Curvature flow connections with Δ_∞

Robert Jensen
rjensen@luc.edu

Department of Mathematics and Statistics
Loyola University Chicago

July 21, 2009

Outline

Outline

Notations and conventions

- A bounded domain in $\mathbb{R}^n$, Ω

Notations and conventions

- A bounded domain in $\mathbb{R}^n$, Ω
- An implicit foliation of Ω, $\mathrm{F} \in \mathcal{C}^\infty(\Omega)$ with $(\mathrm{DF}) \neq \vec{0}$ in Ω where $(\mathrm{DF}) = (\mathrm{F}_{,1}, \ldots, \mathrm{F}_{,n})$ and $\mathrm{F}_{,j} = \dfrac{\partial \mathrm{F}}{\partial \xi_j}$

Notations and conventions

- A bounded domain in $\mathbb{R}^n$, Ω
- An implicit foliation of Ω, $\mathrm{F} \in \mathcal{C}^\infty(\Omega)$ with $(\mathrm{DF}) \neq \vec{0}$ in Ω where $(\mathrm{DF}) = (\mathrm{F}_{,1}, \ldots, \mathrm{F}_{,n})$ and $\mathrm{F}_{,j} = \dfrac{\partial \mathrm{F}}{\partial \xi_j}$
- Leaves of the foliation, $\Gamma_\lambda = \mathrm{F}^{-1}(\{\lambda\})$

Notations and conventions

- A bounded domain in $\mathbb{R}^n$, Ω
- An implicit foliation of Ω, $\mathrm{F} \in \mathcal{C}^\infty(\Omega)$ with $(\mathrm{DF}) \neq \vec{0}$ in Ω where $(\mathrm{DF}) = (\mathrm{F}_{,1}, \ldots, \mathrm{F}_{,n})$ and $\mathrm{F}_{,j} = \dfrac{\partial \mathrm{F}}{\partial \xi_j}$
- Leaves of the foliation, $\Gamma_\lambda = \mathrm{F}^{-1}(\{\lambda\})$
- A vector field on Ω, $\vec{\chi} = (\chi_1, \ldots, \chi_n) \in \mathcal{C}^\infty(\Omega; \mathbb{R}^n)$

Notations and conventions

- A bounded domain in $\mathbb{R}^n$, Ω
- An implicit foliation of Ω, $\mathrm{F} \in \mathcal{C}^\infty(\Omega)$ with $(\mathrm{DF}) \neq \vec{0}$ in Ω where $(\mathrm{DF}) = (\mathrm{F}_{,1}, \ldots, \mathrm{F}_{,n})$ and $\mathrm{F}_{,j} = \dfrac{\partial \mathrm{F}}{\partial \xi_j}$
- Leaves of the foliation, $\Gamma_\lambda = \mathrm{F}^{-1}(\{\lambda\})$
- A vector field on Ω, $\vec{\chi} = (\chi_1, \ldots, \chi_n) \in \mathcal{C}^\infty(\Omega; \mathbb{R}^n)$
- $(\mathrm{D}\vec{\chi}) = \begin{pmatrix} \chi_{1,1} & \cdots & \chi_{1,n} \\ \vdots & \ddots & \vdots \\ \chi_{n,1} & \cdots & \chi_{n,n} \end{pmatrix}$ where $\chi_{i,j} = \dfrac{\partial \chi_i}{\partial \xi_j}$

Notations and conventions

- Implied summation, $a_j b_j = \sum_{j=1}^{n} a_j b_j$

Notations and conventions

- Implied summation, $a_j b_j = \sum_{j=1}^{n} a_j b_j$
- The inner product of vector fields, $\left\langle \vec{\chi}, \vec{\psi} \right\rangle = \chi_j \psi_j$

Notations and conventions

- Implied summation, $a_j b_j = \sum_{j=1}^{n} a_j b_j$
- The inner product of vector fields, $\left\langle \vec{\chi}, \vec{\psi} \right\rangle = \chi_j \psi_j$
- The norm of a vector field, $\|\vec{\chi}\| = \sqrt{\langle \vec{\chi}, \vec{\chi} \rangle}$

Notations and conventions

- Implied summation, $a_j b_j = \sum_{j=1}^{n} a_j b_j$
- The inner product of vector fields, $\left\langle \vec{\chi}, \vec{\psi} \right\rangle = \chi_j \psi_j$
- The norm of a vector field, $\|\vec{\chi}\| = \sqrt{\langle \vec{\chi}, \vec{\chi} \rangle}$
- The Lie bracket of vector fields, $\left[\vec{\chi}, \vec{\psi}\right] = (\{\chi_j \psi_{1,j} - \psi_j \chi_{1,j}\}, \dots, \{\chi_j \psi_{n,j} - \psi_j \chi_{n,j}\})$

Assumptions on the foliation, F, and the vector field, $\vec{\chi}$

- **(A1)** $\|\vec{\chi}\| = 1$ on Ω

Assumptions on the foliation, F, and the vector field, $\vec{\chi}$

- **(A1)** $\|\vec{\chi}\| = 1$ on Ω
- **(A2)** $\langle \vec{\chi}, (\mathrm{DF}) \rangle = 0$ on Ω

Assumptions on the foliation, F, and the vector field, $\vec{\chi}$

- **(A1)** $\|\vec{\chi}\| = 1$ on Ω
- **(A2)** $\langle \vec{\chi}, (\mathrm{DF}) \rangle = 0$ on Ω
- **(A3)** $(\mathrm{D}\vec{\chi})\, \vec{\chi}^{\,t} = -\dfrac{\vec{\chi}\, (\mathrm{D}^2\mathrm{F})\, \vec{\chi}^{\,t}}{\|\mathrm{DF}\|^2} (\mathrm{DF})^t$

