A Mean Field Games model for the choice of insulation technology of households

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Introduction

- problem: how to describe in a continuous period the technological arbitration (insulation choices) of a large population?

- tools: ”Mean field games” (Nash equilibirum with infinite number of players)

- to find dynamical equilibria for this model

Goals

- design a stylised model, describing the arbitration between insulation and non-insulation with positive externality, and the possibility of a technological transition

- develop a numerical method

- simulation

References - Mean field games

• J.-M. Lasry and P.-L. Lions, Jeux à champ moyen. I. Le cas stationnaire. (French), *C. R. Math. Acad. Sci. Paris*, vol. 343 (9), 2006, pp 619-625.

• J.-M. Lasry and P.-L. Lions, Jeux à champ moyen. II. Horizon fini et contrôle optimal. (French), *C. R. Math. Acad. Sci. Paris* , vol. 343 (10), 2006, pp 679-684.

• J.-M. Lasry and P.-L. Lions, Mean field games, *Jpn. J. Math.*, vol. 2 (1), 2007, pp 229-260.

References - Numerical analysis

• Y. Maday, J. Salomon and G. Turinici, Monotonic time-discretized schemes in quantum control, *Num. Math*, vol. 103 (2), 2006, pp 323-338.

• J. Salomon and G. Carlier, A monotonic algorithm for the optimal control of the Fokker-Planck equation, *soumis*, 2008.

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The model - general settings

Framework:

- large economy: continuum of consumer agents
- time period: $[0, T]$
- any household owns exactly one house and cannot move to another one until T

The model - the agents

- arbitration between insulation and heating.
  a generical player (or agent) $\leftrightarrow$ an insulation level

- insulation level x belongs to $[0, 1]$
  ($x = 0$: no insulation, $x = 1$: maximal insulation)

- controlled process of the agent: $dX_t = \sigma dW_t + \alpha_t dt + dN_t(X_t)$
  where α is the control parameter (insulation effort), σ is a data of the model (noise)

- note that X_t is a diffusion process with reflexion, in the above equality, $dN_t(X_t)$ has the form $\chi_{\{0,1\}}(X_t)\vec{n}d\xi_t$ (see Freïdlin)

- initial density: $X_0 \sim m_0(dx)$

The model - the costs

An agent of the economy solves a minimisation problem composed of several terms:

- *Insulation acquisition cost:* $h(\alpha) := \frac{\alpha^2}{2}$
- *Insulation maintenance cost:* $g(t, x, m) := \frac{c_0 x}{c_1 + c_2 m(t,x)}$
- *Heating cost:* $f(t, x) := p(t)(1 - 0,8x)$ where $p(t)$ is the unit heating cost (unit price of energy, say)

Here $m(t, .)$ is the **density of agents at time** t

The model - Insulation maintenance cost

- $g(t, x, m) := \frac{c_0 x}{c_1 + c_2 m(t,x)}$

- increasing in x

- decreasing in $m \leftrightarrow$ economy of scale, positive externality
  The agents should do the same choice, stay together.
  The higher is the number of players having choosen an insulation level, the lower are the related costs.

Recall - Nash equilibrium

- N-players game with a set of action A

- The cost function that Player i wants to minimize is:

$$J_i : (x_1, ..., x_N) \in A^N \longmapsto J_i(x_1, ..., x_N)$$

- We say that $(x_1, ..., x_N)$ is a Nash point if for every $i = 1, ..., N$,

$$J_i(x_1, ..., x_i, ..., x_N) \leq J_i(x_1, ..., x, ..., x_N), \forall x \in A$$

The model - The minimization problem

The minimization problem is:

$$\begin{cases} \inf\limits_{\alpha \text{ adm}} \mathbb{E}\left[\int_0^T h(\alpha(t, X_t^x)) + f(t, X_t^x) + g(t, X_t, m(t, X_t^x))dt\right] \\ dX_t = \alpha_t dt + \sigma dW_t + dN_t(X_t) \\ X_0 = x \end{cases}$$

We can rewrite:

$$\begin{cases} \inf\limits_{\alpha \text{ adm}} \int_0^T \int_0^1 \left[\frac{\alpha(t,x)^2}{2} + f(t,x) + g(t,x,m(t,x))\right] m(t,x) dx dt \\ \partial_t m - \frac{\sigma^2}{2}\Delta m + \text{div}(\alpha m) = 0 \ , \ m(0,.) = m_0(.) \ , \\ m'(.,0) = m'(.,1) = 0 \end{cases}$$

The model - Some observations

- agents are rationals

- rational expectations $\longrightarrow$ every agent uses the global density m for every t as a data

- in the second formulation, the linear constraint is the controlled Fokker-Planck evolution equation (also called Kolmogorov forward equation)

