Bowen factors of Markov shifts and surface diffeomorphisms

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The Mathematical Legacy of Rufus Bowen

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Outline

The classical setting: uniform hyperbolicity

Theorem (Sinai, Bowen)

Any Axiom-A diffeomorphism $f : M \to M$ has a Markov partition with small diameter

$(f, \Omega(f))$ is a Holder continuous factor of a subshift of finite type with good properties: (i) finite-to-one; (ii) described by a relation on the alphabet:

We will now discuss the quotient map π: $\Sigma_A \to \Omega_i$ defined by $\pi(\underline{x}) = \bigcap_{j=-\infty}^{\infty} f^{-j} R_{x_j}$. The Markov partition $\mathcal{C} = \{R_1, \ldots, R_n\}$ is taken so that $2 \max_{1 \le j \le n} \operatorname{diam}(R_j)$ is less than an expansive constant. Define a relation $\sim$ on $\{1, \ldots, n\}$ by $j \sim k$ iff $R_j \cap R_k \neq \emptyset$. Define $\approx$ on Σ_A by $\underline{x} \approx \underline{y}$ iff $x_r \sim y_r$ $\forall r \in Z$. It is easy to prove that $\pi(\underline{x}) = \pi(\underline{y})$ precisely if $\underline{x} \approx \underline{y}$.

p. 13 of Bowen, *On Axiom A diffeomorphisms* (1978); Manning (1971)

Consequences

The factor is an isomorphism wrt any ergodic measure with support $\Omega(f)$

Finitely many ergodic **measures maximizing the entropy (mme)** $\mu_1, \ldots, \mu_r$

each $\mu_i \equiv$ Bernoulli $\times \mathbb{Z}/p_i\mathbb{Z}$

$|\{x = f^n x\}| \sim (p_1 + \cdots + p_r)\, e^{n\, h_{\mathrm{top}}(f)}$ for $n \in \mathrm{lcm}(p_1, \ldots, p_r)\mathbb{Z}$

In fact, $\zeta_f(z) = \prod_{\mathcal{O}} (1 - z^{|\mathcal{O}|})^{-1}$ is rational (Manning)

Goal: generalize to surface diffeomorphisms using Sarig's codings

Almost Borel classification

Definition

$S : X \to X$ and $T : Y \to Y$ are **almost Borel conjugate mod zero entropy** if $\exists$ invariant Borel subsets $X' \subset X$, $Y' \subset Y$ and a Borel isomorphism $\psi : X' \to Y'$ s.t.
(i) $\psi \circ S = T \circ \psi$; (ii) $X \setminus X'$ and $Y' \setminus Y$ carry only measures with zero entropy

Using the Bowen property of Sarig's coding and Hochman's almost Borel classification:

Theorem 1 (Boyle-B)

Any C^{1+}-diffeomorphism of a compact surface is almost Borel conjugate mod zero entropy to a Markov shift

Using "magic word" isomorphisms as between almost conjugate SFTs

Theorem 2 (B)

Let f be a C^∞-diffeomorphism of a compact surface and $0 < \chi < h_{\text{top}}(f)$
Let $\mu_1, \dots, \mu_r$ be mme's, with μ_i isomorphic to Bernoulli$\times$ $\mathbb{Z}/p_i\mathbb{Z}$
Let $p := \text{lcm}(p_1, \dots, p_r)$

$$\lim_{n\to\infty, p|n} |\{x \in M : f^n x = x,\ \chi\text{-hyperbolic}\}| e^{-n h_{\text{top}}(f)} \geq p_1 + \dots + p_r$$

Compare Sarig; Kaloshin; Burguet

Markov shifts

G oriented, countable graph with vertices $\mathcal{V}_G$ and edges $\mathcal{E} \subset \mathcal{V}_G \times \mathcal{V}_G$

Definition

The **Markov shift** defined by G is $S_G : X_G \to X_G$:

$$X_G := \{x \in \mathcal{V}_G^{\mathbb{Z}} : \forall n \in \mathbb{Z}\ x_n \stackrel{G}{\to} x_{n+1}\} \text{ with } S_G : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$$

(will drop indices G whenever possible)

Subshifts of finite type (SFT)

