Nonlinear singular integral equations and approximation of p-Laplace equations

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INTRODUCTION

We consider the Dirichlet problem

$$\text{(DI)} \qquad \begin{cases} M[u](x) = f(x) & \text{in } \Omega, \\ u(x) = g(x) & \text{for } x \in \partial\Omega, \end{cases}$$

Here

$\Omega \subset \mathbb{R}^n$ a bounded domain,

$f \in C(\Omega, \mathbb{R})$, $g \in C(\partial\Omega, \mathbb{R})$ given functions,

$u \in C(\overline{\Omega}, \mathbb{R})$ the unknown function,

$$M[u](x) := \text{p.v.} \int_{B(0,\rho(x))} \frac{p-\sigma}{|z|^{n+\sigma}} \times$$
$$\times\, |u(x+z)-u(x)|^{p-2}(u(x+z)-u(x))\,\mathrm{d}z,$$
$$\rho(x) := \operatorname{dist}(x, \partial\Omega), \qquad 1 < p < \infty, \qquad 0 < \sigma < p.$$

Questions:

The solvability of the Dirichlet problem (DI),

The asymptotic behavior of solutions u_σ of (DI) as $\sigma \to p$.

Viscosity solutions approach to (DI)

Space $\mathcal{T}_p(\Omega)$

- $\mathcal{T}_p(\Omega) = C^2(\Omega)$ if $p \geq 2$.
- $\mathcal{T}_p(\Omega)$ for $1 < p < 2$ denotes the space of functions $\phi \in C^2(\Omega)$ having the property:
  for each compact $R \subset \Omega$ there exist a neighborhood $V \subset \Omega$ of R and constants $\beta > 1/(p-1)$ and $A > 0$ such that for any $y \in R$, if $D\phi$ vanishes at y, then

$$|\phi(x) - \phi(y)| \leq A|x-y|^{\beta+1} \quad \text{for all } x \in V.$$

Any bounded function u in Ω is said to be a (viscosity) subsolution of (DI) if

$$M^+[u^*](x) \geq f(x)$$

whenever $(x, \phi) \in \Omega \times \mathcal{T}_p(\Omega)$ and $u^* - \phi$ has a maximum at x.

- The operator M^+ is defined by

$$M^+[v](x) = \limsup_{\delta \to 0+} \int_{\delta < |z| < \rho(x)} G(v(x+z) - v(x)) K(z)\, \mathrm{d}z,$$

where

$$G(r) := |r|^{p-2} r \quad \text{and} \quad K(z) = \frac{p - \sigma}{|z|^{n+\sigma}},$$

- u^* denotes the upper semicontinuous envelope of u.

- u supersolution

$$M^-[u_*](x) \leq f(x),$$

wherever $\phi \in \mathcal{T}_p(\Omega)$ and $u_* - \phi$ attains a minimum at x, where

$$M^-[v](x) = \liminf_{\delta \to 0+} \int_{\delta < |z| < \rho(x)} G(v(x+z) - v(x))K(z)\,\mathrm{d}z$$

and u_* denotes the lower semicontinuous envelope of u.

- u solultion $\iff$ u subsolution & supersolution.

MAIN RESULTS

- Ω satisifes the uniform exterior sphere (UES for short) condition if and only if

$$\exists R > 0,\ \forall y \in \partial\Omega,\ \exists z \in \mathbb{R}^n,\ B(z, R) \cap \overline{\Omega} = \{y\}.$$

THEOREM 1

Let $f \in C(\overline{\Omega})$. Assume that $0 < \sigma < p$ and that Ω satisfies UES condition. Then there exists a unique solution of (DI).

