Algebraic Groups with Good Reduction and Unramified Cohomology

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(joint work with V. Chernousov and A. Rapinchuk)

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We are interested in analyzing

K-forms of G that have good reduction at all $v \in V$.

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$$G' \otimes_K \overline{K} \simeq G \otimes_K \overline{K} \quad \text{(where } \overline{K} \text{ is a sep. closure of } K\text{).}$$

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If n is *odd*, then these are **all** K-forms.

Otherwise, there may be K-forms coming from hermitian forms over noncommutative division algebras.

Groups with good reduction

- G has *good reduction* at a discrete valuation v of K

- generic fiber $\mathcal{G} \otimes_{\mathcal{O}_v} K_v$ is isomorphic to $G \otimes_K K_v$.

- The special fiber (reduction) $\underline{G}^{(v)} = \mathcal{G} \otimes_{\mathcal{O}_v} K^{(v)}$ is then *a connected simple group of same type as G* (where $K^{(v)}$ is the residue field)

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Examples.

- If G is K-split, then G has a good reduction at *any* v (follows from Chevalley's construction).
- $G = \mathrm{SL}_{1,A}$ has good reduction at v if there exists an Azumaya algebra $\mathcal{A}$ over $\mathcal{O}_v$ such that

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- $G = \mathrm{Spin}_n(q)$ has good reduction at v if

$$q \sim \lambda(a_1x_1^2 + \cdots + a_nx_n^2) \quad \text{with} \quad \lambda \in K_v^{\times}, \ \ a_i \in \mathcal{O}_v^{\times}$$

(assuming that $\mathrm{char}\, K^{(v)} \neq 2$).

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- This situation was first studied in detail by G. Harder (Invent. math. **4**(1967), 165-191)

Basic case $R = \mathbb{Z}$:

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Proposition

Let G be an absolutely almost simple simply connected algebraic group over a number field K, and assume that V contains almost all places of K. **Then** *the number of K-forms of G that have good reduction at all $v \in V$ is finite.*

Case $R = k[x]$, $K = k(x)$, and

$$V = \{ v_{p(x)} \mid p(x) \in k[x] \text{ irreducible} \}.$$

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Theorem (Raghunathan–Ramanathan, 1984)

Let k be a field of characteristic zero, and let G_0 be a connected reductive group over k. If G' is a K-form of $G_0 \otimes_k K$ that has good reduction at all $v \in V$ then

$$G' = G_0' \otimes_k K$$

for some k-form G_0' of G_0.

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This was used to prove conjugacy of Cartan subalgebras in some infinite-dimensional Lie algebras.

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This problem has close connections to:

- Study of simple algebraic groups having same isomorphism classes of maximal tori.
- Finiteness properties of unramified cohomology.
- Hasse principles for algebraic groups.
- Analysis of weakly commensurable Zariski-dense subgps and applications to classical problems on locally symmetric spaces (G. Prasad-A. Rapinchuk).

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What happens if one allows only splitting fields of finite degree, or just maximal subfields?

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(S. Garibaldi - D. Saltman)

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Question 2. *When is* $\mathbf{gen}(D)$ *finite?*

The case of number fields

For K a number (or global) field:

1. $|\mathbf{gen}(D)| = 1$ for any quaternion division algebra D/K;
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Both results rely on Albert - Brauer - Hasse - Noether Theorem:

The natural sequence

$$0 \longrightarrow \mathrm{Br}(K) \longrightarrow \bigoplus_{v \in V^K} \mathrm{Br}(K_v) \xrightarrow{\sum \mathrm{inv}_v} \mathbb{Q}/\mathbb{Z} \to 0,$$

is exact, where V^K *is the set of all valuations of* K.

For quaternion algebras, we consider the 2-torsion:

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Thus, a quaternion algebra D/K is determined by (finite) set of ramification places:

$$\mathbf{Ram}(D) = \{ v \in V^K \mid D \otimes_K K_v \text{ is a } \textit{division} \text{ algebra} \}.$$

Number fields (cont.)

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This can be done using *weak approximation* in conjunction with

$$L = K(\sqrt{d}) \hookrightarrow D \quad \Leftrightarrow \quad d \notin {K_v^{\times}}^2 \quad \text{for all} \quad v \in \mathbf{Ram}(D).$$

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$$D_1 = \left(\frac{-1\,,\,3}{\mathbb{Q}}\right) \quad \text{and} \quad D_2 = \left(\frac{-1\,,\,7}{\mathbb{Q}}\right).$$

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But $10 \equiv 1 (\operatorname{mod} 3)$, so $10 \in {\mathbb{Q}_3^{\times}}^2$ and $L \not\hookrightarrow D_1.$

Theorem 1 (Stability Theorem, C-R-R)

Let $\operatorname{char} k \neq 2$. *If* $|\mathbf{gen}(D)| = 1$ *for every quaternion algebra* D *over* k, *then* $|\mathbf{gen}(D')| = 1$ *for any quaternion algebra* D' *over* $k(x)$.

