Abstracts
Workshop on Regularity Problems in Hydrodynamics

August 3-7, 2009

Mini-Course 1

Examples of hidden convexity in hydrodynamics

***Yann Brenier* (Universite de Nice-Sophie Antipolis)**

We will discuss two examples of hidden convexity:

- if we look for solutions of the Euler equations by using the least action principle, following Arnold's geometric interpretation, we get a minimization problem that has a hidden convex structure (in dimension 3); this has important consequences for the uniqueness and the partial regularity of the pressure field.
- the hydrostatic and semi-geostrophic limits of the Euler and Navier-Stokes equations are examples of very singular limits that cannot be justified on any linear functional spaces but can be addressed successfully on suitable convex functional cones.

Mini-Course 2

Aspects of Navier-Stokes and certain model equations

***Vladimir Sverak* (University of Minnesota)**

This minicourse will consist of four lectures. The topics of the lectures will be as follows:

- Overview
- Kolmogorov's 1941 theory
- Geometry of Euler's equation, Introduction
- Geometry of Euler's equation, Part 2

Lectures 3 and 4 will focus on the Hamiltonian interpretation of Euler's equation. Most time will be devoted to explaining finite-dimensional analogies (as pioneered by V.I.Arnold), and the insights they bring to the study of Euler's equations.

Concentration of the motion inside viscous flows

Lorenzo Brandolese (University of Lyon)

One of the most important questions in mathematical Fluid Mechanics, which is still far from being understood, is to know whether a finite energy, and initially smooth, nonstationary Navier–Stokes flow will always remain regular during its evolution, or can become turbulent in finite time.

As a first step toward the understanding of possible blow-up mechanisms, it is interesting to exhibit examples of smooth and decaying initial data such that, even though the corresponding solutions remain regular for all time, “something strange” happens around a given point (x_0, t_0) in space-time. This is the goal of the present talk.

On the inviscid limit

Peter Constantin (University of Chicago)

I will discuss some of the ideas surrounding the topic of long time, zero viscosity limit, relationship between Euler equations and Navier-Stokes equations and stochastic formulations for the latter.

Basic Constructions and Symmetries of Inviscid Fluid Motion

David Ebin (Stony Brook)

We will present configuration spaces for Compressible and Incompressible flow and discuss symmetries. Then we will derive conservation laws using Noether’s theorem. We will show that there are no self-intersecting flows in the incompressible case, though it seems there are in the compressible case.

Then we will discuss two-dimensional incompressible flows and to what extent they may be integrable and the significance of integrability as well.

Throughout the emphasis will be on material (Lagrangian) coordinates.

Magnitude and gradient estimates for stationary solutions of the Navier-Stokes equations; a computer aided study

Robert Finn **(Stanford)**

We calculate numerically solutions of the stationary Navier-Stokes equations in two dimensions, for a square domain with particular choices of boundary data. The data are chosen to test whether bounded disturbances on the boundary can be expected to spread into the interior of the domain. The results indicate that such behavior indeed can occur, but suggest an estimate of general form for the magnitudes of the solution, analogous to classical bounds for harmonic functions. A corresponding estimate for the derivatives would then follow from earlier work by Finn and Solonnikov.

The qualitative behavior of the solutions we found displayed some striking and unexpected features. As a corollary of the study, we obtain two new examples of non-uniqueness for stationary solutions at large Reynolds number.

Onsager's Conjecture and Turbulent Dissipation

Susan Friedlander **(University of Southern California)**

Onsager conjectured, in the context of the Euler equations, that solutions that are sufficiently smooth conserve energy. However it is possible that "rough" solutions could dissipate energy even in the absence of viscosity. This mechanism is sometimes called turbulent dissipation. We will discuss recent results sharpening the criteria that ensure conservation of energy for the Euler equations. We will describe a model system for the fluid equations that illustrates Onsager's conjecture in both directions and reproduces Kolmogorov's law for turbulent scaling.

The Navier-Stokes Flow with non-Decaying Initial Velocity and its Applications

Yoshikazu Giga **(University of Tokyo)**

Usually, the well-posedness of the Navier-Stokes initial value problem is discussed for initial data decaying at space infinity for example for data of finite kinetic energy and/or of finite enstrophy. However, it is also important to consider nondecaying initial data including for example almost periodic initial data.