Assumptions on the foliation, F, and the vector field, $\vec{\chi}$

- **(A1)** $\|\vec{\chi}\| = 1$ on Ω
- **(A2)** $\langle \vec{\chi}, (\mathrm{DF}) \rangle = 0$ on Ω
- **(A3)** $(\mathrm{D}\vec{\chi})\,\vec{\chi}^{\,t} = -\dfrac{\vec{\chi}\ (\mathrm{D}^2\mathrm{F})\,\vec{\chi}^{\,t}}{\|\mathrm{DF}\|^2}\,(\mathrm{DF})^t$
- **(A4)** $\left\langle \vec{\psi}, \vec{\chi} \right\rangle = \left\langle \vec{\phi}, \vec{\chi} \right\rangle = 0$ in $\Omega \;\Rightarrow\; \left\langle \left[\vec{\phi}, \vec{\psi}\right], \vec{\chi} \right\rangle = 0$ in Ω

Assumptions on the foliation, F, and the vector field, $\vec{\chi}$

- **(A1)** $\|\vec{\chi}\| = 1$ on Ω
- **(A2)** $\langle \vec{\chi}, (\mathrm{DF}) \rangle = 0$ on Ω
- **(A3)** $(\mathrm{D}\vec{\chi})\,\vec{\chi}^{\,t} = -\dfrac{\vec{\chi}\,\left(\mathrm{D}^2\mathrm{F}\right)\vec{\chi}^{\,t}}{\|\mathrm{DF}\|^2}\,(\mathrm{DF})^t$
- **(A4)** $\left\langle \vec{\psi}, \vec{\chi} \right\rangle = \left\langle \vec{\phi}, \vec{\chi} \right\rangle = 0$ in Ω $\Rightarrow$ $\left\langle \left[\vec{\phi}, \vec{\psi}\right], \vec{\chi} \right\rangle = 0$ in Ω
- **(A5)** $\left\langle \dfrac{1}{\|\mathrm{DF}\|^2}\,(\mathrm{DF})\,, \vec{\chi}\,(\mathrm{D}\vec{\chi})^t \right\rangle = 1$ in Ω

Δ_∞ existence theorem

Theorem (1)

If F and $\vec{\chi}$ satisfy **(A1)-(A5)**, *then there is a function, $u \in \mathcal{C}^\infty(\Omega)$ such that*

$$\begin{aligned} (Du) &= e^{-F}\vec{\chi} \quad in \quad \Omega \\ \Delta_\infty u &= 0 \quad in \quad \Omega \end{aligned}$$

Local existence lemma

Lemma (2)

If F and $\vec{\chi}$ satisfy **(A1)-(A4)** *and $\xi_0 \in \Omega$, then there is a neighborhood of ξ_0, $\mathcal{N}_0 \subset \Omega$, and a diffeomorphism $\mathcal{S} : I \times B' \times J \to \mathcal{N}_0$ (where I and J are open intervals and B' is an open ball in $\mathbb{R}^{n-2}$) such that*

$$\left(F(\xi_0), \overline{0}, 0\right) \in I \times B' \times J \tag{1.1}$$
$$\mathcal{S}\left(F(\xi_0), \overline{0}, 0\right) = \xi_0 \tag{1.2}$$
$$\left\langle \vec{\chi}\left(\mathcal{S}\right), \mathcal{S}_{,\lambda}\right\rangle (\lambda, \overline{x}, 0) = 0 \quad \textit{for all} \quad (\lambda, \overline{x}) \tag{1.3}$$
$$F(\mathcal{S}(\lambda, \overline{x}, t)) = \lambda \quad \textit{for all} \quad (\lambda, \overline{x}, t) \tag{1.4}$$
$$\mathcal{S}_{,t} = \vec{\chi}\left(\mathcal{S}\right) \tag{1.5}$$
$$\mathcal{S}_{j,i}, \chi_j\left(\mathcal{S}\right) = 0 \quad \textit{for all} \quad i = 1, \ldots, (n-2) \tag{1.6}$$

Corollary

Corollary (3)

If $\vec{\psi}$ also satisfies **(A1)-(A4)** *with F and if $\vec{\chi} = \vec{\psi}$ on $\Gamma_0 = \Gamma_{F(\xi_0)}$, then*

$$\vec{\psi} = \vec{\chi} \quad on \quad \mathcal{N}_0$$

Proof of Theorem (1)

Using Lemma (2) to prove the local existence of u, let $\mathcal{N}_0$ and $\mathcal{S}\left(\lambda, \overline{x}, t\right)$ be from the Lemma (2). Define u on $\mathcal{N}_0$ implicitly by

$$u\left(\mathcal{S}(\lambda, \overline{x}, t)\right) = te^{-\lambda}$$

Proof of Theorem (1)

Using Lemma (2) to prove the local existence of u, let $\mathcal{N}_0$ and $\mathcal{S}(\lambda, \overline{x}, t)$ be from the Lemma (2). Define u on $\mathcal{N}_0$ implicitly by

$$u\left(\mathcal{S}(\lambda, \overline{x}, t)\right) = te^{-\lambda}$$

It follows that

$$u_{,j}\mathcal{S}_{j,t} = e^{-\lambda} \quad (1.7)$$

$$u_{,j}\mathcal{S}_{j,i} = 0 \quad (1.8)$$

$$u_{,j}\mathcal{S}_{j,\lambda} = -te^{-\lambda} \quad (1.9)$$

Proof of Theorem (1)

From (1.7)-(1.9) we see that at each point of $\mathcal{N}_0$ the vector subspaces of the tangent space
$\operatorname{span}\langle \mathcal{S}_{,t}\rangle, \operatorname{span}\langle \mathcal{S}_{,tt}\rangle$ and $\operatorname{span}\langle \mathcal{S}_{,1}, \ldots, \mathcal{S}_{,n-2}\rangle$
are orthogonal.

