Motion by fractional mean curvature

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Analysis of nonlinear PDEs
and free boundary problems:
Applications to homogenization

Acknowledgment.

The results presented here are published in Interfaces and Free Boundaries.

This paper was written after the series of papers written with R. Monneau and after visiting T. Souganidis in Austin. In particular, I had the opportunity to discuss with L. Caffarelli about this subject.

It is closely related to their joint work about Threshold dynamics associated with non-local diffusions (Archive for Rational Mechanics and Analysis) and to the ongoing work with T. Souganidis about phasefield theory for fractional diffusion-reaction equations. See also the working paper of Caffarelli, Roquejoffre and Savin about non-local minimal surfaces.

Outline

Outline

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Interfaces

In this talk, interface = hypersurface separating two regions of $\mathbb{R}^N$

Examples of interfaces

- In a polycrystalin material,
  2 stable zones (phases) separated by 1 instable zone (transition layer)
- In combustion, burnt region / unburnt region
- In a biology, infected region / sane region
- Linear defect in a crystal

Different problems

- Study of interfaces at equilibrium (ex: minimal surfaces)
- Study of dynamics of interfaces

Outline

Geometrical law

If V = speed along the normal n at point x at time t

$$V = F(t, x, n, S, \dots)$$

where S = curvature tensor of the interface at x

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Strategy for constructing a flow for a given law

- Represent the inner region Ω_t and the interface $\Gamma_t = \partial\Omega_t$ as follows

$$\Gamma_t = \{x \in \mathbb{R}^N : u(t,x) = 0\} \quad \& \quad \Omega_t = \{x \in \mathbb{R}^N : u(t,x) > 0\}.$$

- Exhibit an equation satisfied by u
- Solve the PDE
- Check the invariance principle

Osher-Sethian, Evans-Spruck, Chen-Giga-Goto

Motion by mean curvature

A simple example of geometric motion

$$V = \mathrm{Tr}(S) = \mathrm{Tr}(Dn)$$

where $S = Dn$

The geometric PDE

$$\partial_t u - \Delta u + \frac{D^2 u Du \cdot Du}{|Du|^2} = 0\,.$$

- This motion is local
- Convex sets move faster and faster along their normal
- Lines do not move (minimal surfaces)

Outline

The phasefield approach

Allen-Cahn equation

$$\varepsilon^2 \partial_t u^\varepsilon - \Delta u^\varepsilon + u^\varepsilon((u^\varepsilon)^2 - 1) = 0$$

- $u^\varepsilon(t,x) \to \pm 1$
- $\partial \Omega_t = \partial\{x : u^\varepsilon(t,x) \to 1\}$: moving front

Allen-Cahn, Chen, Evans-Soner-Souganidis

Dislocation dynamics

$$\partial_t u + (-\Delta)^{1/2} u + u(u^2 - 1) = 0$$

Recall: The fractional Laplacian

$$(-\Delta)^{1/2} u = \mathcal{F}^{-1}(|\xi|^1 \mathcal{F} u)$$

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Fractional Allen-Cahn equation

$$\partial_t u + (-\Delta)^{\alpha/2} u + u(u^2 - 1) = 0$$

Recall: The fractional Laplacian

$$(-\Delta)^{\alpha/2} u = \mathcal{F}^{-1}(|\xi|^\alpha \mathcal{F} u) \qquad \text{with } \alpha \in (0,2)$$

Integro-PDEs in applications and literature

elliptic/parabolic + nonlinear + singular integral terms

A increasing litterature

- Continuum mechanics: dislocation dynamics
- Combustion models
- Mathematical finance: stochastic control of jump processes
- fluid mechanics: the quasi-geostrophic model, Boussinesq equation
- statistical mechanics: mean field equation for stochastic Ising models
- Biology, plasmas *etc*

April 2008 (Banff)

