Problems in hyperbolic dynamics

Current Trends in Dynamical Systems and the Mathematical Legacy of Rufus Bowen
Vancouver july 31st – august 4th 2017

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1 Zeta functions (David Ruelle)

See Bowen's problems 86, 95, 124, 125.

Consider a self map f on a space X and define the function

$$\zeta(z) := \exp \sum_{k>0} \frac{\#\{x = f^k(x)\}}{k} z^k.$$

Smale's conjecture. *When f is an Axiom A diffeomorphism of a manifold X, then ζ is rational.*

Initial work by Smale, later on proved in this generality by Manning using Markov partitions, the proof was then improved by Bowen, see Shub *Global stability of dynamical systems.*

Consider now an integer m of the form p^n, p prime. Let F_m be the field with m elements. Its algebraic closure is $\bar{F}_m = \cup_{k>0} F_{m^k}$.

Weil's conjecture. *When f is the Frobenius automorphism $a \mapsto a^p$ acting on an algebraic variety X over $\bar{F}_m$, then ζ is rational.*

Proved by Dwork (and later Deligne).

Observe that the Frobenius automorphism is "hyperbolic" (its derivative vanishes since the characteristic is p).

Problem. *Find a proof of the Weil conjecture by use of a "Markov partition" for the dynamics of the Frobenius automorphism.*

2 Non-uniform hyperbolic attractors with 2-dimensional unstable bundle (Masato Tsujii)

See Bowen's problem 8.

It has been shown that non-uniformly hyperbolic attractors are frequent: Jakobson theorem (for $1D$-endomorphisms), and Benedicks-Carleson theorem (for $2D$-diffeomorphisms) that was later extended by Wang and Young. In these works the unstable direction has dimension 1.

Problem. *Show that non-uniformly hyperbolic attractors with* 2*-dimensional unstable direction are frequent.*

The candidate here is to consider surface endomorphisms as follow. Let $Q_a\colon x \mapsto a - x^2$, where a is close to 2 (which is non-uniformly hyperbolic by the result of Jakobson for a set of parameters a with positive Lebesgue measure).

Let $F_{\varepsilon,a,b} := \Phi_\varepsilon \circ (Q_a \times Q_b)$ acting $\mathbb{R}^2$, where Φ_ε is a coupling between the two coordinates satisfying $\Phi_0 = 0$.

The question is to obtain a set of parameters (ε, a, b) with positive Lebesgue measure, such that the dynamics of $F_{\varepsilon,a,b}$ has a non-uniformly hyperbolic attractor supporting a physical measure with two positive Lyapunov exponents.

An easier case would be to consider an associate Viana map (see Viana IHES 1997).

3 Smooth realizations (Yakov Pesin)

See Bowen's problems 15, 27, 54, 133.

This is a problem in the project of smooth realization of measure preserving transformations. More precisely, given a smooth compact manifold, construct a diffeomorphism with prescribed collection of ergodic properties with respect to a given invariant measure which is usually volume or measure of maximal entropy (MME). Recent results in this direction are constructing a non-uniformly hyperbolic (non-zero Lyapunov exponents) volume preserving diffeomorphism which is Bernoulli. In dimension 2 this was done by Katok, in dimension ≥ 5 by Katok, Feldman, and Brin (in their example one Lyapunov exponent is zero), for all dimensions ≥ 2 (and all Lyapunov exponents non-zero) by Dolgopyat and Pesin.

After this results the next step is to understand asymptotic properties such as the decay of correlations (for Hölder continuous observable functions), the central limit theorem, large deviations, etc. For instance for $C^{1+\alpha}$ (conservative) Anosov diffeomorphisms, the mixing is exponential; but Anosov diffeomorphisms do not exist on every manifold.

Problem. *Given any manifold M, does there exists a volume-preserving, ergodic, non-uniformly hyperbolic diffeomorphism, that is exponentially mixing? polynomially mixing?*
What regularity can we ensure for the diffeomorphism?
What about replacing volume with a measure maximizing the entropy?

The guess is that in the case of volume preserving diffeomorphisms the above answer is yes for polynomial mixing and no for exponential mixing. So there exist manifolds where no volume preserving, ergodic, non-uniformly hyperbolic diffeomorphism has exponential mixing.

For instance, Katok gave an example of a C^∞ non-uniformly hyperbolic diffeomorphism of the disc which is tangent to the identity near the boundary. However, if one allows a polynomial contact at the boundary, then one can obtain a $C^{1+\alpha}$ diffeomorphism with polynomial decay of correlations. Can one obtain better smoothness?

For the case of a measure of maximal entropy, it may be easier to establish exponential decay. It isn't clear how to obtain polynomial decay in this case.

4 Lyapunov regularity with positive volume (Yakov Pesin)

Let f be a $C^{1+\alpha}$ diffeomorphism of a compact manifold M and $\mathcal{R}$ be the set of points that are Lyapunov regular and have non-zero Lyapunov exponents (in particular the Lyapunov exponents for positive and negative iterations coincide for points in $\mathcal{R}$).
Problem. *If the set $\mathcal{R}$ has positive volume, does there exist a hyperbolic absolutely continuous invariant probability measure?*

This is the case when f is an Axiom A diffeomorphism. For instance, note that if μ is an SRB measure obtained by pushing volume forward, then (by the entropy formula) the sum of the positive Lyapunov exponents is less or equal to minus the sum of the negative Lyapunov exponents, hence if the basins of an SRB measure for f and for f^{-1} intersect along a subset of $\mathcal{R}$ with positive Lebesgue measure, then the SRB measures have full stable and unstable dimension, are absolutely continuous with respect to Lebesgue and coincide.

5 Smooth actions of higher-rank abelian groups (Anatole Katok)

The general question:

Are all genuinely higher rank full entropy hyperbolic measure-preserving actions of $\mathbb{Z}^d, d \geq 2$ essentially algebraic?

More specifically, consider a $C^{1+\alpha}$ action of $\mathbb{Z}^d$, with rank $d \geq 2$, on a compact manifold M^n and an ergodic invariant probability measure μ. Then there exists linear forms $\lambda_1, \ldots, \lambda_n \colon \mathbb{Z}^d \to \mathbb{R}$ that associate the Lyapunov exponents of μ to each element of $\mathbb{Z}^d$. Assume that

(1) each element of the suspension (different from the identity) has positive entropy,

(2) each form λ_i is non-vanishing,

(3) a certain "genuine higher rank" condition, i.e. no rank one factors for any finite cover, and

(4) every Lyapunov exponent contributes to entropy in the Ledrappier-Young formula.

Then necessarily $d \leq n-1$. The case of $2 \leq d = n-1$ is called *the maximal rank*.

Algebraic examples are certain affine actions on the torus $\mathbb{T}^n$: each element of the action takes the form $x \mapsto A.x + v$ where $A \in GL(n, \mathbb{Z})$ and more generally, (infra)nilmanifolds.

Problem 1. Under certain additional conditions, i.e. if all exponents are simple, the measure μ is absolutely continuous. In general conditional measure on Lyapunov foliations are smooth mesures on certain submanifolds.

Problem 2. Any action satisfying (1)-(4) restricted to a finite index subgroup has finitely many ergodic components each isomorphic to an algebraic action, the isomorphism is smooth along Lyapunov foliations and is smooth in Whitney sense.

The affirmative answer has been obtained in 2011 by A.K. and F. Rodriguez-Hertz for the maximal rank actions, i.e. when $d = n - 1$ (published 2016, JMD).