Set

$$\nu = \frac{\pi^{\frac{n-1}{2}}\Gamma(\frac{p+1}{2})}{\Gamma(\frac{n+p}{2})},$$

and consider the p-Laplace equation with the Dirichlet data

$$(\mathrm{DpL}) \qquad \begin{cases} \nu\Delta_p u(x) = f(x) & \text{in } \Omega, \\ u = g & \text{on } \partial\Omega. \end{cases}$$

Recall that

$$\Delta_p v(x) = \text{div}\left(|Dv(x)|^{p-2} Dv(x)\right),$$

and

$$K(z) = \frac{p-\sigma}{|z|^{n+\sigma}}, \quad \text{with } \sigma < p.$$

THEOREM 2

Assume that Ω satisfies UES condition. Let u_σ be the solution of (DI). Let v be the solution of (DpL). Then

$$u_\sigma(x) \to v(x) \quad \text{uniformly on } \Omega \quad \text{as } \sigma \to p.$$

- Recently there has been much interest in integro-differential equations. Caffarelli-Silvestre, Silvestre, Barles, Forcadel, Monneau, Imbert,.... Most results are concerned integral operators with Lipschitz continuous G. The paper by Caffarelli-Silvestre has drawn our attention to the convergence question taken in Theorem 2. The generator of Levy processes in mathematical finance, nonlocal front propagations,...
- Regarding Theorem 2, Andreu-Mazon-Rossi-Toledo have studied similar problems with the Dirichlet condition and the Neumann condition. They study integral equations with continuous kernels. The p-Laplace equations appear in the scaling limit as $\varepsilon \to 0+$ and

$$K(z) = \frac{1}{\varepsilon^{p+n}} J(|z|/\varepsilon), \quad \text{where} \quad J \in C_0([0, \infty)).$$

- It is not clear right now if problem (NI), the problem (DI) where the Neumann condition $\partial u/\partial n$ replaces the Dirichlet condition, has a conclusion similar to Theorems 1 and 2.
- If $\sigma > 0$, then (DI) can be solved even with $f \in C(\Omega)$ having the property that $\lim_{x\to\partial\Omega} |f(x)| = \infty$. On the other hand, if $\sigma < 0$, then the solvability of (DI) is guaranteed only when $\lim_{x\to\partial\Omega} f(x) = 0$ with an appropriate convergence rate.
- In the definition of M, one may replace the domain of integration, $B(x, \rho(x))$, by some other choices. For instance, if we replace $B(x, \rho(x))$ by $B(x, \lambda\rho(x))$, with $0 < \lambda < 1$, then we still have the same conclusions as Theorems 1 and 2 except the uniqueness assertion of Theorem 1.

- **It is interesting to see which p-Laplace equation we get when $K(z)$ is replaced by $(p-\sigma)/\|z\|^{p-\sigma}$, where $\|z\|$ is a norm of $\mathbb{R}^n$.**

Associated with the integral equation $M[u] = f$ is the following interacting particle system: fix any $\varepsilon > 0$, set $K^\varepsilon(x) = \varepsilon^n K(\varepsilon x)$, and consider the system of ODE

$$\dot{v}^\varepsilon(k,t) = \sum_{j \in \mathbb{Z}^n \setminus \{0\}} K^\varepsilon(j) G\big(v^\varepsilon(k+j,t) - v^\varepsilon(k,t)\big), \; k \in \mathbb{Z}^n,$$

where $v^\varepsilon : \mathbb{Z}^n \times [0,\infty) \to \mathbb{R}$ is the unknown.
If we define $u^\varepsilon(x,t) = v^\varepsilon(\lfloor x/\varepsilon \rfloor, t)$ and set $u(x,t) = \lim_{\varepsilon \to 0} u^\varepsilon(x,t)$, then the function u should solve the integral equation

$$M[u] = u_t \quad \text{in } \mathbb{R}^n \times (0,\infty).$$

Stability Results

We are concerned with

$$\text{(I)} \qquad M[u](x) = f(x) \quad \text{in } \Omega.$$

THEOREM 3

Let $\mathcal{S}_0$ be a non-empty set of subsolutions of (I). Assume that the family $\mathcal{S}_0$ is uniformly bounded on Ω. Define the function $u : \Omega \to \mathbb{R}$ by

$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}_0\}.$$

Then the function u is a subsolution of (I).