April 2010

Workshop in Bedlewo at Banach Center

The fractional Laplacian: a typical example of (monotone) singular integral operator

$$(-\Delta)^{1/2}u(x) = -C_1 \int \Big(u(x+z) - u(x) \qquad\qquad \Big) \frac{dz}{|z|^{N+1}}$$

()

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- Well defined for $u \in C^2$

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- This operator differentiates α times the function u
- Well defined for $u \in C^2$
- The singularity at 0 is compensated
- The singularity is assumed to be of order less than 2

Outline

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Definition (Fractional mean curvature)

(Caffarelli-Souganidis, CI)

$$\kappa_\alpha[x,\Gamma] = \frac{2C_\alpha}{\alpha}\int_{z:x+z\in\Gamma} \frac{z}{|z|^{N+\alpha}} \cdot n(x+z)\,\sigma(dz)$$

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- Geometrical version

$$\nu(z : x+z\in\Omega_t \qquad\qquad) - \nu(z : x+z\notin\Omega_t \qquad\qquad)$$

where $\nu(dz) = C_\alpha |z|^{-N-\alpha}dz$ with $\alpha\in(0,1)$

- Level-set formulation

$$\nu(z : u(x+z)\geq 0 \qquad\qquad) \qquad - \nu(z : u(x+z)<0 \qquad\qquad)$$

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\kappa_\alpha[x,u] &= \frac{2C_\alpha}{\alpha}\int_{z:u(x+z)=0} \frac{z}{|z|^{N+\alpha}}\cdot \frac{\nabla u}{|\nabla u|}(x+z)\sigma(dz)
\end{aligned}
$$

- Geometrical version

$$\nu(z : x+z\in\Omega_t, n(x)\cdot z\le 0) - \nu(z : x+z\notin\Omega_t, n(x)\cdot z\ge 0)$$

where $\nu(dz) = C_\alpha |z|^{-N-\alpha}dz$ with $\alpha\in(0,1)$

- Level-set formulation

$$\nu(z : u(x+z)\ge 0, \nabla u(x)\cdot z\le 0) \\ - \nu(z : u(x+z) < 0, \nabla u(x)\cdot z \ge 0)$$

Comments on these definitions

- Can be defined for general singular measure
- "convex" part / "concave" part
- On the geometric version, one can see that this operator is elliptic

Another definition for bounded measures

$$\begin{aligned}\kappa &= \nu \star \mathbf{1}_{\Omega_t} - \nu \star \mathbf{1}_{\Omega_t^c} \\ &= \nu \star \mathbf{1}_{\Omega_t} - \nu \star (1 - \mathbf{1}_{\Omega_t}) \\ &= \underbrace{2\nu}_{c_0} \star \mathbf{1}_{\Omega_t} \quad + \quad \underbrace{(-\nu(\mathbb{R}^N))}_{c_1}\end{aligned}$$

$\longrightarrow$ Dislocation dynamics

Later on, we will discuss:

- If ν is singular, then well defined only for regular curves
- In which sense is this mean curvature fractional?

Outline

Fractional MC of a line and a circle

Recall: $\kappa = \int_\Gamma n(y) \cdot \frac{(x-y)}{|x-y|^{N+\alpha}} d\sigma(y)$

For lines: $\kappa \equiv 0$

For a circle of radius $R >$: $\kappa = \frac{C}{R^\alpha}$.

If $|x| = R$, write $x = Rx_0$ and

$$\begin{aligned}
\kappa[x] &= \int_{\mathbb{S}(0,R)} \frac{y}{|y|} \cdot \frac{x-y}{|x-y|^{N+\alpha}} d\sigma(y) \\
&= \frac{R^{1+(N-1)}}{R^{N+\alpha}} \int_{\mathbb{S}(0,1)} \frac{y_0}{|y_0|} \cdot \frac{x_0-y_0}{|x_0-y_0|^{N+\alpha}} d\sigma(y_0) \\
&= \frac{1}{R^\alpha} C
\end{aligned}$$

Outline

Why is it necessary that the curve be regular?

