Gromoll and W. Meyer [GrM1], 1969) *Let (M, g) be a Riemannian manifold such that the Betti numbers $\{b_i(\Lambda M)\}_{i\geq 1}$ are unbounded. Then ${}^{\#}\mathrm{CG}(M, g) = \infty$, where $b_i(\Lambda M) = \mathrm{rank} H_i(\Lambda M, \Lambda^0 M; \mathcal{K})$ for all $i \in \mathbf{N}$ and some field $\mathcal{K}$.*

Motivated by Theorem 12.4, M. Vigué-Poirrier and D. Sullian proved the following remarkable result:

Theorem 12.5. (M. Vigué-Poirrier and Sullivan [ViS1], 1976) *For a compact simply connected Riemannian manifold (M, g), the Betti number sequence $\{b_i(\Lambda M)\}_{i\geq 1}$ is unbounded if and only if the cohomology algebra of M requires at least two generators.*

Note that in 1980, H. Matthias in [Mat1] generalized Theorem 1.4 to Finsler manifolds. Therefore by Theorems 12.4 and 12.5, the most interesting unknown problem on closed geodesics is for Finsler and Riemannian spheres.

Around 1990, V. Bangert (cf. [Ban1], [Ban2]) and J. Franks (cf. [Fra1], [Fra2]) proved the following important result for Riemannian S^2:

Theorem 12.6. (V. Bangert [Ban2] 1993, and J. Franks [Fra1] 1992)

$$ {}^{\#}\mathrm{CG}(S^2, g) = +\infty, \qquad \forall g \in \mathcal{R}(S^2). $$

For the closed geodesic problem on Riemannian (2-dimensional) manifolds, we refer readers to the excellent survey papers [Ban1] and [Tai1].

On the other hand, in 1973 A. Katok constructed remarkable Finsler metrics on S^n which showed that there is a major difference between Riemannian and general Finsler metrics dynamically:

Theorem 12.7. (A. Katok [Kat1], 1973) *For any $n \geq 2$, there exists an irreversible Finsler metric F_{Katok} on S^n which possesses precisely $2[(n + 1)/2]$ distinct prime closed geodesics.*

In fact, for S^2 by [Kat1] and W. Ziller's paper [Zil2], Katok's metric has the form $F_\alpha(x) = \|x\|_g^* + \alpha x(V)$ for any $x \in T^*S^2$, where $\|\cdot\|_g$ is the standard Riemannian metric on S^2, and V is a vector field corresponding to rotations on S^2 along the equatorial direction. Locally in spherical coordinates away from the north and the south poles, F_α has the form:

$$ F_\alpha(q_1, q_2, p_1, p_2) = (p_1^2 \cos^{-2} q_2 + p_2^2)^{1/2} + \alpha p_1. $$

The two closed geodesics on (S^2, F_α) are along the equator and are in fact inverse curves c and c^{-1} to each other. They have lengths $length(c) = 2\pi/(1+\alpha)$ and $length(c^{-1}) = 2\pi/(1-\alpha)$ respectively. The linearized Poincaré map P_c of c is conjugate to the following rotation matrix

$$ R(\theta_c) = \begin{pmatrix} \cos\theta_c & -\sin\theta_c \\ \sin\theta_c & \cos\theta_c \end{pmatrix} \tag{12.1} $$

with $\theta_c = 2\pi/(1+\alpha)$. Similarly $P_{c^{-1}}$ is also conjugate to $R(\theta_{c^{-1}})$ with $\theta_{c^{-1}} = 2\pi/(1-\alpha)$. All iterates of c and c^{-1} are non-degenerate. Then by the precise index iteration formulae of the author proved in [Lon1] of 2000, one can show that the Morse index sequences of iterates of c and c^{-1} counting multiplicity satisfy $\{i(c^m), i(c^{-m})\}_{m\geq 1} = \{1, 3, 3, 5, 5, 7, 7, \ldots\}$.