A Markov shift is C^0-conjugate to some SFT iff X compact iff G can be chosen finite

Theorem (Spectral decomposition)

The non-wandering set of a Markov shift (X, S) *splits into* **transitive components**

$$\Omega(X_G) = \bigsqcup_{i\in I} X_{G_i} \text{ with } S : X_{G_i} \to X_{G_i} \text{ (topologically) transitive}$$

Furthermore,

$$X_{G_i} = \bigsqcup_{j=0}^{p_i-1} S^j(Y_i) \text{ with } S^{p_i} : Y_i \to Y_i \text{ topologically mixing}$$

$(G_i)_{i\in I}$ *are the strongly connected components of* G *and* p_i *are their periods*

Markov shifts – entropy

$h(S,\nu)$ Kolmogorov-Sinai entropy

Theorem (Gurevič)

For a Markov shift S, the Borel entropy

$$h(S) := \sup_{\mu\in\mathbb{P}(S)} h(S,\mu) \in [0,\infty]$$

is the upper growth rate of the periodic orbits through a given vertex

Definition

An **mme** (ergodic invariant probability measure maximizing the entropy) is

$$\mu \in \mathbb{P}_{\mathrm{erg}}(S) \text{ such that } h(S,\mu) = \sup_{\nu\in\mathbb{P}(S)} h(S,\nu)$$

Theorem (Gurevič)

If X is transitive then it has at most one mme μ

In this case, X is called **positive recurrent (PR)** *and μ is (fully-supported, Markov) Parry measure, isomorphic to Bernoulli $\times\ \mathbb{Z}/p\mathbb{Z}$ (or simply:* **periodic-Bernoulli***)*

Markov shifts – periodic points

Markov shift $S : X \to X$ defined by graph G
Assume **transitive** with $h(S) < \infty$ and period p

Classical positive matrix theory yields:

Theorem

$[F] := \{x \in X : x_0 \in F\}$ *for some finite set* $F \neq \emptyset$ *of vertices of* G
$\mathrm{Fix}(S^j, F) := \{x \in X : S^j x = x$ *and* $\mathcal{O}(x) \cap [F] \neq \emptyset\}$
If G *is not PR,* $\lim_{k\to\infty} |\mathrm{Fix}(S^{kp}, F)| e^{-kp\, h(S)} = 0$
If G *is PR,* $\lim_{k\to\infty} |\mathrm{Fix}(S^{kp}, F)| e^{-kp\, h(S)} = p$

Counter-examples

Without restricting to a finite set of vertices:
- $\mathrm{Fix}(S^{kp}, G)$ can be infinite
- $\mathrm{Fix}(S^{kp}, G)$ can be finite but grow arbitrarily fast

Factors of Markov shifts – pathology from loss of entropy

(X, S), (Y, T) selfmaps

Definition

A **factor map** $\pi : (X, S) \to (Y, T)$ is an onto map $\pi : X \to Y$ with $\pi \circ S = T \circ \pi$

$S : X \to X$ is called the extension and $T : Y \to Y$ the factor

Claim: Without additional assumptions, their factors can be very different

Pathology 1 Bad MMEs

MMEs of Markov shifts

A Markov shift has at most countably many mme's and each is periodic-Bernoulli (*1)

Counter-example of Boyle-B

There are continuous factors of mixing SFTs whose mme's include *uncountably* many isomorphic copies of an *arbitrary ergodic automorphism* with positive entropy

Remark (Sarig, applying Ornstein's theory)

Any finite-to-one, C^0 factor of a Markov shift satisfies (*1)

Factors of Markov shifts – pathology of finite-to-one factors

Pathology 2 Finite-to-one factors can still be bad *"at a period"*

μ totally ergodic $\iff$ $\mathrm{Per}(S, \mu) = \{1\}$

Counter-example of Boyle-B

There is a finite-to-one, continuous factor of an SFT which has a unique totally ergodic measure with nonzero entropy

Compare:

Remark

A Markov shift has infinitely many totally ergodic measures with nonzero entropy, or none

The Bowen relation

$\pi : (X, S) \to (Y, T)$ factor map with

(X, S) a symbolic system: $X \subset \mathcal{A}^{\mathbb{Z}}$ and $S : (x_n)_{n\in\mathbb{Z}} \mapsto (x_{n+1})_{n\in\mathbb{Z}}$