See Barles and Georgelin

Fractional mean curvature of a parabola in dimension 2

$$
\begin{aligned}
\kappa[0,P] &= \int_{-\infty}^{+\infty} \int_0^{x^2} \frac{dy}{(x^2+y^2)^{\frac{2+\alpha}{2}}} dx \\
&= \int_{-\infty}^{+\infty} \int_0^{|x|} \frac{|x|}{|x|^{2+\alpha}} \frac{d(y/|x|)}{(1+(y/|x|)^2)^{\frac{2+\alpha}{2}}} dx \\
&= \int_{-\infty}^{+\infty} \frac{|x|}{|x|^{2+\alpha}} \int_0^{|x|} \frac{dz}{(1+z^2)^{\frac{2+\alpha}{2}}} dx \\
&= \int_{-\infty}^{+\infty} \frac{F(|x|)}{|x|^{1+\alpha}} dx
\end{aligned}
$$

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Proposition (Da Lio, Forcadel, Monneau (JEMS'05))

Consider c_0 even, smooth, non-negative and $c_0(z) = |z|^{-N-1}$ if $|z| \geq 1$. If $\nu_\varepsilon(dz) = \frac{1}{\varepsilon^{N+1}} c_0(\frac{z}{\varepsilon}) dz$ then

$$\frac{1}{|\ln \varepsilon|} \kappa^\varepsilon[x,u] \to \kappa_1[x,u] \text{ as } \varepsilon \to 0$$

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Proposition (CI (IFB'09))

$$\lim_{\alpha \to 1, \alpha < 1} (1-\alpha) \kappa_\alpha[x,u] = C \kappa_1[x,u] \,.$$

To be compared with: $\lim_{\alpha \to 2} (2-\alpha)(-\Delta)^{\alpha/2} u = (-\Delta) u$.

Outline

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A question

Bence-Merriman-Osher scheme

Let Ω_0 be an open set of $\mathbb{R}^N$ and h a given paramater (time mesh size).

- Solve the heat equation with initial condition $\mathbf{1}_{\Omega_0}$
- Consider $\Omega_h = \{$ x where the solution at time h is greater than $1/2\}$
- Iterate this process: Ω_{2h}, $\Omega_{3h} \dots$

As $h \to 0$, Ω_{ih} approximates the motion of $\Gamma_0 = \partial\Omega$ by MC

[Caffarelli, Souganidis] If one replaces the heat equation with

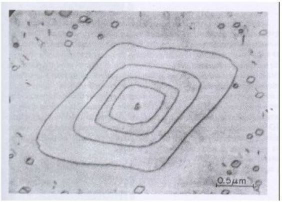
$$\partial_t u + (-\Delta)^{\alpha/2} u = 0 ,$$

what is the new limit as $h \to 0$?

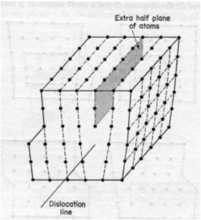
An example: Dislocation dynamics

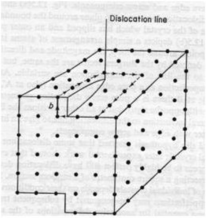
OdG : 1m

OdG : 10^-7 m (1000 atomes)



OdG : 10^-9 (10 atomes)





Example of a geometric law

$$V = c(x) + F(x)$$

where $\left| \begin{array}{lcl} c & = & \text{is a forcing term} \\ F & = & \text{the Peach-Koehler force (self force) at } x \end{array} \right.$

Linear elasticity

$$F(x) = \Delta^{1/2}(\mathbf{1}_{\Omega_t})(x)$$

where Ω_t is such that $\partial \Omega_t$ is the dislocation line

The resulting eikonal equation

$$\partial_t u = (c(x) + \Delta^{1/2}\mathbf{1}_{\Omega_t})(x))|\nabla u|$$

Outline

The level-set equation

$$\partial_t u = \Big(\qquad\qquad \kappa[x, u(t,\cdot)] \Big) |\nabla u|$$

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dislocation dynamics of a single line