12.1 Known multiplicity results

We refer readers to [Ano1] of 1974 ICM report of D. Anosov for comments on A. Katok's Theorem 12.7, where he in fact conjectured that 2 is the minimal number of distinct prime closed geodesics on every Finsler S^2. We note also that in [Zil2], W. Ziller made a similar conjecture for S^n based on Katok's example:

$$^{\#}\mathrm{CG}(S^n, F) \geq n, \qquad \forall F \in \mathcal{F}(S^n). \tag{12.2}$$

We are only aware of a few partial answers to these conjectures for S^n.

In [1] of 1965, Fet proved that there exist at least two distinct closed geodesics on every compact reversible bumpy Finsler manifold (M, F).

Theorem 12.8. (H.-B. Rademacher [Rad1], 1989) *Let F be a bumpy Finsler metric on S^2, i.e., all the closed geodesics and their iterations on (S^2, F) are non-degenerate. Then*

$$^{\#}\mathrm{CG}(S^2, F) \geq 2.$$

In the paper [HWZ2], 2003, H. Hofer, K. Wysocki, and E. Zehnder studied Hamitonian systems on star-shaped hypersurfaces in $\mathbf{R}^4$. Their result can be applied to Finsler 2-spheres to yield:

Theorem 12.9. (H. Hofer-K. Wysocki-E. Zehnder [HWZ2], 2003) *Let F be a bumpy Finsler metric on S^2. Assume that the stable and unstable manifolds at every close geodesic intersect transversally. Then*

$$^{\#}\mathrm{CG}(S^2, F) = 2 \quad \text{or} \quad +\infty.$$

In 1993 and 1997, N. Hingston proved two theorems in [Hin1] and [Hin2] respectively which showed the existence of infinitely many prime closed geodesics on Riemannian manifolds under certain sufficient conditions. Specially Hingston's two theorems can be adapted to Finsler 2-spheres and yield the following theorem:

Theorem 12.10. (N. Hingston [Hin1] of 1993 and [Hin2] of 1997) *Let F be a Finsler metric on S^2 and c is a closed geodesic on (S^2, F) such that $S^1 \cdot c^m$ is isolated as a critical orbit of the energy functional E on ΛS^2 for all $m \geq 1$. Denote by*

$$k_j(c) = \mathrm{rank} H_j(N_c^- \cup \{c\}, N_c^-; \mathbf{Q}), \quad \text{for} \ \ j = 0, 1, 2,$$

where N_c is a slice in $\ker E''(c)$ *transversal to* $S^1 \cdot c$ *and* $N_c^- = \{x \in N_c \,|\, E(x) < E(c)\}$. *Suppose there hold either*

(i) $i(c^m) = m(i(c)+1) - 1$, $\nu(c^m) = \nu(c)$ *for all* $m \geq 1$ *and* $k_0(c) > 0$, *or*

(ii) $i(c^m) + \nu(c^m) = m(i(c) + \nu(c) - 1) + 1$, $\nu(c^m) = \nu(c)$ *for all* $m \geq 1$ *and* $k_{\nu(c)}(c) > 0$.

Then there holds ${}^{\#}\mathrm{CG}(S^2, F) = +\infty$.

Based on the results of W. Klingenberg in 1968 [Kli1], W. Ballmann, G. Thorbergsson, and W. Ziller [BTZ1] in 1982 about closed geodesics on Riemannian spheres under pinching conditions, H.-B. Rademacher generalized their results to Finsler spheres and proved:

Theorem 12.11. (H.-B. Rademacher [Rad3], 2005) *For* $F \in \mathcal{F}(S^2)$ *let*

$$\lambda = \max\{F(-v) \,|\, F(v) = 1, v \in TS^2\}.$$

Suppose the flag curvature K *of* (S^2, F) *satisfies* $\lambda^2(\lambda+1)^{-2} < \delta \leq K \leq 1$. *Then there holds*

$${}^{\#}\mathrm{CG}(S^2, F) \geq 2.$$

Denote the two closed geodesics by c_1 *and* c_2 *with length*$(c_1) \leq$ *length*(c_2). *Then length*$(c_1) \leq 2\pi/\sqrt{\delta}$, *length*$(c_2) \leq \frac{\pi}{\sqrt{\delta}}(\frac{\lambda}{\sqrt{\delta}(\lambda+1)-\lambda} + 3)$, *and* c_1 *is simple.*

