The Geometry of Euler's equation

Introduction

Part 1

Mechanical systems with constraints, symmetries

flexible
joint

fixed length
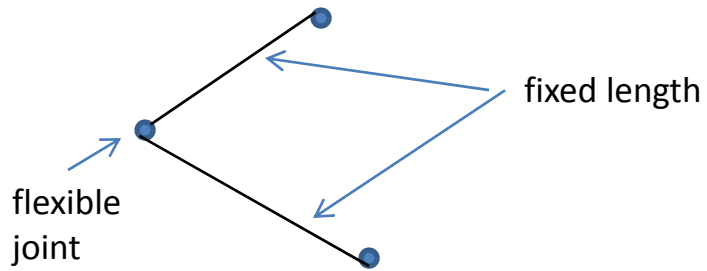
In principle can be dealt with by applying
F=ma, but this can become complicated

Lagrangian mechanics:

M   manifold of possible configurations – "configuration space"

q   points of M (either abstract, or in specific coordinates $q=(q^1, …, q^n)$)

$t \rightarrow q(t)$ trajectories in M (to be computed)

Effective algorithm (due to Lagrange) for   writing the equations for q(t)
without some complicated analysis of the forces:

For a curve q(t) in M (not necessarily a real motion), find the expression for the kinetic energy T and the potential energy V of the system.

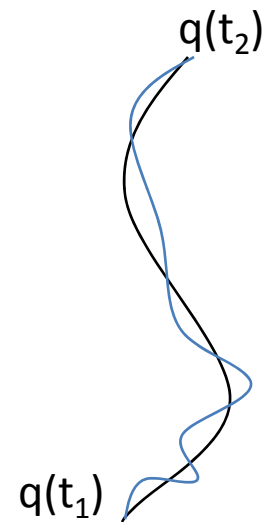Typically $T = T(q, \dot{q})$ with a quadratic expression in $\dot{q}$

and $V = V(q)$

q(t$_2$)

Consider the Lagrangian $\qquad L = L(q, \dot{q}) = T(q, \dot{q}) - V(q)$

Hamilton's principle:

The actual motions are extremals of the "action functional"

q(t$_1$)

$$\int_{t_1}^{t_2} L(q, \dot{q})\, dt$$ in the class of trajectories with fixed endpoints q(t$_1$), q(t$_2$)

Lagrange equations: $\qquad \dfrac{d}{dt}\dfrac{\partial L}{\partial \dot{q}^i} = \dfrac{\partial L}{\partial q^i}$
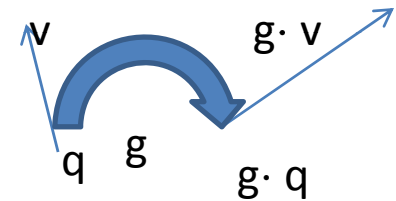
Systems with symmetries

G   group   acting on M $q \in M \to g \cdot q \in M$

Groups of interest: q(t) a solution $\Rightarrow$ $g \cdot q(t)$ a solution

sufficient condition: L is invariant under G

L: $TM \to R$

G acts also on TM: $(q, v) \in T_x(M)$ $\to$ $g \cdot (q, v) = (g \cdot q, Dg(q) \cdot v) \in T_{gx} M$

Invariant L: $L(g \cdot (q, v)) = L(q, v)$

Goal: use the symmetries to simplify the equations

For reducing the equations, we need "continuous symmetry groups" (Lie groups) and the corresponding "infinitesimal symmetries" (Lie algebras).

Infinitesimal transformation of M = vector fields $\xi(q)$ on M. Their fluxes generate a 1-parameter family of transformtions g^t .

$$\frac{d}{dt}\,(g^t \cdot q) = \xi(g^t \cdot q)$$

vector field $\xi(q)$
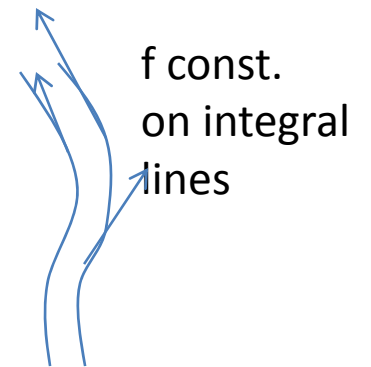and its integral lines
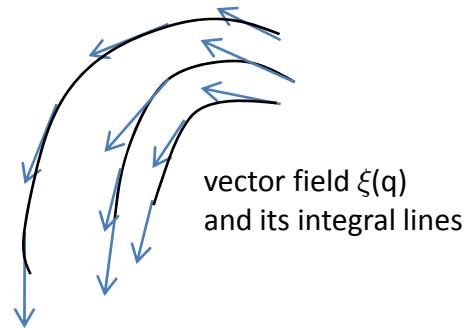
Example:
$$g^t \cdot x = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