Proof of Theorem (1)

From (1.7)-(1.9) we see that at each point of $\mathcal{N}_0$ the vector subspaces of the tangent space
$\mathrm{span}\left\langle \mathcal{S}_{,t}\right\rangle, \mathrm{span}\left\langle \mathcal{S}_{,tt}\right\rangle$ and $\mathrm{span}\left\langle \mathcal{S}_{,1}, \ldots, \mathcal{S}_{,n-2}\right\rangle$
are orthogonal.
Consequently,

$$(\mathrm{D}u\left(\mathcal{S}\right)) = \alpha \mathcal{S}_{,t} + \beta \mathcal{S}_{,tt}$$

Proof of Theorem (1)

From (1.7)-(1.9) we see that at each point of $\mathcal{N}_0$ the vector subspaces of the tangent space
$\mathrm{span}\left\langle \mathcal{S}_{,t} \right\rangle, \mathrm{span}\left\langle \mathcal{S}_{,tt} \right\rangle$ and $\mathrm{span}\left\langle \mathcal{S}_{,1}, \ldots, \mathcal{S}_{,n-2} \right\rangle$
are orthogonal.
Consequently,

$$(\mathrm{D}u\,(\mathcal{S})) = \alpha \mathcal{S}_{,t} + \beta \mathcal{S}_{,tt}$$

and by (1.7)

$$\begin{aligned} \alpha &= u_{,j}\mathcal{S}_{j,t} \\ &= e^{-\lambda} \end{aligned}$$

Proof of Theorem (1)

We now have

$$(\mathrm{D}u) = e^{-\lambda}\mathcal{S}_{,t} + \beta\mathcal{S}_{,tt}$$

Proof of Theorem (1)

We now have

$$(\mathrm{D}u) = e^{-\lambda}\mathcal{S}_{,t} + \beta\mathcal{S}_{,tt}$$

By applying (1.9) and **(A5)** we obtain

$$\begin{aligned} -te^{-\lambda} &= u_{,j}\left(\mathcal{S}\right)\mathcal{S}_{j,\lambda} \\ &= e^{-\lambda}\left\langle \mathcal{S}_{,t}, \mathcal{S}_{,\lambda}\right\rangle + \beta\left\langle \mathcal{S}_{,tt}, \mathcal{S}_{,\lambda}\right\rangle \\ &= e^{-\lambda}\left\langle \mathcal{S}_{,t}, \mathcal{S}_{,\lambda}\right\rangle + \beta \end{aligned}$$

Proof of Theorem (1)

We now have

$$(\mathrm{D}u) = e^{-\lambda}\mathcal{S}_{,t} + \beta\mathcal{S}_{,tt}$$

By applying (1.9) and **(A5)** we obtain

$$\begin{aligned} -te^{-\lambda} &= u_{,j}(\mathcal{S})\,\mathcal{S}_{j,\lambda} \\ &= e^{-\lambda}\langle\mathcal{S}_{,t},\mathcal{S}_{,\lambda}\rangle + \beta\langle\mathcal{S}_{,tt},\mathcal{S}_{,\lambda}\rangle \\ &= e^{-\lambda}\langle\mathcal{S}_{,t},\mathcal{S}_{,\lambda}\rangle + \beta \end{aligned}$$

And after rewriting

$$\beta e^{\lambda} = t + \langle\mathcal{S}_{,t},\mathcal{S}_{,\lambda}\rangle \tag{1.10}$$

Proof of Theorem (1)

Differentiating (1.10) wrt t,

$$\begin{aligned} \beta_{,t}e^{\lambda} &= 1 + \langle \mathcal{S}_{,tt}, \mathcal{S}_{,\lambda}\rangle + \langle \mathcal{S}_{,t}, \mathcal{S}_{,\lambda t}\rangle \\ &= 1 - 1 + \frac{1}{2}\frac{\partial}{\partial\lambda}\left(\langle \mathcal{S}_{,t}, \mathcal{S}_{,t}\rangle\right) \\ &= 0 \end{aligned}$$

Proof of Theorem (1)

Differentiating (1.10) wrt t,

$$\begin{aligned} \beta_{,t}e^{\lambda} &= 1 + \langle \mathcal{S}_{,tt}, \mathcal{S}_{,\lambda}\rangle + \langle \mathcal{S}_{,t}, \mathcal{S}_{,\lambda t}\rangle \\ &= 1 - 1 + \frac{1}{2}\frac{\partial}{\partial\lambda}\left(\langle \mathcal{S}_{,t}, \mathcal{S}_{,t}\rangle\right) \\ &= 0 \end{aligned}$$

Since β is independent of t, evaluating (1.10) at $t = 0$ combined with (1.3) yields

$$\begin{aligned} \beta e^{\lambda} &= \langle \mathcal{S}_{,t}, \mathcal{S}_{,\lambda}\rangle (\lambda, \overline{x}, 0) \\ &= 0 \end{aligned}$$