12.2 Known stability results

As usual, denote by P_c the linearized Poincaré map of a closed geodesic c on a manifold M and $\mathbf{U} = \{z \in \mathbf{C} \,|\, |z| = 1\}$. Then c is *hyperbolic* if $\sigma(P_c) \cap \mathbf{U} = \emptyset$, is *elliptic* if $\sigma(P_c) \subset \mathbf{U}$, and is non-degenerate if $1 \notin \sigma(P_c)$. Note that we have $P_{c^m} = P_c^m$ for all $m \geq 1$. For a closed geodesic c on a 2-dimensional surface, c is *irrationally elliptic* or *rationally elliptic* if P_c is conjugate to a rotation matrix (0.1) with $\theta_c/\pi \in \mathbf{R} \setminus \mathbf{Q}$ or $\theta_c/\pi \in \mathbf{Q}$ respectively,

In 1982-83, W. Ballmann, G. Thorbergsson, and W. Ziller studied the stability of closed geodesics on S^n, specially they proved:

Theorem 12.12. (W. Ballmann, G. Thorbergsson, and W. Ziller, [BTZ2], 1983) *For* $g \in \mathcal{R}(S^n)$, *if the sectional curvature satisfies* $\frac{4}{9} \leq K \leq 1$, *there exist at least two elliptic closed geodesics on* (S^n, g).

In 1989 and 2005, H.-B. Rademacher studied Finsler spheres S^n, and proved

Theorem 12.13. (H.-B. Rademacher [Rad1], 1989) *Let* $F \in \mathcal{F}(S^n)$ *be bumpy and satisfy* ${}^{\#}\mathrm{CG}(S^2, F) < +\infty$. *Then there exists at least two irrationally elliptic closed geodesics on* (S^2, F).

Theorem 12.14. (H.-B. Rademacher [Rad3], 2005) *Let* $F \in \mathcal{F}(S^n)$ *satisfy*

$$\lambda < 2, \quad \left(\frac{3\lambda}{2(\lambda+1)}\right)^2 < \delta \leq K \leq 1,$$

where λ is defined in Theorem 1.11. Then there exists at least one elliptic closed geodesic on (S^n, F).

Note that in 2000 the author proved a related stability result for closed characteristics on convex compact hypersurfaces in $\mathbf{R}^4$:

Theorem 12.15. (Y. Long [Lon1], 2000) *Let Σ be a convex compact C^2 hypersurface in $\mathbf{R}^4$ with precisely two closed characteristics on Σ. Then both of them are elliptic.*

13 Main new multiplicity and stability results

Recently, V. Bangert and the author proved the following result which settles Anosov's conjecture for the lower bound of the number of distinct prime closed geodesics as well as (12.2) for S^2 positively. This theorem was first reported in July 2004 in the conference to celebrate Professor I. Ekeland's 60th birthday.

Theorem 13.1. (V. Bangert and Y. Long [BaL1], 2005)

$$^{\#}\mathrm{CG}(S^2, F) \geq 2, \qquad \text{for every Finsler metric on } S^2.$$

Motivated by Theorem 13.1 and those mentioned in Subsections 12.2, recently my student Wei Wang and myself proved the following result:

Theorem 13.2. (Y. Long and W. Wang [LoW1]) *Let F be an irreversible Finsler metric on S^2 satisfying $^{\#}\mathrm{CG}(S^2, F) < +\infty$. Then there exists at least two irrationally elliptic closed geodesics on (S^2, F).*

As a consequence of this theorem we obtain

Corollary 13.3. (Y. Long and W. Wang [LoW1] and [LoW2]) *Let F be an irreversible Finsler metric on S^2 satisfying $^{\#}\mathrm{CG}(S^2, F) = 2$. Then both of the two closed geodesics c_1 and c_2 on (S^2, F) are irrationally elliptic with rotation angles $\theta_1 = 2\pi/(1+\alpha)$ and $\theta_2 = 2\pi/(1-\alpha)$ for some $\alpha \in (0,1) \setminus \mathbf{Q}$.*

It is rather surprising that if Conjecture 1 in Section 15 holds, then Corollary 13.3 yields that whenever $^{\#}\mathrm{CG}(S^2, F)$ is finite, there are precisely two distinct prime irrationally elliptic closed geodesics and they behave analytically like those two prime closed geodesics of Katok's metric, i.e., whose iterations possess the same Morse indices and nullities. Then their local critical modules are all the same as the two of Katok. Note that here these two prime closed geodesics may not be inverse curves of each other as the two of Katok.