THEOREM 4

Let $\{u_n\}$ be a sequence of subsolutions of (I). Assume that the collection $\{u_n\}$ is uniformly bounded on Ω. Define $u : \Omega \longrightarrow \mathbb{R}$ by

$$u(x) = \lim_{k\to\infty} \sup\{u_n(y) \,|\, y \in B(x, k^{-1}),\ n \geq k\}.$$

Then u is a subsolution of (I).

Let

$$\psi^- \in \mathcal{S}^-(\Omega) \cap \mathrm{LSC}(\Omega) \quad \text{and} \quad \psi^+ \in \mathcal{S}^+(\Omega) \cap \mathrm{USC}(\Omega).$$

Here

$$\begin{aligned} \mathcal{S}^- &= \text{ the space of subsolutions ,} \\ \mathcal{S}^+ &= \text{ the space of supersolutions.} \end{aligned}$$

Assume that $\psi^- \leq \psi^+$ in Ω. Set

(P)

$$u(x) = \sup\{v(x) \mid v \in \mathcal{S}^-(\Omega),\ \psi^- \leq v \leq \psi^+ \ \text{ in } \ \Omega\}.$$

Note that $u : \Omega \longrightarrow \mathbb{R}$ is bounded.

THEOREM 5

The function u given by (P) is a solution of (I).

LEMMAS

Let $0 < \delta < \rho(x)$ and set

$$M_\delta^+[\phi](x) = \limsup_{\varepsilon \to 0+} \int_{\varepsilon < |z| < \delta} G(\phi(x+z) - \phi(x)) K(z)\, \mathrm{d}z.$$

LEMMA 1

Assume that $p \geq 2$ and that there are a vector $q \in \mathbb{R}^n$ and a constant $C > 0$ such that

$$u(x+z) - u(x) \leq q \cdot z + C|z|^2 \qquad \text{for } z \in B(0,\, \delta).$$

Then there is a constant $C_1 > 0$, depending only on n, such that

$$M_\delta^+[u](x) \leq C_1 C(|q| + \delta C)^{p-2} \delta^{p-\sigma}.$$

LEMMA 2

Assume that $1 < p < 2$ and there are a vector $q \in \mathbb{R}^n \setminus \{0\}$ and a constant $C > 0$ such that

$$u(x+z) - u(x) \le q \cdot z + C|z|^2 \quad \text{for } z \in B(0,\, \delta).$$

Then there is a constant $C_1 > 0$, depending only on p and n, such that

$$M_\delta^+[u](x) \le C_1 C |q|^{p-2} \delta^{p-\sigma}.$$

Let $1 < p < 2$ and $\beta > 1/(p-1)$. Set

$$\phi(x) = |x|^{\beta+1}.$$

LEMMA 3

There is a constant $C_1 > 0$ depending only on β, p and n such that for any $x \in B(0,\delta)$,

$$M_\delta^+[\phi](x) \leq C_1 \delta^{(\beta+1)(p-1)-\sigma}.$$

Remark that $\phi \in \mathcal{T}_p(\mathbb{R}^n)$ and

$$(\beta+1)(p-1) - \sigma > 1 + (p-1) - \sigma = p - \sigma > 0.$$

LEMMA 4

Let $\delta > 0$, $\{x_k\} \subset \Omega$ and $x_0 \in \Omega$. Let $\{u_k\}$ be a sequence of bounded measurable functions on Ω and u a bounded measurable function on Ω. Assume that $\{u_k\}$ is uniformly bounded on Ω and $(x_k, u_k(x_k)) \to (x_0, u(x_0))$ as $k \to \infty$. Assume that for $z \in \Omega$,

$$\lim_{j\to\infty} \sup\{u_k(y) \mid y \in B(z, j^{-1}) \cap \Omega,\ k \geq j\} \leq u(z).$$