$$\xi(x) = \begin{pmatrix} -x_2 \\ x_1 \end{pmatrix}$$
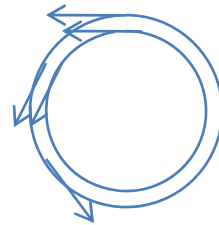
Functions f on M invariant under an infinitesimal transformation $\xi(q)$

f const.
on integral
lines

$\xi \cdot f = d/d\epsilon\, f(q+\epsilon\, \xi(q))_{\,\epsilon\,=\,0} = 0$ (derivative in the ξ direction vanishes)

Infinitesimal transformations ξ on M extend to infinitesimal transformations of TM,
(still denoted by ξ)

$q \rightarrow q+\epsilon\, \xi(q)$

original transformation

$(q,v) \rightarrow (q+\epsilon\xi(q), v + \epsilon D\xi(q)\, v\ )$

extension

just linearize
$g\cdot(q, v) = (g\cdot q, Dg(q) \cdot v)$

Lagrangian **L: TM $\rightarrow$ R** invariant under an infinitesimal transformation ξ

$$\xi \cdot L = 0$$

the derivative of L in the direction
of the extended ξ

A simple version of Noether's theorem:

M   configuration space,   L: TM→ R a Lagrangian,
ξ a vector field on M which is an "infinitesimal symmetry" of L (i.e. $\xi \cdot L = 0$)

Then the quantity $\xi^i(q)\dfrac{\partial L}{\partial \dot{q}^i}$ is conserved, i.e. $\dfrac{d}{dt}\ \xi^i(q)\dfrac{\partial L}{\partial \dot{q}^i} = 0$
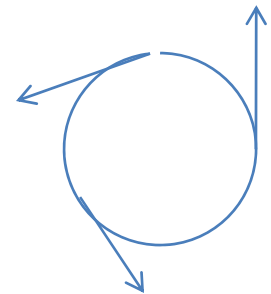
Proof: Chain rule, the Lagrange equations, the assumption $\xi \cdot L = 0$

Example: A motion in a radial potential (in R^2 or R^3) conserves the
          angular momentum.

$$L(q, \dot{q}) = \frac{1}{2}m|\dot{q}|^2 - V(|x|)$$

$$\xi(q) = \begin{pmatrix} -q^2 \\ q^1 \end{pmatrix}$$

$$\xi^i\frac{\partial L}{\partial \dot{q}^i} = m(q^1\dot{q}^2 - q^2\dot{q}^1)$$

<u>Hamiltonian mechanics</u> (gives a more geometric picture):

Instead of working with TM, work with the co-tangent space T^*M.

Additional structures on T^*M: canonical $1 -$ form α and the symplectic form $d\alpha$

Example: Let X be a 1-d linear space (it is of course $\sim$ R, but with the ambiguity of choosing a fixed vector)

Let e be a basis of X and e* the basis of X* dual to e, i.e. <e*,e> = 1.

q   coordinate in X with respect to e
p   coordinate in X* with respect to e*
(p,q)   coordinates in X*$\times$X

pq is independent of the choice of e (by the very construction)

α = p dq   1-form on X*$\times$ X - it is "canonical" (independent of the "arbitrary" choice of e)
$d\alpha$ = ω = dp$\wedge$dq   canonical 2-form on X*$\times$ X

Conclusion: X*$\times$X has a canonical volume element   (unlike X$\times$X)

The situation in T*M is very similar:

$q = (q^1, \ldots, q^n)$ coordinates in M

$\frac{\partial}{\partial q^1}, \ldots, \frac{\partial}{\partial q^n}$ basis of $T_q M$ (at a given q)

$dq^1, \ldots, dq^n$ basis of $T_q^* M$ dual to the above basis of $T_q M$

$(q^1, \ldots, q^n, \xi^1, \ldots, \xi^n)$ coordinates in TM,

                           ξ^i being the coordinates in the basis $\frac{\partial}{\partial q^i}$

$(q^1, \ldots, q^n, p_1, \ldots, p_n)$ coordinates in $T^* M$

                           p_i being the coordinates in the basis dq^i

$\alpha = p_i dq^i$ canonical 1-form in $T^* M$

$\omega = d\alpha = dp_i \wedge dq^i$ symplectic form on $T^* M$

In particular we have a canonical volume form $\omega \wedge \ldots \wedge \omega$ (n times) on T*M, in addition to the anti-symmetric form ω. (And the forms $\omega \wedge \omega$, $\omega \wedge \omega \wedge \omega$, etc.)