When the metric is bumpy, recently the following results were proved:

Theorem 13.4. (H. Duan and Y. Long [DuL1], H.-B. Rademacher [Rad4]) *Every bumpy irreversible Finsler metric on S^n possesses at least two distinct prime closed geodesics for any integer $n \geq 2$.*

Note that recently H.-B. Rademacher ([Rad5]) proved a similar result for $\mathbf{CP}^2$.

Despite of lots of efforts on closed geodesics on Riemannian manifolds, it seems that so far, it is still not known whether there exist always at least two geometrically distinct closed geodesics on every Riemannian 3-sphere. After studies [LoW3] and [DuL2] on S^3, very recently H. Duan and the author in [LDu1] answered this question positively.

In [LDu1], we introduced the following defintion:

Definition 13.5. (Y. Long and H. Duan [LDu1], 2008) *A prime closed geodesic c on a (reversible or irreversible) Finsler manifold (M, F) is* **irrational** *if in the basic normal form decomposition (4.7) of the linearized Poincaré map P_c contains no matrix $R(\theta)$ with $\theta/\pi \in \mathbf{R} \setminus \mathbf{Q}$, and* **irrational** *otherwise.*

Then we proved the following results:

Theorem 13.6. (Y. Long and H. Duan [LDu1], 2008) *For any integer $n \geq 2$, let (S^n, F) be a Finsler sphere with ${}^{\#}\mathrm{CG}(S^n, F) = 1$. Then the prime closed geodesic c can not be rational.*

Together with our Theorem 13.4 and the mean index identity of Rademacher ([Rad1]), it yields

Theorem 13.7. (Y. Long and H. Duan [LDu1], 2008) ${}^{\#}\mathrm{CG}(S^3, F) \geq 2$ *holds always for every reversible Finsler metric F on S^3. Specially this holds for every Riemannian metric on S^3.*

Our method works also for irreversible Finlser metrics with some minor modification and yields:

Theorem 13.8. (Y. Long and H. Duan [LDu1], 2008) ${}^{\#}\mathrm{CG}(S^3, F) \geq 2$ *holds always for every irreversible Finsler metric F on S^3.*

Here I should mention a related recent result of H. Duan and myself:

Theorem 13.9. (H. Duan and Y. Long [DuL2]) *Let F be a bumpy and irreversible Finlser metric on S^3. Then either there exist precisely two non-hyperbolic prime closed geodesics, or there exist at least three distinct prime closed geodesics.*

Note that Theorem 13.9 does not claim that the existence of precisely two non-hyperbolic prime closed geodesics on (S^3, F) must happen for some F.

14 Main ideas in the proof of Theorem 13.1

The conditions (F1)-(F3) for Finsler metrics were introduced by P. Finsler for the local existence and uniqueness of geodesics connecting two nearby points on a manifold. The problem of closed geodesics is global. Thus our proof of the Theorem 2.1 is naturally topological and variational, and does not depends on geometrical properties of each individual Finsler metric such as curvatures.

The main ideas in the proof of Theorem 13.1 of V. Bangert and the author are contained in the following four steps, where we explain more on the rationally elliptic case. Here to make explanations shorter, topics related to the smoothness of the energy functional E on the free loop space on S^2 are all omitted. We concentrate on topological facts and variational arguments which are related to the multiplicity.

Fix an $F \in \mathcal{F}(S^2)$. Assuming that there exists precisely one prime closed geodesic c on (S^2, F), we proceed as follows to reach a contradiction. In this section we use homological modules with **Q**-coefficients only.