Then

$$\limsup_{k\to\infty} \int_{B(0,\rho(x_k))\setminus B(0,\delta)} G(u_k(x_k+z) - u_k(x_k))K(z)\,\mathrm{d}z \leq \int_{B(0,\rho(x_0))\setminus B(0,\delta)} G(u(x_0+z) - u(x_0))K(z)\,\mathrm{d}z.$$

COMPARISON

THEOREM 6

Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be a subsolution and a supersolution of (I), respectively. Assume that $u \leq v$ on $\partial\Omega$ and u and v are bounded on $\overline{\Omega}$. Then $u \leq v$ in Ω.

- **PROOF**

We suppose $m := \max(u - v) > 0$ and get a contradiction. Let $x \in \Omega$ be a maximum point. We have

$$(u - v)(z) \leq m = (u - v)(x) \quad \text{for } z \in \Omega$$

and hence

$$u(z) - u(x) \leq v(z) - v(x) \quad \text{for } z \in \Omega.$$

Let $U = \{y \in \Omega \mid \operatorname{dist}(y, \partial\Omega) < \varepsilon\}$, with a small $\varepsilon > 0$.
We may assume that

$$(u - v)(z) \leq \frac{m}{2} \left(= (u - v)(x) - \frac{m}{2} \right) \quad \text{for } z \in U.$$

Consequently,

$$u(z) - u(x) \leq v(z) - v(x) - \frac{m}{2} \quad \text{for } z \in U.$$

Then, formally,

$$\begin{aligned}
f(y) &\leq \int_{B(y,\rho(y))} G(u(z)-u(y))K(y-z)\,\mathrm{d}z \\
&= \left(\int_{B(y,\rho(y))\cap U} + \int_{B(y,\rho(y))\setminus U} \right) \cdots \,\mathrm{d}z \\
&\leq \int_{B(y,\rho(y))\cap U} G(v(z)-v(y)-m/2)K(y-z)\,\mathrm{d}z \\
&\quad + \int_{B(y,\rho(y))\setminus U} G(v(z)-v(y))K(y-z)\,\mathrm{d}z \\
&< \int_{B(y,\rho(y))} G(v(z)-v(y))K(y-z)\,\mathrm{d}z = f(y).
\end{aligned}$$

BARRIER

THEOREM 7

Let $\sigma > 0$ and $f \in C(\overline{\Omega})$. Assume that Ω satisifes UES condition. There exist functions $\psi^+ \in \mathrm{USC}(\overline{\Omega})$ and $\psi^- \in \mathrm{LSC}(\overline{\Omega})$ such that ψ^+ (resp., ψ^-) is a supersolution (resp., subsolution) of (I), $\psi^- \leq \psi^+$ on $\overline{\Omega}$ and $\psi = g$ on $\partial\Omega$.

Remark. **For fixed p and $0 < \sigma_0 < p$, as far as $\sigma_0 < \sigma < p$, the functions $\psi^{\pm}$ can be chosen independently of σ.**

- **When $g \in C^2(\overline{\Omega})$, we construct the functions $\psi^{\pm}$ by setting**

$$\psi^{\pm}(x) = g(x) \pm A \,\mathrm{dist}\,(x, \partial\Omega)^{\varepsilon}$$

near the boundary, $\partial\Omega$, where $\varepsilon > 0$ is chosen sufficiently small and $A > 0$ sufficiently large.

CONVERGENCE

We consider the p-Laplace equation

$$\text{(pL)} \qquad \nu\Delta_p u(x) = f(x) \quad \text{in } \Omega.$$

THEOREM 8

Assume that Ω satisfies UES condition. Then there is a (unique) weak solution $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\overline{\Omega})$ of (DpL).