We first survey several results on the well-posedness when initial velocity does not decay at space infinity not only for the usual Navier-Stokes equations but also the Navier-Stokes-Coriolis system.

We then give an application to non blow-up criteria. In fact we show that continuous vorticity alignment implies non blow-up without assuming that the kinetic energy is finite. This provides a different view point for a famous result of Constantin-Fefferman in 1993, where integral estimates plays a key role. The last part is a recent joint work with H. Miura of Osaka University.

Regularity questions for the Navier-Stokes equations in non-smooth domains, and a conjectured estimate for solutions of the Stokes equations

John Heywood (University of British Columbia)

There are a number of questions regarding the regularity of solutions of the Navier-Stokes equations that depend upon making an estimate of the form

$$\int_\Omega \mathbf{u}\cdot\nabla\mathbf{u}\cdot\tilde{\triangle}\mathbf{u}\,dx \le c\,\|\nabla\mathbf{u}\|^{\frac{3}{2}}\left\|\tilde{\triangle}\mathbf{u}\right\|^{\frac{3}{2}}, \tag{1}$$

where the integral and L^2-norms are taken over a domain $\Omega\subset R^3$, and $\tilde{\triangle}$ is the Stokes operator. If the boundary of Ω is non-Lipschitzian (for example, it might be smooth except for a single cusp-like reentrant corner), the validity of such an estimate is unknown. Nevertheless, we have reason to believe that it holds, for any three-dimensional domain, with the constant $c = 1/\sqrt{3\pi}$.

Indeed, let Ω be an arbitrary open subset of R^3, and let

$\mathbf{D}(\Omega) = \{\phi\in\mathbf{C}_0^\infty(\Omega) : \nabla\cdot\phi = 0\}$,
$\mathbf{J}(\Omega) =$ Completion of $\mathbf{D}(\Omega)$ in the $\mathbf{L}^2$-norm $\|\phi\|$, and
$\mathbf{J}_0(\Omega) =$ Completion of $\mathbf{D}(\Omega)$ in Dirichlet-norm $\|\nabla\phi\|$.

Then, for any $\mathbf{f}\in\mathbf{J}(\Omega)$, there is at most one $\mathbf{u}\in\mathbf{J}_0(\Omega)$ such that

$$\int_\Omega \nabla\mathbf{u}\cdot\nabla\phi\,dx = -\int_\Omega \mathbf{f}\cdot\phi\,dx\,,\quad \text{for all } \phi\in\mathbf{D}(\Omega). \tag{2}$$

For such $\mathbf{u}$ and $\mathbf{f}$ one customarily writes $\tilde{\triangle}\mathbf{u} = \mathbf{f}$, thereby defining the Stokes operator $\tilde{\triangle}$ on a certain subspace of $\mathbf{J}_0(\Omega)$. We believe that the inequality

$$\sup_\Omega |\mathbf{u}|^2 \le \frac{1}{3\pi}\|\nabla\mathbf{u}\|\left\|\tilde{\triangle}\mathbf{u}\right\| \tag{3}$$

should be valid for all $\mathbf{u}\in\mathbf{J}_0(\Omega)$ and $\mathbf{f}\in\mathbf{J}(\Omega)$ satisfying (2).

Wenzheng Xie has proven this inequality for $\Omega = R^3$, and shown for all domains that the constant $1/3\pi$, if it is valid at all, cannot be improved upon. Considering the Poisson equation as a model problem, he also proved an analogue of (3) for the Laplacian, valid for arbitrary open sets, with the constant $1/2\pi$. Further, he showed his proof of this generalizes to the Stokes operator, except for his use of the maximum principle in estimating the L^2-norm of the Green's function for the Helmholtz operator by the L^2-norm of its fundamental singularity. He conjectured an analogous estimate in the Stokesian case.

This lecture concerns our efforts to complete Xie's attempted proof of (3), by a variational argument that does not depend on his conjecture. However, our reasoning comes to rest on several new conjectures, for which we provide suggestive evidence. Perhaps happily, these new conjectures have analogues for the Laplacian which are also not yet proven, suggesting that for the matter at issue, the Laplacian remains a useful model problem.