- we use the Mean field games framework

The model - Mean Field Equilibrium

- A Mean field equilibrium (Nash equilibrium with an infinite number of players) corresponds to a solution of the following system:

$$\begin{cases} \partial_t m - \nu \Delta m + \text{div}(\alpha m) = 0 \ , \ m|_{t=0} = m_0 \\ -\nabla v = \alpha \\ \partial_t v + \nu \Delta v + \frac{\alpha^2}{2} + \alpha \cdot \nabla v = -\phi'(m) \ , \ v|_{t=T} = 0 \end{cases} \tag{1}$$

- $\phi(m) := \int_0^1 \left(p(t)(1 - 0,8x) + \frac{c_0 x}{c_1 + c_2 m(t,x)} \right) m(t,x) dx$ is the aggregate cost function and $\nu := \frac{\sigma^2}{2}$

- coupling 2 PDEs backward-forward (numerical difficulties)

The model - Externality & scale effect

The MFG framework is interesting to describe a situation which lives between two economical ideas: positive externality and economy of scale

- positive externality: positive impact on any agent utility NOT INVOLVED in a choice of an insulation level by a player

- economy of scale: economies of scale are the cost advantages that a firm obtains due to expansion (unit costs decrease)

The model - Defaults

- stylised from the "industrial" point of view

- not realistic (heating price, maintenance...)

- and not a simplification of statistical data

The model - Qualities

- transition effect (continuous time,continuous space)

- atomised agent (his action has no influence on the global density, micro-macro approach)

- non-cooperative equilibrium with rational expectations

Numerical simulations (with G. Turinici and J. Salomon)

- Define $J(\alpha) := \int_0^T \int_0^1 \left[\frac{\alpha(t,x)^2}{2} + f(t,x) + g(t,x,m)\right] m(t,x) dx dt$

- Case of a concave map on m and a linear constraint (Fokker-Planck evolution equation) $\longrightarrow$ we can't use a big part of gradient methods

- we used a "monotonic scheme" algorithm, which gives a strategy to find α^{k+1} given α^k.

Numerical simulations - Monotonic scheme

- The introduction of the adjoint leads to a factorization:

$$J(\alpha^{k+1}) - J(\alpha^k) \le \int_0^T \int_0^1 (\alpha^{k+1} - \alpha^k) H(\alpha^{k+1}, \alpha^k)$$

- The monotonic scheme gives a choice of α^{k+1} such that:

$$(\alpha^{k+1} - \alpha^k) H(\alpha^{k+1}, \alpha^k) \le 0$$

- problem: if m becomes numerically negative, the monotonic scheme explodes: $\longrightarrow$ be carefull choosing the discretization of:

$$\partial_t m - \nu \Delta m + \mathrm{div}(\alpha m) = 0$$

Numerical simulations - Godunov scheme

- some very high order schemes do not satisfy the condition $m_j^i > 0$ for all i, j

- We have choosen a Godunov scheme:

$$m_j^{i+1} = m_j^i + \nu \frac{dt}{dx^2}(m_{j+1}^i - 2m_j^i + m_{j-1}^i) - \frac{dt}{dx}(m_{j+\frac{1}{2}}^i \alpha_{j+\frac{1}{2}}^i - m_{j-\frac{1}{2}}^i \alpha_{j-\frac{1}{2}}^i)$$

- choosing this scheme, we get a discretization for the adjoint equation

Some numerical results

- the costs:
  · heating: $\rightarrow f(t,x) = p(t)(1 - 0,8x)$
  · insulation: $\rightarrow g(t,x,m) = \frac{x}{0.1+m(t,x)}$

- *1st example*: $p(t)$ constant / same choices

- *2d example*: $p(t)$ reaching a peak (non constant) / irreversibility of the insulation investment

- *3d example*: $p(t)$ reaching a peak (non constant) / multiplicity of equilibria