⟨1⟩ By the author's precise index iteration formulae proved in Section 3 of [Lon1] of 2000 (cf. Section 8.1 of [Lon2]), there are 9 possibilities for the closed geodesic c depending on the eigenvalues of the linearized Poincaré map P_c which is a 2×2 symplectic matrix. Here the first three cases are for eigenvalue 1, the next three cases are for eigenvalue -1, the 7th is for the rationally elliptic case, the 8th is for the irrationally elliptic case, and the 9th is for the hyperbolic case. Note that the closed geodesic c and all of its iterates are non-degenerate in the last two cases in which two closed geodesics were found by H.-B. Rademacher in [Rad1] of 1989. Therefore we only need to study the first seven degenerate cases.

Here we choose a Riemannian metric g on S^2, and define $\Lambda = \Lambda S^2$ to be the free loop space of $H^1(S^1, S^2)$, where a curve c is H^1, if it is absolutely continuous and $\dot{c}(t)$ is square integrable in g as in the Chapter 1 in [Kli2].

Denote by $E(c) = \int_0^1 F^2(\dot{c}(t))dt$ and $\Lambda^a = \{\gamma \in \Lambda \,|\, E(\gamma) \le a\}$ for $a \in \mathbf{R}$.

⟨2⟩ In order to apply Morse theory, using the arguments of W. Ziller in [Zil1] of 1977, we obtain the Betti numbers as follows (cf. V. Bangert and Y. Long [BaL1], 2005):

$$H_q(\Lambda, \Lambda^0) \quad = \quad \begin{cases} 0, & \textit{if } q \le 0 \ \textit{ or } \ q = 2, \\ \mathbf{Q}, & \textit{if } q = 1 \ \textit{ or } \ q \ge 3. \end{cases} \tag{14.1}$$

$$b_q \equiv \mathrm{rank} H_q(\Lambda, \Lambda^0) \quad = \quad \begin{cases} 0, & \textit{if } q \le 0 \ \textit{ or } \ q = 2, \\ 1, & \textit{if } q = 1 \ \textit{ or } \ q \ge 3. \end{cases} \tag{14.2}$$

As usual the Morse type number M_k for all $k \ge 0$ is defined by

$$M_k = \sum_{\substack{1 \le j \le q \\ m \ge 1}} \mathrm{rank} H_k(\Lambda(c^m) \cup \{S^1 \cdot c^m\}, \Lambda(c^m)), \qquad \forall k \ge 0,$$

where $\Lambda(c^m) = \{\gamma \in \Lambda \mid E(\gamma) < E(c^m)\}$. Then for every integer $k \ge 1$, it is well known that the following Morse inequalities hold:

$$M_k \quad \ge \quad b_k, \tag{14.3}$$

$$\begin{aligned} M_k - M_{k-1} + M_{k-2} \quad - \cdots \quad & +(-1)^{k-1} M_1 + (-1)^k M_0 \\ \ge \quad & b_k - b_{k-1} + b_{k-2} - \cdots + (-1)^{k-1} b_1. \end{aligned} \tag{14.4}$$

$\langle 3\rangle$ For each one of the first six cases, using index iteration formulae in [Lon6] we obtain precisely all the Morse indices and nullities of all iterations c^m of c. Together with techniques of D. Gromoll and W. Meyer in [GrM1] of 1969, we are able to compute all the local critical modules of the energy functional $E(c) = \int_0^1 F^2(\dot{c}(t))dt$ near c^m for all $m \geq 1$ in Λ. Then we find that either the Morse inequalities yield already a contradiction which implies the existence of at least two distinct prime closed geodesics on (S^2, F), or the Morse inequalities lift up the dimension of a certain local homology group. Then Hingston's Theorem 12.10 becomes applicable and yields infinitely many distinct prime closed geodesics on (S^2, F) which completes the proof for this case.

For example, when $P_c = -I$, by the Morse inequality $M_1 \geq b_1$, we obtain $i(c) = 1$ and $\nu(c) = 0$. By Theorem 8.1.5 of [Lon2], we obtain

$$i(c^m) = m - \frac{1 + (-1)^m}{2}, \;\; \nu(c^m) = 1 + (-1)^m, \;\; \forall m \geq 1. \tag{14.5}$$

We denote by

$$\hat{k}_j(c^m) = \mathrm{rank} H_j(N_{c^m}^- \cup \{c^m\}, N_{c^m}^-)^{\mathbf{Z}_m}, \quad for \;\; j = 0, 1, \ldots, \nu(c^m).$$