=motion by fractional mean curvature flow

The level-set equation

$$\partial_t u = \mu(\frac{Du}{|Du|})\left(c(x) + \kappa[x, u(t,\cdot)]\right)|\nabla u|$$

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Geometric equation

If u is a solution and ϕ is non-decreasing, then $\phi(u)$ is a solution

Viscosity solution for the level-set equation

Super-solution

A lsc function u is a super-solution of (f-MCM) if, for any bounded test function ϕ touching u at x from below

$$\begin{cases} \partial_t \phi(t,x) \geq \kappa_*[x, \phi(t,\cdot)] |\nabla \phi(t,x)| & \text{if } \nabla \phi(t,x) \neq 0 \\ \partial_t \phi(t,x) \geq 0 & \text{if not} \end{cases}$$

Solution = super-solution AND sub-solution

Technical difficulty: to get stability

Main difficulties to overcome

- The fractional mean curvature is neither continuous in x nor in t.
- The most difficult results: stability and strong comparison principle

Notion of relaxed semi-limits (Barles and Perthame)

Discontinuous stability (Barles-CI'08, CI'09)

Let $(u_\alpha)_\alpha$ be a family of super-solutions of (f-MCM) uniformly bounded from below. Then the infimum of this family is a super-solution of (f-MCM).

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See also the recent preprint by Ishii and Matsumura

Outline

Consider $\nu(dz) = |z|^{-N-\alpha} dz$

Theorem (CI'09)

Consider $u_0 \in W^{1,\infty}$. There then exists a unique bounded continuous viscosity solution of (f-MCM).

Theorem (The invariance principle — Forcadel CI Monneau'08, CI'09)

If $u_0, v_0 \in W^{1,\infty}$ satisfy

$$\{u_0 = 0\} = \{v_0 = 0\} \quad \text{and} \quad \{u_0 > 0\} = \{v_0 > 0\}$$

then the corresponding solutions u and v satisfy

$$\{u(t,\cdot) = 0\} = \{v(t,\cdot) = 0\} \quad \text{and} \quad \{u(t,\cdot) > 0\} = \{v(t,\cdot) > 0\}$$

Outline

Anomalous diffusion-reaction equations

Fractional Allen-Cahn equation (joint work with Souganidis)

Scaling properly the fractional AC eq'n makes appear a front moving

- by anisotropic mean curvature if $\alpha \geq 1$,
- by fractional mean curvature if $\alpha < 1$ (to be finished).

Application : Scaling mean field equation in statistical mechanics

Conclusion

- Interfaces moving by fractional mean curvature and corresponding non-local minimal surfaces (Caffarelli, Roquejoffre, Savin) appear in different situations: dislocations, combustion, statistical mechanics

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- The ellipticity of the operator permits to construct a geometric flow after the onset of singularities

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- Interfaces moving by fractional mean curvature and corresponding non-local minimal surfaces (Caffarelli, Roquejoffre, Savin) appear in different situations: dislocations, combustion, statistical mechanics

- A new (good?) formulation of the geometrical problem
- This formulation relies on the idea of compensating the singular measure in a geometrical way
- The ellipticity of the operator permits to construct a geometric flow after the onset of singularities

- This permits to get homogenization results of moving fronts in the regular case
- Repeated games to approximate the flow (with Serfaty)

References

- *Fractional mean curvature flows.*
  Interfaces and Free Boundaries (2009)
- **With N. Forcadel and R. Monneau.**
  Homogenization of some particle systems with two-body interactions and of dislocation dynamics.
  Journal of Differential Equations (2009)

Preprints available here `http://www.ceremade.dauphine.fr/∼imbert`