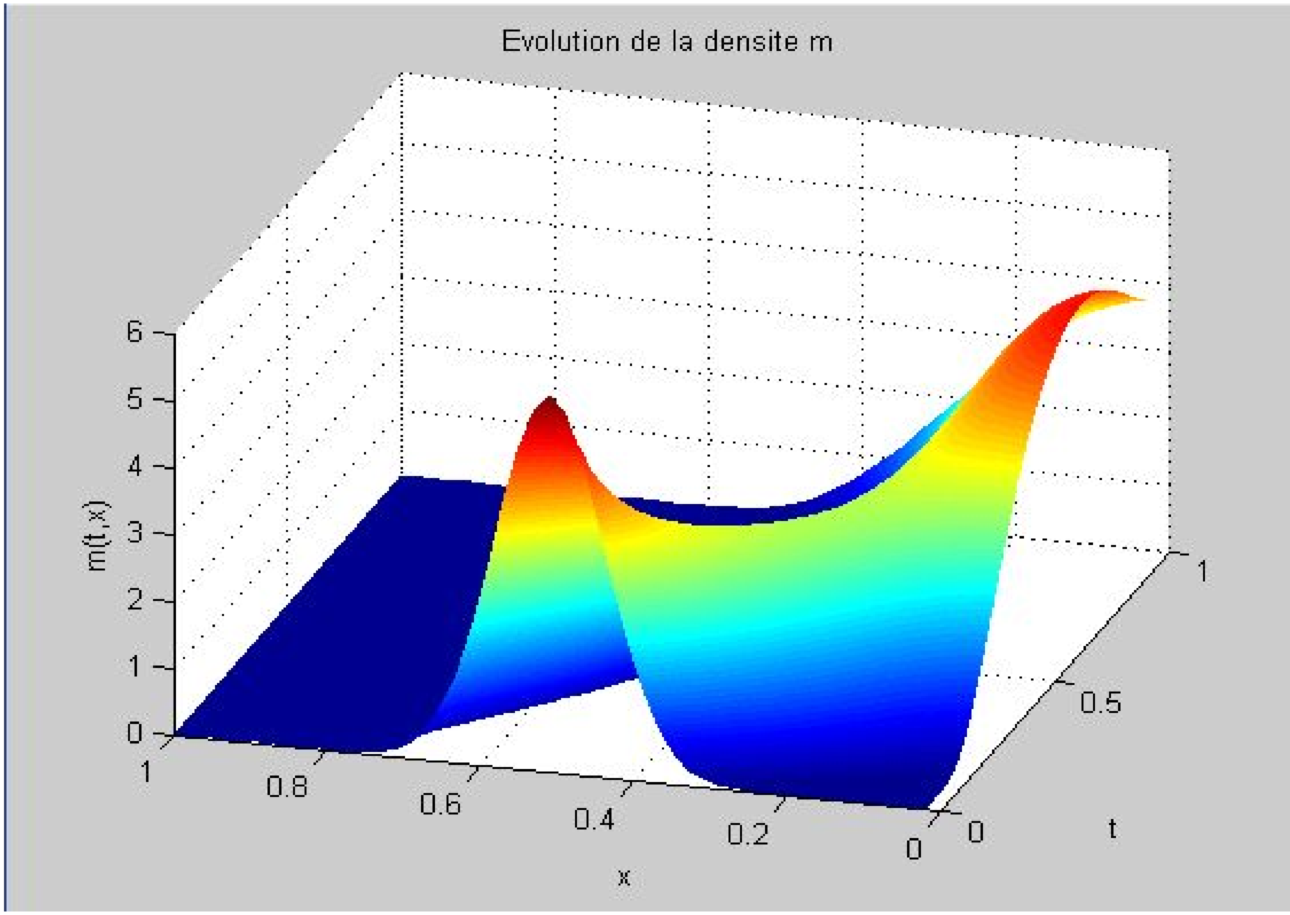
Some numerical results - First case

- the initial density of the householders is a gaussian centered in $\frac{1}{2}$