Proof of Theorem (1)

We conclude

$$(\mathrm{D}u\,(\mathcal{S})) = e^{-\lambda}\mathcal{S}_{,t}$$

Proof of Theorem (1)

We conclude

$$(\mathrm{D}u(\mathcal{S})) = e^{-\lambda}\mathcal{S}_{,t}$$

and consequently

$$\begin{aligned} u_{,i}u_{,j}u_{,ij}(\mathcal{S}) &= e^{-2\lambda}u_{,ij}\mathcal{S}_{i,t}\mathcal{S}_{j,t} \\ &= e^{-2\lambda}u_{,ij}\mathcal{S}_{i,t}\mathcal{S}_{j,t} + e^{-2\lambda}u_{,i}\mathcal{S}_{i,tt} \\ &= e^{-2\lambda}\frac{\partial^2}{\partial t^2}(u(\mathcal{S})) \\ &= 0 \end{aligned}$$

QED

Proof of Lemma (2)

By assumption **(A4)** there is a submanifold $\mathcal{W} \subset \Omega$ such that

$$\begin{aligned} \xi_0 &\in \mathcal{W} \\ \vec{v}_\xi \cdot \vec{\chi}(\xi) &= 0 \quad \text{for all } \xi \in \mathcal{W} \text{ and } \vec{v}_\xi \in T\mathcal{W}_\xi \end{aligned}$$

Proof of Lemma (2)

By assumption **(A4)** there is a submanifold $\mathcal{W} \subset \Omega$ such that

$$\begin{aligned} \xi_0 &\in \mathcal{W} \\ \vec{v}_\xi \cdot \vec{\chi}(\xi) &= 0 \quad \text{for all } \xi \in \mathcal{W} \text{ and } \vec{v}_\xi \in T\mathcal{W}_\xi \end{aligned}$$

Define the submanifold, $\mathcal{Z}$, by $\mathcal{Z} = \Gamma_0 \cap \mathcal{W}$ where $\Gamma_0 = \Gamma_{\mathrm{F}(\xi_0)}$.

Proof of Lemma (2)

By assumption **(A4)** there is a submanifold $\mathcal{W} \subset \Omega$ such that

$$\begin{aligned} \xi_0 &\in \mathcal{W} \\ \vec{v}_\xi \cdot \vec{\chi}(\xi) &= 0 \quad \text{for all } \xi \in \mathcal{W} \text{ and } \vec{v}_\xi \in T\mathcal{W}_\xi \end{aligned}$$

Define the submanifold, $\mathcal{Z}$, by $\mathcal{Z} = \Gamma_0 \cap \mathcal{W}$ where $\Gamma_0 = \Gamma_{\mathrm{F}(\xi_0)}$.

Let $\Phi : B' \to \mathcal{Z}$ be a coordinate system for some open neighborhood of ξ_0 with $\Phi(\bar{0}) = \xi_0$; and for $G \subset \Omega \times \mathbb{R} \times \mathbb{R}$ let $\mathcal{H} : G \to \Omega$ be the solution of the first order ODE initial value problem

$$\begin{aligned} \mathcal{H}(\xi, s, \sigma) &= \xi \quad \text{if } s = \sigma \\ \mathcal{H}_{,\sigma} &= \frac{1}{\|\mathrm{DF}\|^2}(\mathrm{DF}) \end{aligned} \qquad (1.11)$$

Proof of Lemma (2)

By (1.11) and the definition of $\mathcal{W}$ we conclude

$$\begin{aligned} \frac{\partial}{\partial \sigma}\left(\mathrm{F}\left(\mathcal{H}\right)\right) &= 1 \\ \mathcal{H}(\xi, s, \sigma) &\in \mathcal{W} \quad \text{if } \xi \in \mathcal{Z} \end{aligned}$$

Proof of Lemma (2)

Now for $H \subset \Omega \times \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$ let $\mathcal{T} : H \to \Omega$ be the solution of the 2nd order ODE initial value problem

$$\begin{aligned} \mathcal{T}(\xi, \vec{\eta}, s, \sigma) &= \xi \quad \text{if } s = \sigma \\ \mathcal{T}_{,\sigma}(\xi, \vec{\eta}, s, \sigma) &= \vec{\eta} \quad \text{if } s = \sigma \\ \mathcal{T}_{,\sigma\sigma} &= -\left[\frac{\mathcal{T}_{i,\sigma} \mathrm{F}_{,ij}(\mathcal{T})\, \mathcal{T}_{j,\sigma}}{\|\mathrm{DF}\|^2}\right] (\mathrm{DF})(\mathcal{T}) \end{aligned}$$

Proof of Lemma (2)

Now for $H \subset \Omega \times \mathbb{R}^{n-2} \times \mathbb{R} \times \mathbb{R}$ let $\mathcal{T} : H \to \Omega$ be the solution of the 2nd order ODE initial value problem

$$\begin{aligned}
\mathcal{T}(\xi, \vec{\eta}, s, \sigma) &= \xi \quad \text{if } s = \sigma \\
\mathcal{T}_{,\sigma}(\xi, \vec{\eta}, s, \sigma) &= \vec{\eta} \quad \text{if } s = \sigma \\
\mathcal{T}_{,\sigma\sigma} &= -\left[\frac{\mathcal{T}_{i,\sigma}\mathrm{F}_{,ij}\left(\mathcal{T}\right)\mathcal{T}_{j,\sigma}}{\|\mathrm{DF}\|^2}\right](\mathrm{DF})\left(\mathcal{T}\right)
\end{aligned}$$