Eventually we wish to work in function spaces, and the expressions in local coordinates will not really be suitable, but it is important to understand the finite dimension first.

Nice structures on T*M, but the natural evolution quantity d/dt q(t) (generalized velocity) undoubtedly belongs to TM.

We need a natural map between TM and T*M.
We have the Lagrangian L to provide it!

$$(q, \dot{q}) \in TM \quad \rightarrow \quad (q, p) \in T^*M$$

$$p_i = \frac{\partial L}{\partial \dot{q}^i}$$

inversion

$$(q, p) \in T^*M \rightarrow (q, \dot{q}) \in TM$$

Hamiltonian

(Legendre trasnform
of L (at a fixed q))

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

$$H(p, q) = \inf_v (p_i v^i - L(q, v)) = p_i \dot{q}^i - L(q, \dot{q})$$

The equations in p,q, H

$$\dot{q}^i = \frac{\partial H}{\partial p_i}$$

$$\dot{p}_i = -\frac{\partial H}{\partial q^i}$$

Our system now "lives" completely in T*M, where we have the benefit of the canonical geometric structures. In particular, the space of parameters which describe the state of our system, has a canonical volume (and much more)

Another view: the form ω = $dp_i \wedge dq^i$ on X=T*M $\sim$ {(p,q)} provides an isomorhism J: T*(X) and T(X) by

compare:
Riemannian metric: use the symmetric form g_{ij} to identify T*M and TM; raise and lower indices with g_{ij} or g^{ij} (inverse of g_{ij})

$$< \beta, \xi >= \omega(\xi, J\beta)$$

Symplectic form: use the anti-symmetric form ω_{ij} to indetify T*M and TM; raise and lower indices with ω_{ij} or ω^{ij} (inverse of ω_{ij})

Letting x=(p,q), we can write the equations

$$\dot{x} = J \, dH(x)$$

raise H_i = $\partial H/\partial x_i$ by J $\sim$ ω^{ij}:
d/dt x^i = $\omega^{ij} H_j$

Riemannian geometry $\sim g_{ij}$, symplectic geometry $\sim \omega_{ij}$, T*M comes with a natural ω_{ij}

Transformations of X=T*M leaving the equations invariant – same as transformations leaving the form ω_{ij} invariant - "canonical transformations", or "symplectic transformations"

Every transformation $\qquad q \to \tilde{q} = \phi(q) \qquad$ of the configuration space
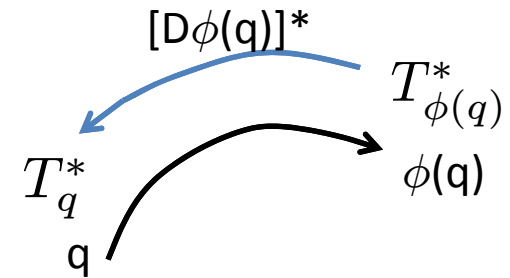
$$\underset{\underset{\uparrow}{\omega_{ij} \text{ - preserving}}}{}$$

extends to a canonical (= symplectic) transformation of the phase space

$$(p, q) \to (\tilde{p}, \tilde{q}) = (\ ([D\phi(q)]^*)^{-1}p, \phi(q)\ )$$

$[D\phi(q)]^*$

$T^*_{\phi(q)}$

$\phi(q)$

T^*_q

q

In 1d one can see easily how this is volume-preserving: possible squeezing in q is compensated by stretching in p and vice versa
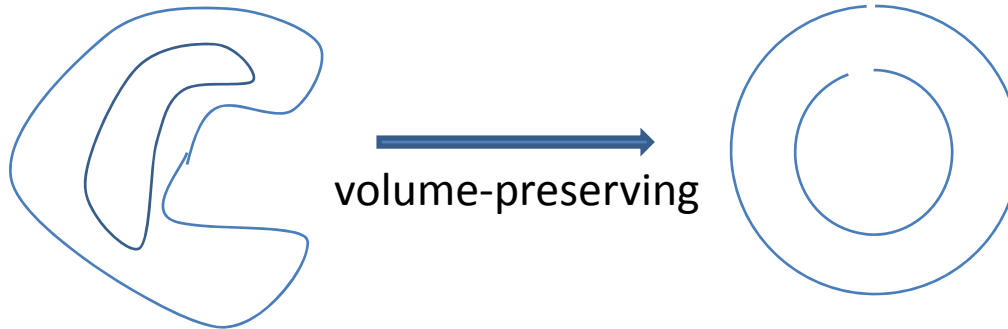
linear in p