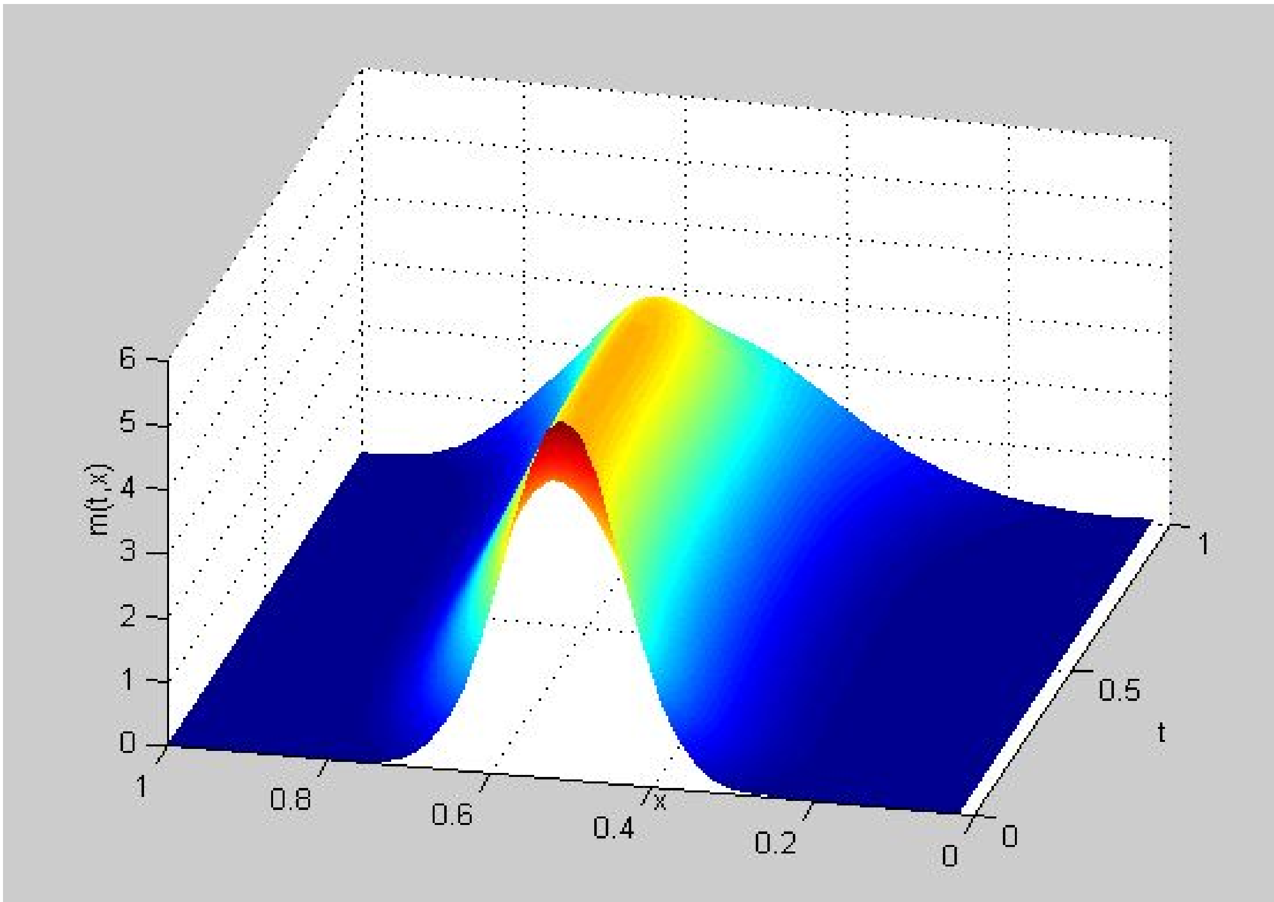
- the time period and the noise are respectively $T = 1$ and $\nu = 0.07$

- the energy price is constant (in the three next graphics, $p(t) \equiv$ 0, 3.2 and 10)

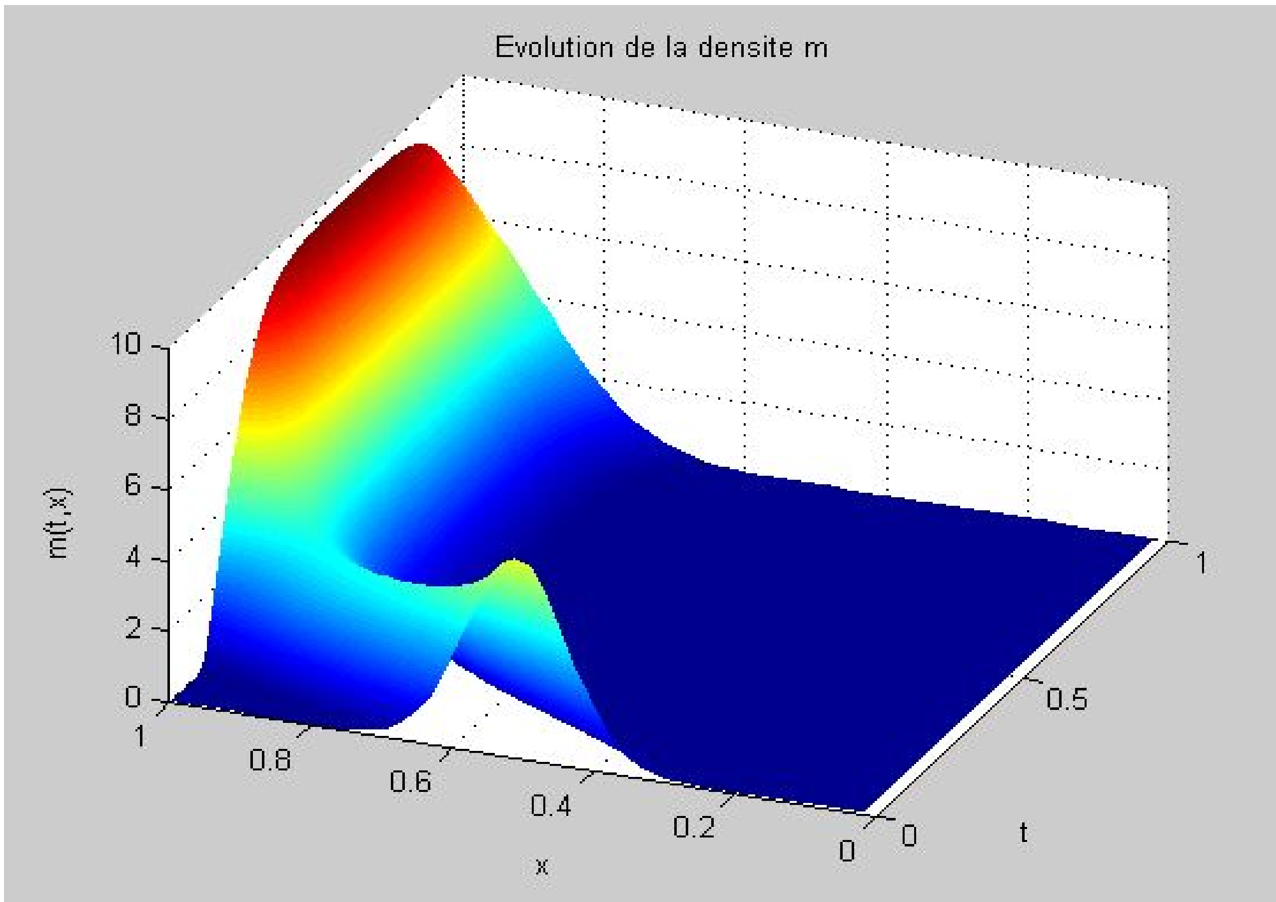
Some numerical results - $p(t) \equiv 0$