Consequently if $\vec{\eta} = \vec{\chi}$, then

$$\mathcal{T}_{,\sigma} = \vec{\chi}\left(\mathcal{T}\right)$$

Proof of Lemma (2)

Set

$$\mathcal{S}(\lambda, \overline{x}, t) = \mathcal{T}\left(\mathcal{H}\left(\Phi\left(\overline{x}\right), \lambda_0, \lambda\right), \vec{\chi}\left(\mathcal{H}\left(\Phi\left(\overline{x}\right), \lambda_0, \lambda\right)\right), 0, t\right)$$

Proof of Lemma (2)

Set

$$\mathcal{S}(\lambda, \overline{x}, t) = \mathcal{T}\left(\mathcal{H}\left(\Phi\left(\overline{x}\right), \lambda_0, \lambda\right), \vec{\chi}\left(\mathcal{H}\left(\Phi\left(\overline{x}\right), \lambda_0, \lambda\right)\right), 0, t\right)$$

Corollary (3) follows because if $\vec{\psi}$ is a another vector field satisfying **(A1)-(A4)** with F, if $\vec{\chi} = \vec{\psi}$ on Γ_0, and if $\mathcal{S}_{\vec{\psi}}$ is the map corresponding to $\vec{\psi}$ then by our construction

$$\mathcal{S}_{\vec{\psi}} = \mathcal{S}$$

Proof of Lemma (2)

Set

$$\mathcal{S}(\lambda, \overline{x}, t) = \mathcal{T}\left(\mathcal{H}\left(\Phi\left(\overline{x}\right), \lambda_0, \lambda\right), \vec{\chi}\left(\mathcal{H}\left(\Phi\left(\overline{x}\right), \lambda_0, \lambda\right)\right), 0, t\right)$$

Corollary (3) follows because if $\vec{\psi}$ is a another vector field satisfying **(A1)-(A4)** with F, if $\vec{\chi} = \vec{\psi}$ on Γ_0, and if $\mathcal{S}_{\vec{\psi}}$ is the map corresponding to $\vec{\psi}$ then by our construction

$$\mathcal{S}_{\vec{\psi}} = \mathcal{S}$$

By construction $\mathcal{S}$ is smooth and consequently it is locally a diffeomorphism.

Proof of Lemma (2)

Now, (1.1) is clear from the choice of Φ, and (1.2) follows similarly because

$$\begin{aligned}
\mathcal{S}\left(\lambda_0, \bar{0}, 0\right) &= \mathcal{T}\left(\mathcal{H}\left(\Phi\left(\bar{0}\right), \lambda_0, \lambda_0\right), \vec{\chi}\left(\mathcal{H}\left(\Phi\left(\bar{0}\right), \lambda_0, \lambda_0\right)\right), 0, 0\right) \\
&= \mathcal{H}\left(\Phi\left(\bar{0}\right), \lambda_0, \lambda\right) \\
&= \Phi\left(\bar{0}\right) \\
&= \xi_0
\end{aligned}$$

Proof of Lemma (2)

Now, (1.1) is clear from the choice of Φ, and (1.2) follows similarly because

$$\begin{aligned} \mathcal{S}\left(\lambda_0, \bar{0}, 0\right) &= \mathcal{T}\left(\mathcal{H}\left(\Phi\left(\bar{0}\right), \lambda_0, \lambda_0\right), \vec{\chi}\left(\mathcal{H}\left(\Phi\left(\bar{0}\right), \lambda_0, \lambda_0\right)\right), 0, 0\right) \\ &= \mathcal{H}\left(\Phi\left(\bar{0}\right), \lambda_0, \lambda\right) \\ &= \Phi\left(\bar{0}\right) \\ &= \xi_0 \end{aligned}$$

(1.5) in turn follows from

$$\begin{aligned} \mathcal{S}_{,t} &= \mathcal{T}_{,t} \\ &= \vec{\chi}\left(\mathcal{T}\right) \\ &= \vec{\chi}\left(\mathcal{S}\right) \end{aligned}$$

Proof of Lemma (2)

$$\mathcal{S}(\lambda, \overline{x}, 0) = \mathcal{H}(\Phi(\overline{x}), \lambda_0, \lambda)$$

which implies

$$\begin{aligned}
\mathcal{S}_{,\lambda}(\lambda, \overline{x}, 0) &= \mathcal{H}_{,\sigma}(\Phi(\overline{x}), \lambda_0, \lambda) \\
&= \frac{1}{\|\mathrm{DF}(\mathcal{H}(\Phi(\overline{x}), \lambda_0, \lambda))\|^2}(\mathrm{DF})(\mathcal{H}(\Phi(\overline{x}), \lambda_0, \lambda)) \\
&= \frac{1}{\|\mathrm{DF}(\mathcal{S}(\lambda, \overline{x}, 0))\|^2}(\mathrm{DF})(\mathcal{S}(\lambda, \overline{x}, 0))
\end{aligned}$$

which establishes (1.4) for $t = 0$.