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- $|\mathbf{gen}(D)| > 1$ if D is not of exponent 2.

- $\mathbf{gen}(D)$ can be infinite: for any prime p, there exist division algebras of degree p with infinite genus over certain very LARGE fields (suggested by Rost, Wadsworth, Schacher...).

Quaternion algebras with nontrivial genus: outline

Start with *nonisomorphic* quaternion division algebras

$$\Delta_1^{(0)} \quad \text{and} \quad \Delta_2^{(0)}$$

over a field $k^{(0)}$ of char $\neq 2$, having a *common quadratic subfield*.

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There exists an extension $K/k^{(0)}$ **such that**

$$D_1 = \Delta_1^{(0)} \otimes_{k^{(0)}} K \quad \text{and} \quad D_2 = \Delta_2^{(0)} \otimes_{k^{(0)}} K$$

are *nonisomorphic* division algebras with *same quadratic subfields*.

Some further details

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- One takes care of other quadratic subfields *similarly*.

- This process generates a *tower* of extensions:

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- Construction can be adapted to produce algebras of degree p having infinite genus (J. Meyer for $p = 2$, S. Tikhonov for general p).

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Let K be a finitely generated *field. Then for any central division K-algebra D the genus* $\mathbf{gen}(D)$ *is finite.*

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Proofs of both theorems use analysis of ramification and information about unramified Brauer group.

Ramification

For a discrete valuation v of K, we set

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- If $(n\,, \mathrm{char}\, K^{(v)}) = 1$ or $K^{(v)}$ is perfect, there is a *residue map*

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Then $x \in {}_n\mathrm{Br}(K)$ is unramified at v $\Leftrightarrow$ $r_v(x) = 0$.

Ramification (cont.)

• Given a set V of discrete valuations of K, one defines corresponding *unramified Brauer group:*

$$\mathrm{Br}(K)_V \;=\; \{\, x \in \mathrm{Br}(K) \mid x \ \text{unramified} \ \ \text{at} \ \ \text{all} \ \ v \in V \,\}.$$

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• If residue maps r_v exist for all $v \in V$, then we have

$$ {}_n\mathrm{Br}(K)_V \;=\; \bigcap_{v \in V} \ker r_v .$$

KEY FACT: *Let v be a discrete valuation of K, and n be prime to characteristic of residue field $K^{(v)}$.*

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- If D is of degree n with $(n\,,\,\mathrm{char}\,K) = 1$, then we have

$$|\,\mathbf{gen}(D)\,| \leqslant |\,{}_n\mathrm{Br}(K)_V\,| \cdot \varphi(n)^r,$$

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- Explicit estimates on size of genus available in certain cases.

Question. *Does there exist a quaternion division algebra D over $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, such that*

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- The answer is **not** known for any finitely generated K.
- One can construct examples where ${}_2\mathrm{Br}(K)_V$ is "large."

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- Let G be an absolutely almost simple K-group.

$\mathbf{gen}_K(G) =$ set of isomorphism classes of K-forms G' of G having same K-isomorphism classes of maximal K-tori as G.

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Conjecture. (1) *For* $K = k(x)$, k *a number field, and* G *an absolutely almost simple simply connected* K*-group with* $|Z(G)| \leqslant 2$, *we have* $|\mathbf{gen}_K(G)| = 1$;

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(2) *If* G *is an absolutely almost simple group over a finitely generated field* K *of "good" characteristic, then* $\mathbf{gen}_K(G)$ *is finite.*

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Assume that $K^{(v)}$ is finitely generated, and G has good reduction at v.

Then *every* $G' \in \mathbf{gen}_K(G)$ *has good reduction at v, and reduction* $\underline{G}'^{(v)} \in \mathbf{gen}_{K^{(v)}}(\underline{G}^{(v)})$.

Let K be a finitely generated field equipped with a set V of discrete valuations such that:

(I) for any $a \in K^{\times}$, set $V(a) := \{v \in V \mid v(a) \neq 0\}$ is finite;

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Corollary.

Let G be an absolutely almost simple simply connected K-group.

There exists a finite subset $S \subset V$ (depending on G) such that every $G' \in \mathbf{gen}_K(G)$ has good reduction at all $v \in V \setminus S$.

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Question.