Further light on our conjectures is provided by considering the hypotheses required of a sequence $\{a_n\}$ in order that the inequality

$$\left(\sum_{n=1}^{m} a_n c_n\right)^2 \leq 3\pi \left(\sum_{n=1}^{m} n^{2/3} c_n^2\right)^{1/2} \left(\sum_{n=1}^{m} n^{4/3} c_n^2\right)^{1/2} \tag{4}$$

should hold for any second sequence $\{c_n\}$, and any positive integer m. The hypotheses under which we have proven (4) are analogous to two assertions concerning the eigenfunctions of the Stokes operator, one proven and the other conjectured, which together imply (3). We mention that the hypotheses for the $\{a_n\}$ are satisfied if $a_n = 1$, for all n, in which case (4) provides a simple but interesting inequality for sequences, in which the constant 3π is optimal.

On the singularity formation of a 3D model for Incompressible Euler and Navier-Stokes equations

Tom Hou (California Institute of Technology)

We study the singularity formation of a recntly proposed 3D model for the incompressible Euler and Navier-Stokes equations. This 3D model is derived from the axisymmetric Navier-Stokes equations with swirl using a set of new variables. The model preserves almost all the properties of the full 3D Euler or Navier-Stokes equations except for the convection term which is neglected. If we add the convection term back to our model, we would recover the full Navier-Stokes equations. We will present numerical evidence which supports that the 3D model may develop a potential finite time singularity. We will also analyze the mechanism that leads to these singular events in the new 3D model and how the convection term in the full Euler and Navier-Stokes equations destroys such a mechanism, thus preventing the singularity from forming in a finite time. Finally, we prove rigorously that the 3D model develops finite time singularities

for a class of initial data with finite energy and appropriate bounadry conditions. This work may shed interesting light into the stabilizing effect of convection for 3D incompressible Euler and Navier-Stokes equations.

TBA

Dong Li **(Institute for Advanced Study)**

Dependence on initial data for solutions to the Euler equations

Gerard Misiolek **(University of Notre Dame)**

Conjugate points on the volumorphism group

Stephen Preston **(University of Colorado at Boulder)**

Solutions of the ideal Euler equations can be thought of as geodesics on the volumorphism group, and the global structure of them is thus related to the global properties of the Riemannian exponential map. On finite-dimensional manifolds, the structure of the singularities of this map gives a lot of information about the global structure of the manifold. For the infinite-dimensional volumorphism group, things can get much more complicated. I will describe the general structure of these singularities for the volumorphism group of a general three-dimensional manifold, along with some explicit examples.

Two basic problems on the weak solutions of 3-d Euler equations

Alexander Shnirelman **(Concordia University)**

There are, roughly speaking, four classes of weak solutions of 3-d Euler equations: (1) Classical regular solutions; (2) Moderately irregular solutions, which are Hölder continuous in space-time with the Hölder exponent more than 1/3; (3) Hölder continuous solutions with the Hölder exponent less or equal than 1/3; (4) Everywhere discontinuous and, possibly, unbounded, square integrable weak solutions.

Solutions of the class 2 enjoy the energy conservation, and, possibly, even the uniqueness property (with fixed initial velocity). So, they are too regular to describe the developed turbulence. Solutions of the class 4 are too flexible (highly non-unique), and have other nonphysical features (like the absence of pressure), which makes them poor candidates for realistic description of turbulence. As for solutions of class 3, they are the least studied. There are two basic problems about these solutions:

1. Construct an example of such solution with monotonically decreasing energy and nontrivial pressure;

2. Given an initial, Hölder continuous velocity field, prove that there exists a global in time weak solution with decreasing energy.

In the talk I'll tell what I know about these (and some related) problems.

Higher derivatives estimate for the 3D Navier-Stokes equation

Alexis Vasseur (University of Texas at Austin)

A non linear family of spaces, based on the energy dissipation, is introduced. This family bridges an energy space (containing weak solutions to Navier-Stokes equation) to a critical space (invariant through the canonical scaling of the Navier-Stokes equation). This family is used to get uniform estimates on higher derivatives to solutions to the 3D Navier-Stokes equations. Those estimates are uniform, up to the possible blowing-up time. The proof uses blow-up techniques. It is based on a local parabolic regularization result obtained via De Giorgi techniques.