Proof of Lemma (2)

To extend (1.4) for $t \neq 0$ note,

$$\begin{aligned} \mathcal{S}_{,t} &= \mathcal{T}_{,t} \\ &= \vec{\chi}(\mathcal{T}) \\ &= \vec{\chi}(\mathcal{S}) \end{aligned}$$

which implies

$$\begin{aligned} \frac{\partial}{\partial t}\mathrm{F}(\mathcal{S}) &= \langle \mathcal{S}_{,t}, (\mathrm{DF})(\mathcal{S})\rangle \\ &= 0 \end{aligned}$$

Proof of Lemma (2)

Considering again the case $t = 0$ we have as noted before that

$$\begin{aligned} \mathcal{S}(\lambda, \overline{x}, 0) &= \mathcal{H}(\Phi(\overline{x}), \lambda_0, \lambda) \\ \text{and consequently} & \\ \mathcal{S}(\lambda, \overline{x}, 0) &\in \mathcal{W} \\ \text{which implies that for } t = 0 & \\ \mathcal{S}_{j,i}, \chi_j(\mathcal{S}) &= 0 \quad \text{for all} \quad i = 1, \ldots, (n-2) \\ \text{and} & \\ \mathcal{S}_{j,\lambda}, \chi_j(\mathcal{S}) &= 0 \end{aligned}$$

which proves (1.3) and which proves (1.6) for $t = 0$.

Proof of Lemma (2)

To finish the proof of (1.6) for $t \neq 0$ we calculate

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\mathcal{S}_{j,i}\chi_j\left(\mathcal{S}\right)\right) &= \frac{\partial}{\partial t}\left(\mathcal{S}_{j,i}\mathcal{S}_{j,t}\right) \\
&= \mathcal{S}_{j,ti}\mathcal{S}_{j,t} + \mathcal{S}_{j,i}\mathcal{S}_{j,tt} \\
&= \frac{1}{2}\frac{\partial}{\partial x_i}\left(\mathcal{S}_{j,t}\mathcal{S}_{j,t}\right) + \mathcal{S}_{j,i}\mathcal{S}_{j,tt} \\
&= \mathcal{S}_{j,i}\mathcal{S}_{j,tt} \\
&= -\left[\frac{\mathcal{S}_{l,t}\mathrm{F}_{,lk}\left(\mathcal{S}\right)\mathcal{S}_{k,t}}{\left\|\mathrm{DF}\left(\mathcal{S}\right)\right\|^2}\right]\mathcal{S}_{j,i}\mathrm{F}_{,j}\left(\mathcal{S}\right) \\
&= 0 \quad \text{by } (1.4)
\end{aligned}$$

which proves (1.6) for $t \neq 0$.

QED

More notation and conventions

- A solution of $\Delta_\infty w = 0$ on $\Omega \subset \mathbb{R}^2$, u

More notation and conventions

- A solution of $\Delta_\infty w = 0$ on $\Omega \subset \mathbb{R}^2$, u
- A subdomain of Ω with compact closure, Ω_c

More notation and conventions

- A solution of $\Delta_\infty w = 0$ on $\Omega \subset \mathbb{R}^2$, u
- A subdomain of Ω with compact closure, Ω_c
- A connected component of the interior of a level set of $\|\mathrm{D}u\|$, $\Omega_\lambda = \{\overline{x} \in \Omega \mid \|\mathrm{D}u(\overline{x})\| < \lambda\}$

More notation and conventions

- A solution of $\Delta_\infty w = 0$ on $\Omega \subset \mathbb{R}^2$, u
- A subdomain of Ω with compact closure, Ω_c
- A connected component of the interior of a level set of $\|\mathrm{D}u\|$, $\Omega_\lambda = \{\overline{x} \in \Omega \mid \|\mathrm{D}u(\overline{x})\| < \lambda\}$
- The boundary of Ω_λ, Γ_λ

More notation and conventions

- A solution of $\Delta_\infty w = 0$ on $\Omega \subset \mathbb{R}^2$, u
- A subdomain of Ω with compact closure, Ω_c
- A connected component of the interior of a level set of $\|\mathrm{D}u\|$, $\Omega_\lambda = \{\overline{x} \in \Omega \mid \|\mathrm{D}u(\overline{x})\| < \lambda\}$
- The boundary of Ω_λ, Γ_λ
- The open disk with center $\overline{y}$ and radius r, $B(\overline{y}, r)$ or B for short

More notation and conventions

- A solution of $\Delta_\infty w = 0$ on $\Omega \subset \mathbb{R}^2$, u
- A subdomain of Ω with compact closure, Ω_c
- A connected component of the interior of a level set of $\|\mathrm{D}u\|$, $\Omega_\lambda = \{\overline{x} \in \Omega \mid \|\mathrm{D}u(\overline{x})\| < \lambda\}$
- The boundary of Ω_λ, Γ_λ
- The open disk with center $\overline{y}$ and radius r, $B(\overline{y}, r)$ or B for short
- The boundary of the open disk with center $\overline{y}$ and radius r, $\partial B(\overline{y}, r)$ or ∂B for short

Known results

Theorem (4) (Evans, Savin, Wang, Yu)

Du is continuous.

Known results

Theorem (4) (Evans, Savin, Wang, Yu)

Du is continuous.