Note that there holds

$$\hat{k}_j(c^m) \leq k_j(c^m), \qquad\qquad \forall m \geq 1.$$

Using the method of D. Gromoll and W. Meyer in [GrM1], all the local critical modules can be computed out and we obtain

$$M_0 = 0, \; M_1 = 1 + \hat{k}_0(c^2), \; M_2 = 1 + \hat{k}_0(c^2) + \hat{k}_1(c^2), \; M_2 = 1 + \hat{k}_1(c^2) + \hat{k}_2(c^2).$$

Therefore by the Morse inequality we obtain

$$1 + k_2(c^2) \geq 1 + \hat{k}_2(c^2) = M_3 - M_2 + M_1 \geq b_3 - b_2 + b_1 = 2. \tag{14.6}$$

Let $d = c^2$. Then (3.5) and (3.6) yield condition (ii) of Theorem 12.10 and thus ${}^{\#}\mathrm{CG}(S^2, F) = +\infty$.

$\langle 4\rangle$ For the 7th case, P_c is conjugate to the matrix $R(\theta)$ in (12.1) for some rotation angle $\theta \in (0, 2\pi) \cap (\pi\mathbf{Q} \setminus \{\pi\})$, i.e., c is rationally elliptic.

By the Morse inequality $M_1 \geq b_1 = 1$, we obtain $i(c) = 1$. Thus by Theorem 8.1.7 of [Lon8] there holds

$$i(c^m) = 2[\frac{m\theta}{2\pi}] + 1, \;\; \nu(c^m) = 0, \qquad \text{if} \;\; m\theta \neq 0 \bmod 2\pi, \tag{14.7}$$

$$i(c^m) = 2[\frac{m\theta}{2\pi}] - 1, \;\; \nu(c^m) = 2, \qquad \text{if} \;\; m\theta = 0 \bmod 2\pi. \tag{14.8}$$

Therefore there is a unique minimal integer $n \geq 3$ such that $\nu(c^n) = 2$.

By (14.8), both the iteration formulae in (i) and (ii) of Theorem 12.10 hold for the iterates of c^n. Thus we obtain ${}^{\#}\mathrm{CG}(S^2, F) = +\infty$ by Theorem 1.10 whenever $k_0(c^n) + k_2(c^n) > 0$. Therefore we need only study case 7 under the condition

$$k_0(c^n) = k_2(c^n) = 0. \tag{14.9}$$

Let $\kappa_m = E(c^m)$ for all $m \geq 1$. Then we have $\kappa_m \to +\infty$ as $m \to +\infty$ and

$$0 < \kappa_1 < \kappa_2 < \cdots < \kappa_i < \kappa_{i+1} < \cdots.$$

As an example we consider the case of $\theta = \pi/3$, $n = 6$, and $k_0(c^6) = k_1(c^6) = k_2(c^6) = 0$. Then we obtain $M_k = 5$ for all $k \geq 1$. Thus the left hand side of the Morse inequality (14.4) is always 5 and the right hand side of (14.4) is at most 2. Therefore (14.4) always holds and the Morse inequality yields no information. Likewise Hingston's Theorem 1.10 does not apply because $k_0(c^6) = k_2(c^6) = 0$. Thus this case needs some new ideas and a new approach. Here we make the following comparison on long exact sequences for the triple $(\Lambda, \Lambda^{\kappa_\tau}, \Lambda^0)$:

$$\begin{array}{ccccccccc}
0 & & 0 & & \mathbf{Q} & & \mathbf{Q} & & 0 \\
\| & & \| & & \| & & \| & & \| \\
H_2(\Lambda, \Lambda^0) & \longrightarrow & H_2(\Lambda, \Lambda^{\kappa_\tau}) & \longrightarrow & H_1(\Lambda^{\kappa_\tau}, \Lambda^0) & \longrightarrow & H_1(\Lambda, \Lambda^0) & \longrightarrow & H_1(\Lambda, \Lambda^{\kappa_\tau}) \\
\| & & \| & & \| & & \| & & \| \\
0 & & 0 & & \mathbf{Q}^\tau & & \mathbf{Q} & & 0.
\end{array} \tag{14.10}$$

Here the top line gives the case of Katok's example with $\tau = 1$ which matches up perfectly. The bottom line is for our sample of $\theta = \pi/3$ with $\tau = 5$ which yields a contradiction. This comparison yields an important idea for dealing with the general case. Here our crucial observation is that the alternative summation in the Morse inequality indicates how the higher dimensional local critical groups kill the lower dimensional local critical groups at the dimension level. This is a too rough understanding of the mutual relations among these local homology groups. To understand it further we need to study them more carefully at the homological levels.