THEOREM 9

Any weak subsolution (reps., supersolution) $u \in W^{1,p}_{\text{loc}}(\Omega) \cap C(\Omega)$ of (pL) is a viscosity subsolution (resp., supersolution) of (pL).

THEOREM 10

Assume that Ω satisfies UES condition. Let $u \in \mathrm{USC}(\overline{\Omega})$ and $v \in \mathrm{LSC}(\overline{\Omega})$ be, respectively, viscosity sub and supersolutions of (pL). Assume that $u \leq v$ on $\partial\Omega$. Then $u \leq v$ in Ω.

- **OUTLINE OF THE PROOF OF CONVERGENCE**

The half relaxed limits $u^{\pm}$ of u_σ are defined by

$$u^{+}(x) = \lim_{\varepsilon \to 0+} \sup\{u_\sigma(y) \mid y \in B(x,\varepsilon) \cap \overline{\Omega},\ p - \varepsilon < \sigma < p\},$$

$$u^{-}(x) = \lim_{\varepsilon \to 0+} \inf\{u_\sigma(y) \mid y \in B(x,\varepsilon) \cap \overline{\Omega},\ p - \varepsilon < \sigma < p\}.$$

We show that u^+ and u^- are sub and supersolution of (pL), respectively. The existence of barrier functions for (DpL) yields

$$u^+ = u^- \quad \text{on } \partial\Omega.$$

By comparison (Theorem 10), we find that $u^+ = u^-$ on $\overline{\Omega}$, which implies that the uniform convergence of u_σ to the unique solution of (DpL).

- **THE CONSTANT ν**

Suppose that $u_\sigma(x) \approx \phi(x)$ near $x = 0 \in \Omega$ for a fixed $\phi \in C^2$ and that $D\phi(0) = (0, ..., 0, q)$. Set $A = (a_{ij}) := D^2\phi(0)$. For $z \approx 0$, compute that

$$
\begin{aligned}
G(\phi(z) - \phi(0)) \approx & G(qz_n + \frac{1}{2}Az \cdot z) = G(qz_n)G(1 + \frac{Az \cdot z}{2qz_n}) \\
\approx & G(qz_n)(1 + G'(1)\frac{Az \cdot z}{2qz_n}) \\
= & G(qz_n) + \frac{p-1}{2}|qz_n|^{p-2}Az \cdot z,
\end{aligned}
$$

For $\delta > 0$ sufficiently small, we compute

$$M[u_\sigma](0) \approx M[\phi](0)$$
$$\approx \int_{B(0,\delta)} (G(qz_n) + \frac{p-1}{2}|qz_n|^{p-2} Az \cdot z) K(z)\, \mathrm{d}z$$

by symmetry,

$$= \frac{(p-1)(p-\sigma)|q|^{p-2}}{2} \sum_{i=1}^{n} \int_{B(0,\delta)} a_{ii} z_i^2 |z_n|^{p-2} |z|^{n+\sigma}\, \mathrm{d}z.$$

Thus,

$$M[u_\sigma](0) \approx \nu |q|^{p-2} \delta^{p-\sigma} \Big(\sum_{i<n} a_{ii} + (p-1) a_{nn} \Big).$$

Sending $\sigma \to p$, we get

$$M_\sigma[u_\sigma](0) \to M_\sigma[\phi](0) = \nu|q|^{p-2}\Big(\sum_{i<n} a_{ii} + a_{nn}\Big).$$

On the other hand, we have

$$\begin{aligned}\Delta_p\phi(0) =& |q|^{p-2}\Delta\phi(0) + (p-2)|q|^{p-4}q^2 a_{nn}\\ =& |q|^{p-2}\sum_i a_{ii} + (p-2)|q|^{p-2}a_{nn}\\ =& |q|^{p-2}\Big(\sum_{i<n} a_{ii} + (p-1)a_{nn}\Big).\end{aligned}$$