When can a finitely generated field K be equipped with V that satisfies (Φ)? Does a divisorial V satisfy (Φ)?

Algebraic groups vs. division algebras

Additional challenges for arbitrary algebraic groups:

- It is not known how to classify forms by cohomological invariants.
- Even when such description is available (e.g. for type G_2), one needs to prove finiteness of unramified cohomology in degrees > 2, which is a difficult problem.

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Inner forms of type A

- Finiteness results for unramified Brauer groups imply that divisorial V does satisfy (Φ) for inner forms of type A_{n-1} for any finitely generated K such that $\operatorname{char} K \nmid n$.

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Theorem 5.

(1) *Let D be a central division algebra of exponent 2 over $K = k(x_1, \ldots, x_r)$ where k is a number field or a finite field of characteristic $\neq 2$. Then for $G = \mathrm{SL}_{m,D}$ $(m \geqslant 1)$, we have $|\mathbf{gen}_K(G)| = 1$.*

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(2) *Let $G = \mathrm{SL}_{m,D}$, where D is a central division algebra over a finitely generated field K. Then $\mathbf{gen}_K(G)$ is finite.*

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Theorem 6.

Let C be a smooth geometrically integral curve over a number field k, $K = k(C)$, and V divisorial set of places. Fix $n \geqslant 5$.

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Let $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, and set $G = \mathrm{Spin}_n(q)$.

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Theorem 7.

Let $K = k(C)$, where C is a smooth geometrically integral curve over a number field k, and set $G = \mathrm{Spin}_n(q)$. If either $n \geqslant 5$ is odd, or $n \geqslant 10$ is even and q is isotropic, then $\mathbf{gen}_K(G)$ is finite.

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Then *set of* K*-isomorphism classes of* $\mathrm{Spin}_n(q)$ *with good reduction at all* $v \in V$ *is finite.*

- Using Milnor's conjecture, we reduce the argument to finiteness of unramified cohomology $H^i(K, \mu_2)_V$, for $i \geqslant 1$, where $\mu_2 = \{\pm 1\}$.
- We establish finiteness of $H^i(K, \mu_2)_V$ for all $i \geqslant 1$.

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Let C be a smooth geometrically integral curve *over a number field k, $K = k(C)$, and V divisorial set of places. Fix $n \geqslant 5$.*

Then *set of K-isomorphism classes of $\mathrm{Spin}_n(q)$ with good reduction at all $v \in V$ is finite.*

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  Key case: $i = 3$. Involves Kato's and Jannsen's results on cohomological Hasse principle for H^3.

Main definitions

There exist residue maps

$$r_v^i \colon H^i(K, \mu_n^{\otimes j}) \to H^{i-1}(K^{(v)}, \mu_n^{\otimes (j-1)})$$

for all $i \geqslant 1$, all j, whenever $(n, \operatorname{char} K^{(v)}) = 1$.

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For such V, we define *Picard group* $\operatorname{Pic}(V) = \operatorname{Div}(V)/\operatorname{P}(V)$,
$\operatorname{Div}(V)$ = free abelian group on $v \in V$,
$\operatorname{P}(V)$ = subgp of "principal divisors" $\sum_{v\in V} v(a)v$, $a \in K^\times$.

Good reduction and unramified cohomology

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Consider a field K *equipped with set* V *of discrete valuations satisfying* (I) *such that* $(\operatorname{char} K^{(v)}, 2) = 1$ *for all* $v \in V$. *Let* $n \geqslant 5$.

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Set-up: Global-to-local map in Galois cohomology

Let

- K be a field
- V a set of discrete valuations of K
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Kernel of $\theta_{G,V}$ is called *Tate-Shafarevich set*

$$Ш(G,V) := \ker \theta_{G,V}.$$

Hasse principle over number fields

Let k = *number field*, V = set of all places of k.

- If G is *simply-connected* or *adjoint* alg. K-group, then

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We show that global-to-local map is proper for groups over $K = k(C)$ in certain situations.

Properness of $\theta_{G,V}$ for special orthogonal groups

Theorem 10

Let C be a smooth geometrically integral curve over a number field k, $K = k(C)$, and V divisorial set of places. Fix $n \geqslant 5$.

- Using Milnor's conjecture, reduce proof to finiteness of
$$\Omega_i = \ker\left(H^i(K, \mu_2) \to \prod_{v \in V} H^i(K_v, \mu_2)\right)$$
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Further results and a conjecture

For $K = k(C)$ and $V =$ divisorial set of valuations, we also establish properness of $\theta_{G,V}$ for:

- G of type G_2
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Then *for any absolutely almost simple alg. group G over K,*

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Results over some other classes of fields

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