On the other hand, there are many more symplectic   transformation than this:

Example:
dim M = 1
dim T*M = 2

volume-preserving

Symmetries in the Hamiltonian picture

1. The Lagrangian picture (confinuration space, etc.) is invarian under the change of coordinates in the configuration space.

   (Example: polar or cartesian coordinates give equivalent equations)

2. The Hamiltonian picture is invariant under transformations of the phase space which preserve the symplectic form:

$$(\tilde{p}, \tilde{q}) = (\tilde{p}(p, q), \tilde{q}(p, q))$$

$$d\tilde{p}_i \wedge d\tilde{q}^i = dp_i \wedge dq^i \longleftarrow$$

"canonical transformation"
also called symplectic trans.

$$\tilde{H}(\tilde{p}, \tilde{q}) = H(p, q)$$

$$(p(t), q(t)) \text{ solve for Hamiltonian } H \ \Rightarrow \ (\tilde{p}(t), \tilde{q}(t)) \text{ solve for Hamiltonian } \tilde{H}.$$

$x=(p,q)$, $X = T^*M$, $\omega = dp_i \wedge dq^i$, J: $T^*X \to TX$ induced by ω

Infinitesimal symplectic tranformations:
vector field $\xi(x)$ on X such whose flux is a 1 parameter family of symplectic transformations

Alternatively: $x \to x + \epsilon\xi(x)$ is symplectic modulo $O(\epsilon^2)$, or
$\xi \cdot \omega = 0$ (Lie derivative of ω in the direction ξ)

For any smooth f on X, $\xi(x) = J\,df$ is an infinitesimal symplectic transformation
it is just the vector field generating the evolution by the hamiltonian f

Vice-versa, for any infinitesimal symplectic transformation ξ, there
locally exists a function f such that J df = ξ

"generating functions"
generate the usual
stream functions in 2d

infinitesimal symplectic transformations $\longleftrightarrow$ **functions**
locally

$J^{-1}\,\xi$

Proof: We must check that d $J^{-1}\,\xi$ = 0. We can either do it
by direct calculation, or use Cartan's formula $\quad \xi \cdot \omega = i_\xi d\omega + d(i_\xi \omega)$

=0 by assumptions
(ω is infinitesima sympl. tr.)

=0 for the
sympl. form

Example: ξ vector field on M

Extend ξ to T*M: $(p,q) \rightarrow (p - \epsilon (D\xi(q))^* p , q + \epsilon \xi(q))$

This vector field is generated by   **f(p,q) = p_i ξ^i(q)**,
the quantity from Noether's theorem:

J df = the extension of ξ to T*M

Noether's theorem:
Assumption: the Hamiltonian H is invariant under (the extension of) ξ
which is the same as: H is invariant under the flow generated by f=p_i ξ^i
Conclusion: f is invariant under the flow generated by H

Proof: $$(J\,dH) \cdot f = H_{p_i} f_{q^i} - H_{q_i} f_{p^i} = -(J\,df) \cdot H$$

Or: $(\omega)_{ij}$ (anti-sym form on vectors) is non-singular and it also
gives $(\omega)^{ij} = [(\omega)_{ij}]^{-1}$, anti-sym. form on co-vectors.

Definition: f,g smooth functions on T*M

$$\{f, g\} = f_{p_i} g_{q^i} - g_{p_i} f_{q^i}$$ is called the **Poisson bracket**

Can be defined on any symplectic manifold by {f,g}= (J df) g = ω^{ij} f$_j$ g $_i$

If H is a Hamiltonian, the evolution of any quantity f is

df/dt = {H,f}   (Taking f=p$_i$, or q^i, we get the canonical equations)

Noether's theorem is now clear:
{H,f}=0 means that f is a conserved quantity, but it also means (equivalently)
that H is invariant under the infinitesimal transformations generated by f

Properties of the Poisson bracket

$$\{f, g\} = -\{g, f\}$$

$$\{fg, h\} = f\{g, h\} + g\{f, h\} \qquad \text{(it is a derivative)}$$

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0 \qquad \text{(Jacobi identity)}$$