(Y, T) arbitrary

Let $[a] := \{x \in X : x_0 = a\}$ for $a \in \mathcal{A}$

Bowen relation (Boyle-B)

The **Bowen relation** of π is the symmetric relation over $\mathcal{A}$ defined by:

$$a \sim b \iff \pi([a]) \cap \pi([b]) \neq \emptyset$$

It is of **finite type** if $|\{b \in \mathcal{A} : a \sim b\}| < \infty$ for each $a \in \mathcal{A}$

The factor $\pi : X \to Y$ has the **Bowen property** if, for all $x, x' \in X$

$$\pi(x) = \pi(x') \iff \forall n \in \mathbb{Z}\ x_n \sim x'_n$$

Recall Bowen *On Axiom A diffeomorphisms* (1978):

We will now discuss the quotient map $\pi\colon \Sigma_A \to \Omega_i$ defined by $\pi(\underline{x}) = \bigcap_{j=-\infty}^{\infty} f^{-j} R_{x_j}$. The Markov partition $\mathcal{C} = \{R_1, \ldots, R_n\}$ is taken so that $2 \max_{1\le j\le n} \operatorname{diam}(R_j)$ is less than an expansive constant. Define a relation $\sim$ on $\{1, \ldots, n\}$ by $j \sim k$ iff $R_j \cap R_k \neq \emptyset$. Define $\approx$ on Σ_A by $\underline{x} \approx \underline{y}$ iff $x_r \sim y_r$ $\forall r \in Z$. It is easy to prove that $\pi(\underline{x}) = \pi(\underline{y})$ precisely if $\underline{x} \approx \underline{y}$.

Bowen factors of finite type

Examples

1. The coding of an Axiom A diffeomorphism induced by **Markov partitions** defines a finite-to-one Bowen factor

2. Expansive continuous factors, in particular of SFTs (ie, Fried's **finitely presented systems**)

3. Any **one-block code** between two symbolic systems is a Bowen factor.
Note: need not preserve entropy, even if of finite type.

$\#$-recurrent set (Sarig)

$X^{\#}$ is the set of $x \in X$ s.t. $|\{n \leq 0 : x_n = a\}| = |\{n \geq 0 : x_n = b\}| = \infty$ for some a, b

Theorem (Sarig)

Given a surface C^{1+}-diffeomorphim, and $\chi > 0$, let $\pi_\chi : X_\chi \to M_\chi$ be Sarig's Hölder continuous factor map with X_χ a Markov shift and M_χ its χ-hyperbolic part

π_χ restricted to $X_\chi^{\#}$ is finite-to-one **and a Bowen factor of finite type** (Boyle-B)

Almost Borel conjugacy to a Markov shift

(X, S) a Markov shift and $(X_i)_{i \in I}$ be its spectral decomposition

Theorem (Boyle-B)

Let $\pi : (X, S) \to (Y, T)$ be a Borel factor of a Markov shift with $h(S) < \infty$

Assume for all $i \in I$: the restriction $\pi | X_i^{\#}$ is finite-to-one with the Bowen property

Then (Y, T) is almost Borel conjugate modulo zero entropy to a Markov shift

Main ingredients of the proof

- Hochman's almost Borel generator theorem
- Countable unions of Markov shifts are almost Borel conjugate to Markov shifts, etc.
- Low entropy part (injectivity from marker lemma)
- Top entropy part (a.e. injectivity a la Manning)

Theorem (Boyle-B)

Any C^{1+}-diffeomorphism of a compact surface is almost Borel conjugate mod zero entropy to a Markov shift

Almost Borel classification of C^{1+}-diffeos

We are reduced to the classification of Markov shifts (Boyle-B)

We need the **set of periods** of a measure

$$\mathrm{Per}(f,\mu) := \{p \geq 1 : e^{2i\pi/p} \in \sigma_{rat}(f,\mu)\} \text{ for } \mu \in \mathbb{P}_{\mathrm{erg}}(f,\mu)$$

and to maximize entropy *at a period*:

Corollary (Boyle-B)

For each $p \geq 1$ let:

- $H(p) := \sup^+\{h(f,\mu) : \mu \in \mathbb{P}_{\mathrm{erg}}(f),\ \max \mathrm{Per}(f,\mu)|p\}$
- $M(p) := |\{\mu \in \mathbb{P}_{\mathrm{erg}}(f) : \max \mathrm{Per}(f,\mu) = p,\ h(f,\mu) = H(p)\}|$

Then (H, M) is a complete invariant of almost Borel conjugacy mod zero entropy among C^{1+}-diffeomorphisms f of compact surfaces

Almost Borel classification for C^∞ diffeos

f C^∞-diffeo of a compact surface with $h_{\text{top}}(f) > 0$: **purely topological** invariant

Definition

The **homoclinic class** of a hyperbolic periodic orbit $\mathcal{O}$ is

$$HC(\mathcal{O}) := \overline{W^s(\mathcal{O}) \pitchfork W^u(\mathcal{O})}$$

It has **period** $p \geq 1$ if $HC(\mathcal{O}) = \bigcup_{k=0}^{p-1} f^k(A)$ and

$\text{int}_{HC(\mathcal{O})}(A \cap f^k(A)) = \emptyset$ for $0 < k < p$ and $f^p : A \to A$ topologically mixing

Corollary of B-Crovisier-Sarig

Let $(HC(\mathcal{O}_j))_{j\in J}$ be the distinct homoclinic classes and p_j their periods
The previous complete invariant (H, M) of almost Borel conjugacy mod zero entropy satisfies

- $H(p) := \sup^+_{j:p_j|p} h_{\text{top}}(f, HC(\mathcal{O}_j))$
- $M(p) := |\{j \in J : h_{\text{top}}(f, HC(\mathcal{O}_j)) = H(p), p_j = p\}|$

Example

Among top. mixing surface C^∞ diffeos, the topological entropy is a complete invariant for almost Borel conjugacy mod zero entropy (in fact Borel conjugacy mod periodic points)

Lower bounds on periodic points

As for almost conjugacy between SFTs, we have *magic word isomorphisms*

Theorem (B, in preparation)

Let $\pi : (X, S) \to (Y, T)$ be a Borel factor of a transitive Markov shift with $h(T) < \infty$
Assume the restriction $\pi|X^{\#}$ is finite-to-one and a Bowen factor of finite type
Then $\exists$ X-word w s.t. $X_w := \{x \in X : w$ occurs i.o. in the past and future$\}$ satisfies:

$$\exists 1 \leq d < \infty \; \forall x \in X_w \; |\pi^{-1}(\pi(x)) \cap X_w| = d$$

Proof: $\deg_\pi(v_1 \dots v_n, i) := |\{u_i : u \in \mathcal{A}^n, \; u_1 \sim v_1, \dots, u_n \sim v_n\}|$

Theorem (B)

Let f be a C^∞-diffeomorphism of a compact surface and $0 < \chi < h_{\text{top}}(f)$
Let $\mu_1, \dots, \mu_r$ be its mme's, with μ_i isomorphic to Bernoulli$\times \mathbb{Z}/p_i\mathbb{Z}$
Let $p := \text{lcm}(p_1, \dots, p_r)$

$$\lim_{n\to\infty, p|n} |\{x \in M : f^n x = x, \; \chi\text{-hyperbolic}\}| e^{-n h_{\text{top}}(f)} \geq p_1 + \dots + p_r$$

Conclusion

The Bowen property gives good control of the (necessary?) failure of injectivity

Number of periodic points

Are there f compact surface C^∞-diffeo and $\chi > 0$ s.t.

$$\limsup_{n\to\infty} |\{x = f^n x, \chi\text{-hyperbolic}\}| e^{-nh_{\text{top}}(f)} = \infty?$$

True Borel conjugacy

For a C^∞-diffeo, each $HC(\mathcal{O})$ is almost Borel conjugate *mod zero entropy* to a transitive Markov shift

Can this be strengthened to almost Borel *mod periodic points*?

The dream coding

Is the (χ??)hyperbolic part of a surface diffeomorphism a factor of a Markov shift s.t.

1. the factor is Hölder-continuous
2. each "piece" (say homoclinic class) is coded by a single transitive component
3. the factor is 1-to-1 (wrt ergodic measures fully-supported on their "piece"??)