Some numerical results - $p(t) \equiv 3.2$



Some numerical results - $p(t) \equiv 10$



Some numerical results - Second case

- the initial density of the agents is a gaussian centered in 0.25 (*i.e* agents are not equipped in insulation material)

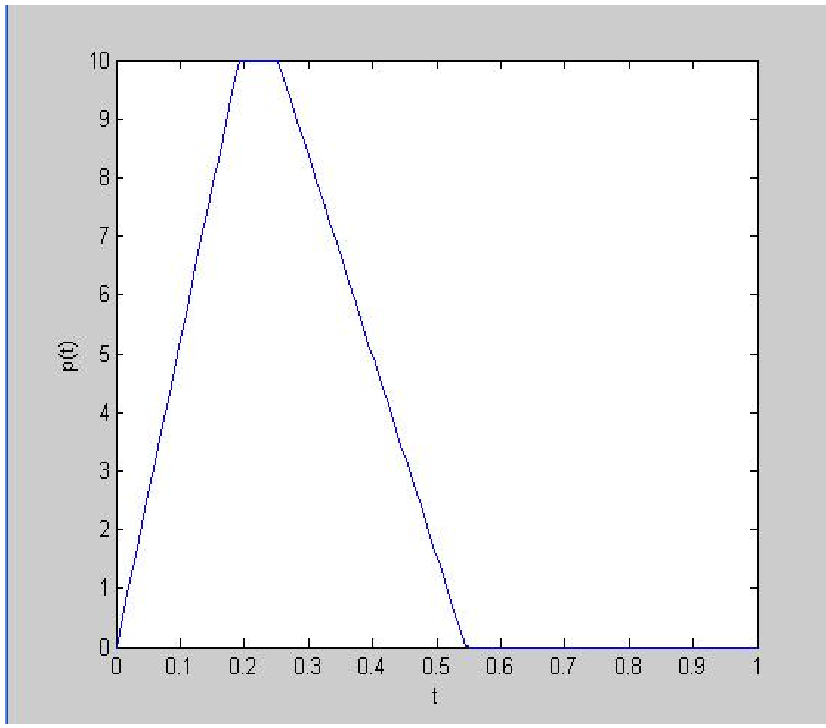
- the energy price is not a constant parameter, we look at the following case: the price first reaches a peak and then decreases to his first level.