Theorem (5) (Barron, Crandall, Gariepy, J)

u is a solution of $\Delta_\infty w = 0$ in Ω if and only if for each $\bar{y} \in \Omega$ there is a curve $\zeta : [a, b] \to \overline{\Omega}$ such that

$$0 \in (a, b),\ \zeta(0) = \bar{y};\ \textit{and}\ \ \zeta(a), \zeta(b) \in \partial\Omega \quad (2.1)$$

$$\|\zeta_{,t}\| \leq 1 \quad a.e. \quad (2.2)$$

$$\frac{d}{dt}(u(\zeta)) \geq \|Du(\bar{y})\| \quad a.e. \quad (2.3)$$

$$\frac{d}{dt}(u(\zeta))(0) = \|Du(\bar{y})\| \quad (2.4)$$

New results

Assume

$$\|\mathrm{D}u(x)\| \geq k_1 > 0 \quad \text{for all} \quad x \in \Omega_c \tag{2.5}$$

New results

Assume

$$\|\mathrm{D}u(x)\| \geq k_1 > 0 \quad \text{for all} \quad x \in \Omega_c \tag{2.5}$$

For $\overline{y} \in \Omega$ and $r < \mathrm{distance}(\overline{y}, \partial\Omega)$ set

$$\mathcal{A}(\overline{y}, r) = \left\{ \overline{x} \in \partial B(\overline{y}, r) \middle| \, \|\mathrm{D}u(\overline{x})\| = \max_{B(\overline{y}, r)} \|\mathrm{D}u\| \right\} \tag{2.6}$$

New results

Assume

$$\|\mathrm{D}u(x)\| \geq k_1 > 0 \quad \text{for all} \quad x \in \Omega_c \tag{2.5}$$

For $\overline{y} \in \Omega$ and $r < \mathrm{distance}(\overline{y}, \partial\Omega)$ set

$$\mathcal{A}(\overline{y}, r) = \left\{ \overline{x} \in \partial B(\overline{y}, r) \Big| \, \|\mathrm{D}u(\overline{x})\| = \max_{B(\overline{y},r)} \|\mathrm{D}u\| \right\} \tag{2.6}$$

Theorem (6)

There exists a constant $r_1 > 0$ such that if

1. $\overline{y} \in \Omega_c$ *and* $r < r_1$
2. $B(\overline{y}, r) \subset \Omega_\lambda \quad \textit{where} \quad \lambda = \max_{B(\overline{y},r)} \|Du\|$

then

$$|\mathcal{A}(\overline{y}, r)| \leq 2$$

New results

Theorem (7)

For the r_1 *from Theorem (6) there is an increasing function* $\rho \in \mathcal{C}([0, r_1]; [0, \infty))$ *with* $\rho(0) = 0$ *such that if*

1. $\overline{y} \in \Omega_c$ *and* $r < r_1$
2. $B(\overline{y}, r) \subset \Omega_\lambda \quad \text{where} \quad \lambda = \max_{B(\overline{y},r)} \|Du\|$
3. $|\mathcal{A}(\overline{y}, r)| = 2$

then

$$(\overline{x}_1 - \overline{y}) \cdot (\overline{x}_2 - \overline{y}) \leq -(1 - \rho(r))\, r^2 \quad \text{for every} \quad \overline{x}_1, \overline{x}_2 \in \mathcal{A}(\overline{y}, r)$$

New results

Theorem (8)

If $\overline{x} \in \Omega_c \cap \partial\Omega_\lambda \cap \partial B(\overline{y}, r)$ *and* $B(\overline{y}, r) \subset \Omega_\lambda$ *for some* $\overline{y} \in \Omega_\lambda$ *and* $r > 0$ *then there is* $\hat{r} > 0$ *such that*

$$\boxed{B(\overline{x}, \hat{r}) \setminus \Omega_\lambda \quad \textit{is convex}}$$

New results

Theorem (8)

If $\overline{x} \in \Omega_c \cap \partial\Omega_\lambda \cap \partial B(\overline{y}, r)$ *and* $B(\overline{y}, r) \subset \Omega_\lambda$ *for some* $\overline{y} \in \Omega_\lambda$ *and* $r > 0$ *then there is* $\hat{r} > 0$ *such that*

$$B(\overline{x}, \hat{r}) \setminus \Omega_\lambda \quad \textit{is convex}$$

Corollary (9)

Under the assumptions of Theorem (8)
$\partial\Omega_\lambda \cap B(\overline{x}, \hat{r})$ *is a* C^1 *curve.*

Claims

Lemma (10)

For all but a finite number of points, $\overline{x} \in \Omega_c \cap \partial\Omega_\lambda$*, there is a* $\overline{y} \in \Omega_\lambda$ *and* $r > 0$ *such that*

$$\overline{x} \in \partial B(\overline{y}, r) \subset \Omega_\lambda$$

Claims

Lemma (10)

For all but a finite number of points, $\overline{x} \in \Omega_c \cap \partial\Omega_\lambda$, there is a $\overline{y} \in \Omega_\lambda$ and $r > 0$ such that

$$\overline{x} \in \partial B(\overline{y}, r) \subset \Omega_\lambda$$

Corollary (11)

$\partial\Omega_\lambda \cap B(\overline{x}, \hat{r})$ consists of a finite number of C^1 curves.

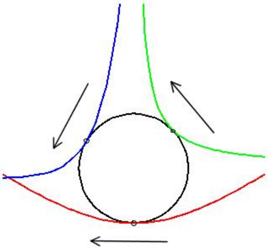
Proof of Theorems 6 and 7

Observe that the curves from Theorem (5), which we will call generalized flow curves, translate and scale invariantly. I.e., (2.1) - (2.4) remain true for the translated and scaled curve.

Proof of Theorems 6 and 7

Observe that the curves from Theorem (5), which we will call generalized flow curves, translate and scale invariantly. I.e., (2.1) - (2.4) remain true for the translated and scaled curve.