For the general case, let $\tau = \max\{j \geq 1 \,|\, j\theta < 2\pi\}$. Then we have the following three important claims:

Claim 1. $2 \leq \tau \leq n - 1$.

Claim 2. $H_1(\Lambda^{\kappa_\tau}, \Lambda^0) = \oplus_{m=1}^{\tau} H_1(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) = \mathbf{Q}^\tau$.

Claim 3. *$H_2(\Lambda, \Lambda^{\kappa_\tau}) = 0$ when $\tau < n - 1$, or $H_2(\Lambda, \Lambda^{\kappa_\tau}) = \mathbf{Q}^a$ for some $a \in [0, n - 3]$ when $\tau = n - 1$.*

Assuming these three claims for the moment, we continue our study of case 7 under the condition (14.9). Suggested by (14.10) we consider the long exact sequence for the triple $(\Lambda, \Lambda^{\kappa_\tau}, \Lambda^0)$. By (14.2) and D. Gromoll-W. Meyer's technique in [GrM1] for computing local homological modules we obtain

$$\begin{array}{ccccccccc} H_2(\Lambda,\Lambda^0) & \longrightarrow & H_2(\Lambda,\Lambda^{\kappa_\tau}) & \longrightarrow & H_1(\Lambda^{\kappa_\tau},\Lambda^0) & \longrightarrow & H_1(\Lambda,\Lambda^0) & \longrightarrow & H_1(\Lambda,\Lambda^{\kappa_\tau}) \\ \| & & \| & & \| & & \| & & \| \\ 0 & & \mathcal{H}_2 & & \mathbf{Q}^\tau & & \mathbf{Q} & & 0. \end{array} \tag{14.11}$$

When $\tau < n-1$, we have $\mathcal{H}_2 = 0$ by Claim 3. Thus (14.11) yields

$$\mathbf{Q}^\tau = 0 \oplus \mathbf{Q} = \mathbf{Q}.$$

This contradicts the fact $\tau \geq 2$ in Claim 1.

When $\tau = n-1$, we have $\mathcal{H}_2 = \mathbf{Q}^a$ for some $a \in [0, n-3]$ by Claim 3. Thus (14.11) yields

$$\mathbf{Q}^{n-1} = \mathbf{Q}^\tau = \mathbf{Q}^a \oplus \mathbf{Q} = \mathbf{Q}^{a+1}.$$

This contradicts to the fact $a \leq n-3$ in Claim 3.

Therefore we are reduced to the proofs of Claims 1 to 3.

To prove Claim 1, we use the condition (14.9) and an identity satisfied by the mean index $\hat{i}(c) \equiv \lim_{m\to+\infty} i(c^m)/m = \theta/\pi$ of c, and derive an important estimate $0 < \theta < \pi$. It implies Claim 1.

In general the homological groups on different level sets may not be additive. We are only aware of two papers [BoS1] of R. Bott and H. Samelson in 1958 and [Zil1] of W. Ziller in 1977 who studied such homological addition properties of level sets in the loop spaces for compact globally symmetric spaces. But our (S^2, F) is not a globally symmetric space in general and their techniques do not apply. For the proof of Claim 2, we carry out precise computations on the connecting homomorphisms between level sets and prove the following vanishing property in the long exact sequence for the triple $(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}, \Lambda^0)$:

$$\partial_2 = 0 : H_2(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}) \to H_1(\Lambda^{\kappa_{m-1}}, \Lambda^0).$$

Here the precise understanding (14.7) and (14.8) of the Morse indices and nullities of c^m with $1 \leq m \leq \tau$ are crucial. This yields

$$H_1(\Lambda^{\kappa_m}, \Lambda^0) = H_1(\Lambda^{\kappa_{m-1}}, \Lambda^0) \oplus H_1(\Lambda^{\kappa_m}, \Lambda^{\kappa_{m-1}}), \qquad \forall m = 1, \ldots, \tau,$$

which yields Claim 2.