- We change the set of the admissible controls:

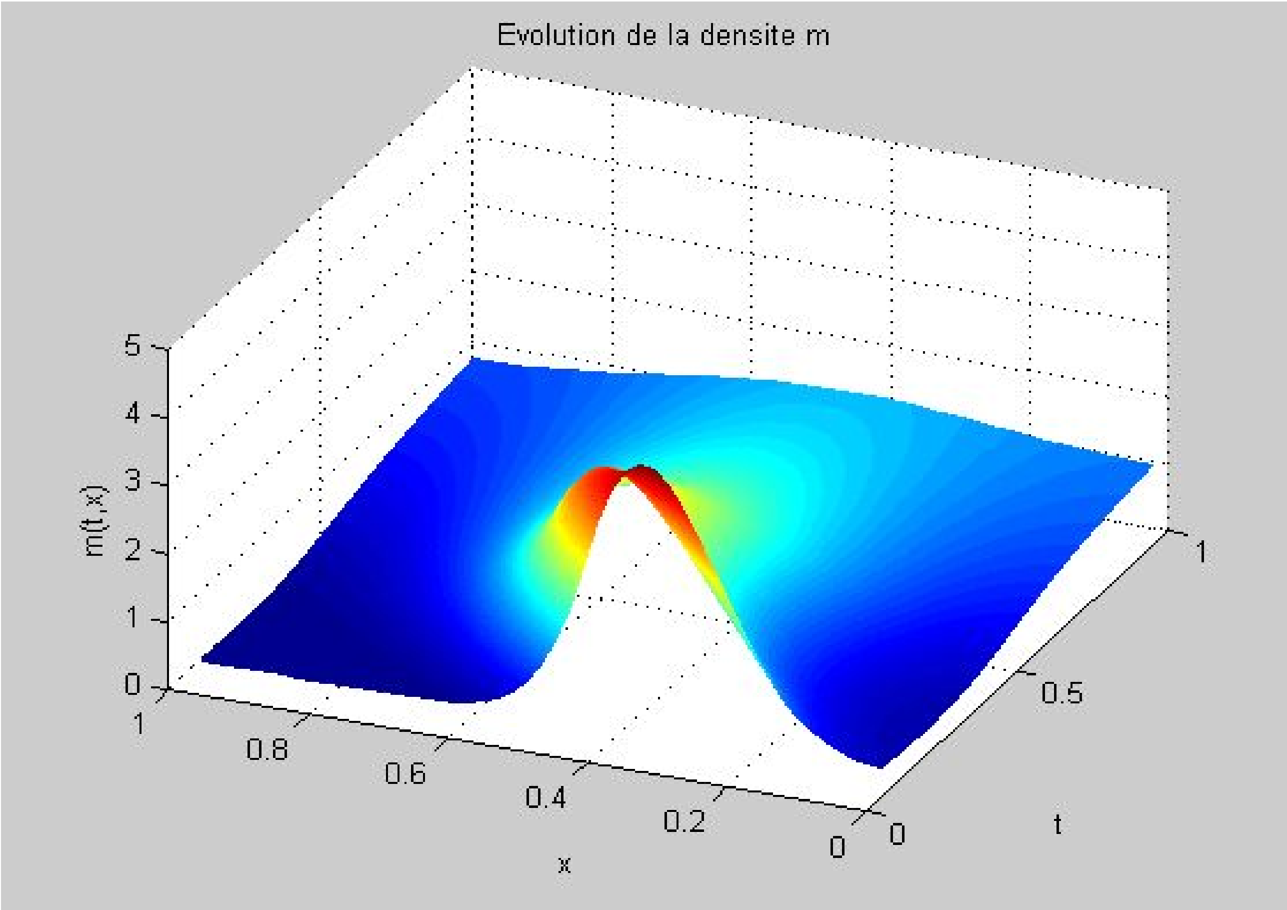
$$0 \leq \alpha(t, x) \leq C = 10$$

it means an investment irreversibility in insulation material

Some numerical results - $p(t)$



Some numerical results - Second case

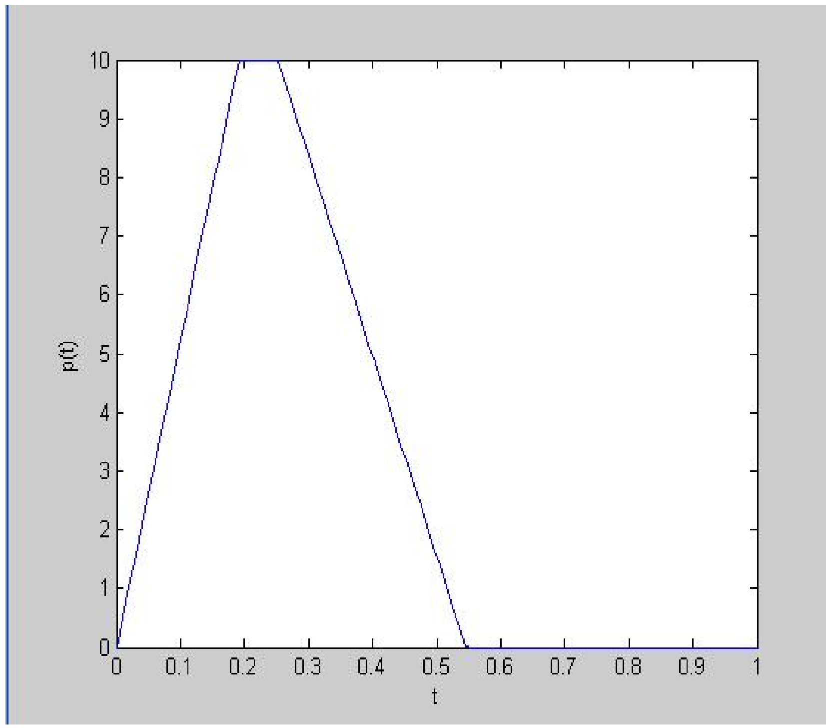


Some numerical results - Third case