Arguing by contradiction, assume $\mathcal{A}(\bar{y}, r) \geq 3$ and thus



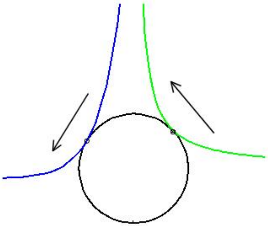
Proof of Theorems 6 and 7

The important point to note is that two of the generalized flow curves (in this case the blue and the green curves) must have the same orientation with respect to the circle.

Proof of Theorems 6 and 7

The important point to note is that two of the generalized flow curves (in this case the blue and the green curves) must have the same orientation with respect to the circle.

We will derive a contradiction to such a configuration - as depicted below - for disks with sufficiently small radii.



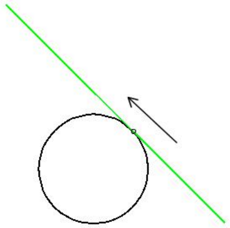
Proof of Theorems 6 and 7

Suppose we have a sequence of such disks where the radii $r_j \longrightarrow 0$ as $j \longrightarrow \infty$. Consider the corresponding scaled solutions u_j, and blue and green generalized flow curves.

Proof of Theorems 6 and 7

Suppose we have a sequence of such disks where the radii $r_j \to 0$ as $j \to \infty$. Consider the corresponding scaled solutions u_j, and blue and green generalized flow curves.

We may assume that the solutions u_j converge to a linear function and consequently the blue and green generalized flow curves must coalesce to the same straight line.

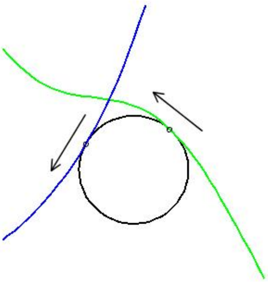


Proof of Theorems 6 and 7

We claim that the blue and green curves cannot coalesce.

Proof of Theorems 6 and 7

We claim that the blue and green curves cannot coalesce.
If they did, then for some radius we would have the following picture.

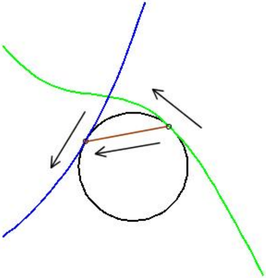


Proof of Theorems 6 and 7

This leads to a contradiction as depicted in the following picture.

Proof of Theorems 6 and 7

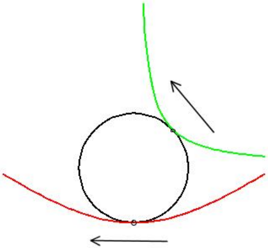
This leads to a contradiction as depicted in the following picture.



QED Theorem (6)

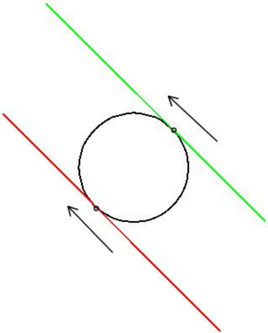
Proof of Theorems 6 and 7

If $\mathcal{A}(\bar{y}, r) = 2$, then we could have the picture



Proof of Theorems 6 and 7

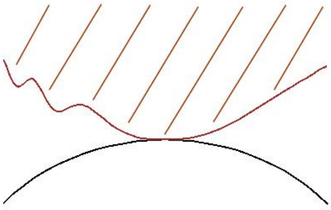
While the red and green curves don't have to coalesce, looking at the limit of scalings leads us to



QED Theorem (7)

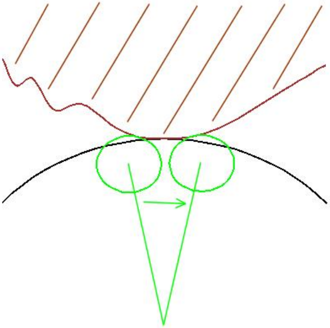
Proof of Theorem (8) and Corollary (9)

Here is the picture implied by the assumptions for Theorem (8).
The shaded area at the top is part of $\Omega \setminus \Omega_\lambda$



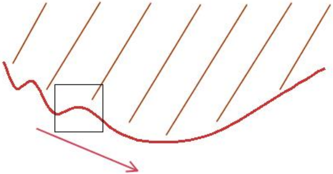
Proof of Theorem (8) and Corollary (9)

It is easy to see that the contact point of Theorem (8) cannot be on a corner, and by Theorems (6) and (7), near the contact point we can roll a disk of small radius along $\partial\Omega_\lambda$ so that the disk remains in Ω_λ and so that it only contacts $\partial\Omega_\lambda$ at only one point.



Proof of Theorem (8) and Corollary (9)

Since $\partial\Omega_\lambda$ is a $\mathcal{C}^1$ curve near the contact point of Theorem (8) it can itself be viewed as a generalized flow curve and if we assume for the sake of contradiction that $B(\overline{x}, \hat{r}) \setminus \Omega_\lambda$ is not convex then we have the situation obtaining in the boxed region of the picture below.

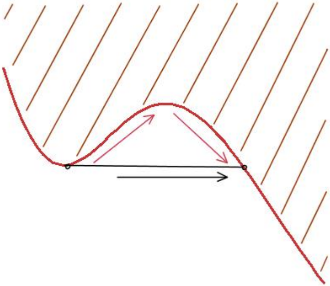


Proof of Theorem (8) and Corollary (9)

However, this leads to the contradiction depicted below.

Proof of Theorem (8) and Corollary (9)

However, this leads to the contradiction depicted below.



QED Theorem (8)