When $\tau < n-1$ by direct computation we obtain $\mathcal{H}_2 = 0$ in (14.11).

When $\tau = n-1$, together with the mean index identity mentioned above we obtain $2\pi = n\theta = (n-1-\hat{k}_1(c^n))\pi$, which implies $\hat{k}_1(c^n) = n-3$. By the long exact sequence for the triple $(\Lambda, \Lambda^{\kappa_\tau+1}, \Lambda^{\kappa_\tau})$ we obtain

$$\mathbf{Q}^{\hat{k}_1(c^n)} = H_2(\Lambda^{\kappa_\tau+1}, \Lambda^{\kappa_\tau}) \longrightarrow H_2(\Lambda, \Lambda^{\kappa_\tau}) \equiv \mathcal{H}_2 \longrightarrow H_2(\Lambda, \Lambda^{\kappa_\tau+1}) = 0.$$

Therefore Claim 3 holds.

This completes our study for the case 7 and the proof of Theorem 13.1.

15 Open problems

For further problems on closed characteristics, we refer readers to [Lon10] and [Lon11].

Based on what we already known for closed geodesics, the following problems seem to be interesting and important for further studies about closed geodesic problem on Finsler as well as Riemannian spheres.

Combining Theorems 12.9 and 13.1, it is natural to make the following conjecture:

Conjecture 1. *There holds*

$$^{\#}\mathrm{CG}(S^2, F) = 2 \quad or \quad +\infty, \qquad \forall F \in \mathcal{F}(S^2).$$

Conjecture 2. *For every $F \in \mathcal{F}(S^n)$, there exist two integers $2 \le p_n \le q_n$ satisfying $p_n \to +\infty$ as $n \to +\infty$ such that*

$$\{\,^{\#}\mathrm{CG}(S^n, F) \,|\, F \in \mathcal{F}(S^n)\} = \{k \in \mathbf{N} \,|\, p_n \le k \le q_n\} \cup \{+\infty\}.$$

Specially we suspect that

Conjecture 3. $\{\,^{\#}\mathrm{CG}(S^3, F) \,|\, F \in \mathcal{F}(S^3)\} = \{p_3, \cdots, 4\} \cup \{+\infty\}$ for some $p_3 \in \{2, 3, 4\}$.

For this conjecture, very few is known other than the case of Katok's metric F which satisfies $^{\#}\mathrm{CG}(S^3, F) = 4$ (cf. [Kat1] and [Zil2]), our Theorem 13.8 which yields $p_3 \ge 2$, and Theorem 13.9.

Conjecture 4. $^{\#}\mathrm{CG}(S^n, g) = +\infty$ *for every $g \in \mathcal{R}(S^n)$ with $n \ge 3$.*

Note that our Theorem 13.7 $^{\#}\mathrm{CG}(S^3, g) \ge 2$ holds for all $g \in \mathcal{R}(S^3)$.

Conjecture 5. *There exists at least one elliptic closed geodesic on (S^n, F) for every $F \in \mathcal{F}(S^n)$ with $n \ge 2$.*

For S^2, by our Theorem 13.2, it is only necessary to study Conjecture 5 when $^{\#}\mathrm{CG}(S^2, F) = +\infty$. But in this case it seems unfortunately that there is no effective method available yet without pinching conditions. On the other hand, in an interesting paper [Grj1] of 1980, A. Grjuntal proved the existence of Riemannian metrics on S^2 with positive curvature whose all closed non-selfintersecting geodesics are hyperbolic.

Conjecture 6. *For every Finsler metric F on S^n with ${}^\#\mathrm{CG}(S^n, F) < +\infty$, all the prime closed geodesics are irrationally elliptic.*

Because our proofs of Theorems 13.1 to 13.9 are variational, we hope that they may help at least in the study of some of the above conjectures for Finsler (as well as Riemannian) spheres.

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