- the initial density of the agents is an approximation of a Dirac in 0.1 (*i.e* agents are not equipped in insulation material)

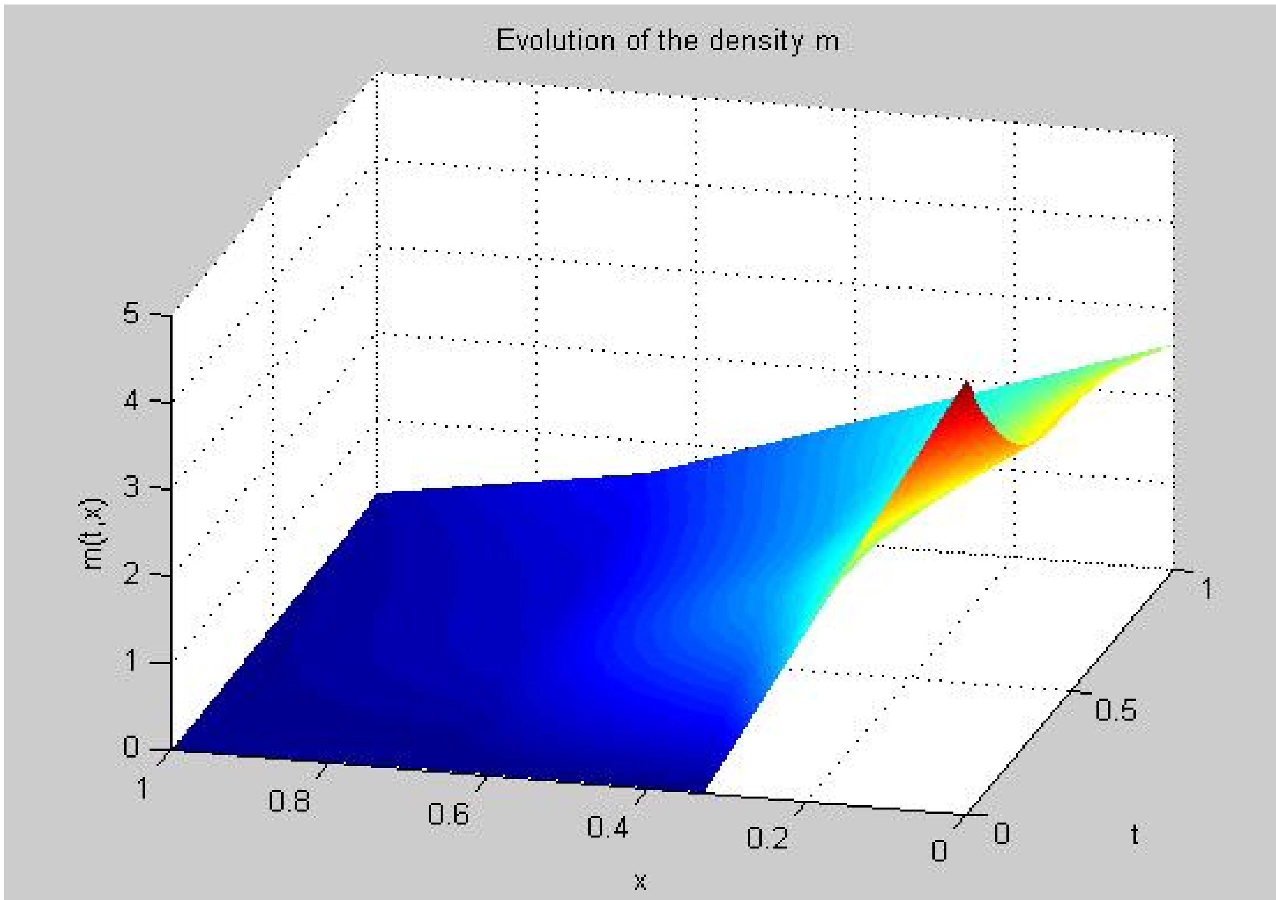
- the energy price is not a constant parameter, we look at the following case: the price first reaches a peak and then decreases to his first level.

Some numerical results - $p(t)$

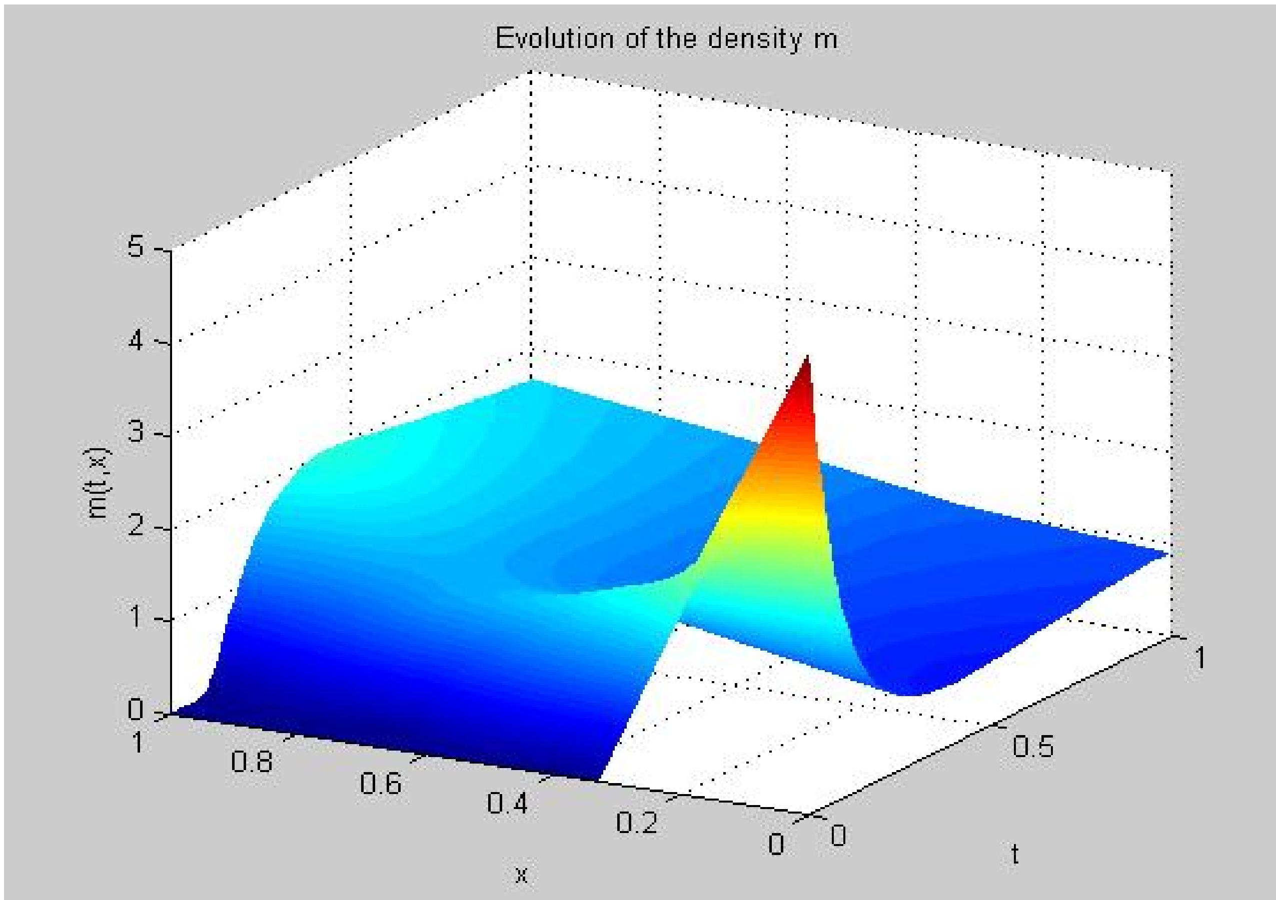


Question: In such a case, can we find two Mean Field Equilibria, the first related to the expectation of a higher insulation level, the second to the expectation of heating ?

Some numerical results - Two equilibria



Some numerical results - Two equilibria



Multiplicity of equilibria - Incentive policy

- we found an insulation-equilibrium and an energy consumption-equilibrium

- we can imagine incentive public policies in order to get the best equilibrium from a certain point of view

- from the ecological point of view: the best is the insulation-equilibrium

Concluding remarks and targets

- dynamical model (continuous time, transition) with infinite number of agents

- Mean Field Equilibirum (Nash, rational expectations) with economy of scale and positive externality

- numerical tool to solve such a problem

- multiplicity of equilibria, incentive policies

- main goal: theoretical proof of convergence (in preparation) and calibration

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