Corollary: H is a Hamiltonian, f,g are conserved $\Rightarrow$ {f,g} is also conserved:

Proof: {H,{f,g}}={{H,f},g}+{f,{H,g}} = 0

The Lie bracket and the Poisson bracket

ξ , η vector fields on any manifold
we can differentiate functions in those direction: f $\rightarrow$ $\xi \cdot$ f = D$_\xi$ f (often also denoted by L$_\xi$ F)

The Lie bracket [ξ,η] of ξ , η is a vector field given by

$\xi \cdot$ ($\eta \cdot$ f) - $\eta \cdot$ ($\xi \cdot$ f) = [ξ,η]$\cdot$ f

Remark: [ξ,η]=0
iff the flows given by
ξ and η commute

In coordinates $$[\xi, \eta]^i = \xi^j \eta^i_{,j} - \eta^j \xi^i_{,j}$$

Relation between the Lie bracket and the Poisson bracket

$$[J\, df\, , \, J\, dg] = J\{f, g\}$$

the corrections are
of course important
for symplectic geometry,
Lie alg. representations,
etc., see e.g. the book
of A.A. Kirillov

Conceptually, and modulo some corrections which will not be important for us here

**infinitesimal symplectic
transformations, Lie bracket** $\sim$ **functions, Poisson bracket**

symmetries   conserved quantities

X=T*M, Hamiltonian H, group G of symmetries (i.e. leaving invariant H and ω)

use G to simplify the system on X: project X $\rightarrow$ Y = G \ X, follow trajectories in Y

"manifold of G-orbits" (not quite
a smooth manifold, but often close)

instead working with Y, work with

e.g. smooth near
most points

C^∞(G \X) smooth functions invariant under G
            g· f = f, g$\in$ G, where g· f(x)= f(g^{-1}· x)

{g· f_1, g· f_2}= g·{f_1,f_2}   (because G preserves ω)

The algebra C^∞(G \X) is closed under the Poisson bracket

We have the Poisson bracket on C^∞(Y) inherited from X .

no in general
(for example,
dim Y can be odd),
but the "correction"
is in fact beneficial!

Does the Poisson bracket on Y come from a symplectic form,
i.e. does Y also inherit the symplectic structure?

Example: motion in a central field ($\sim$ radial potential) in R^3

$X = T^*M = R^3 \times R^3$, coordinates (p,q), $H = |p|^2/2m + V(|q|)$
$G = SO(3)$, $g \cdot (p,q) = (g \cdot p, g \cdot q)$,

(infinitesimal rotations about the q_3 axis
are generated by $q_1 p_2 - q_2 p_1$,
similar for the other axes)

Noether's theorem: $q \wedge p$ is conserved

This is enough for integrating the equations, but it is instructive to look also at $G \setminus X$

The invariant functions $C^\infty(G \setminus X)$ can be obtained as functions of

$y_0 = p \cdot q$, $y_1 = \frac{1}{2}|q|^2$,   and $y_2 = \frac{1}{2}|p|^2$

Calculate the Poisson brackets $\{y_i, y_j\}$

	y_0	y_1	y_2
y_0	0	$2y_1$	$-2y_2$
y_1	$-2y_1$	0	$-y_0$
y_2	$2y_2$	y_0	0

This table also defines
the Lie algebra sl(2,R)

For general functions f , g of y_0, y_1, y_2 $\{ f , g \} = f_{y_i} g_{y_j} \{y_i, y_j\}$

Equations for y_0, y_1, y_2 are

$$d/dt \ y_j = \{H, y_j\}$$

The "manifold" Y with the coordinates y_j and the bracket $\{ f , g \}$
cannot be a symplectic manifold - dim Y = 3.

A special function C on Y :

$$C(y_0, y_1, y_2) = 4y_1 y_2 - y_0{}^2$$

satisfies $\{ C , f \} = 0$ for each f   (check that $\{ C , y_j \} = 0$,   j=0,1,2)

The evolution by *any* Hamiltonian always preserves C

So the evolution $d/dt \, y_j = \{H, y_j\}$ takes place on C=const.,
and we are dealing with a system with 1 degree of freedom.

The solutions curves are given by C=const. and H=const.,
time-dependence is calculated by integrating along the curves.

C=c_1, H = c_2

surfaces C = const.

each of them is a symplectic
manifold of dimension 2
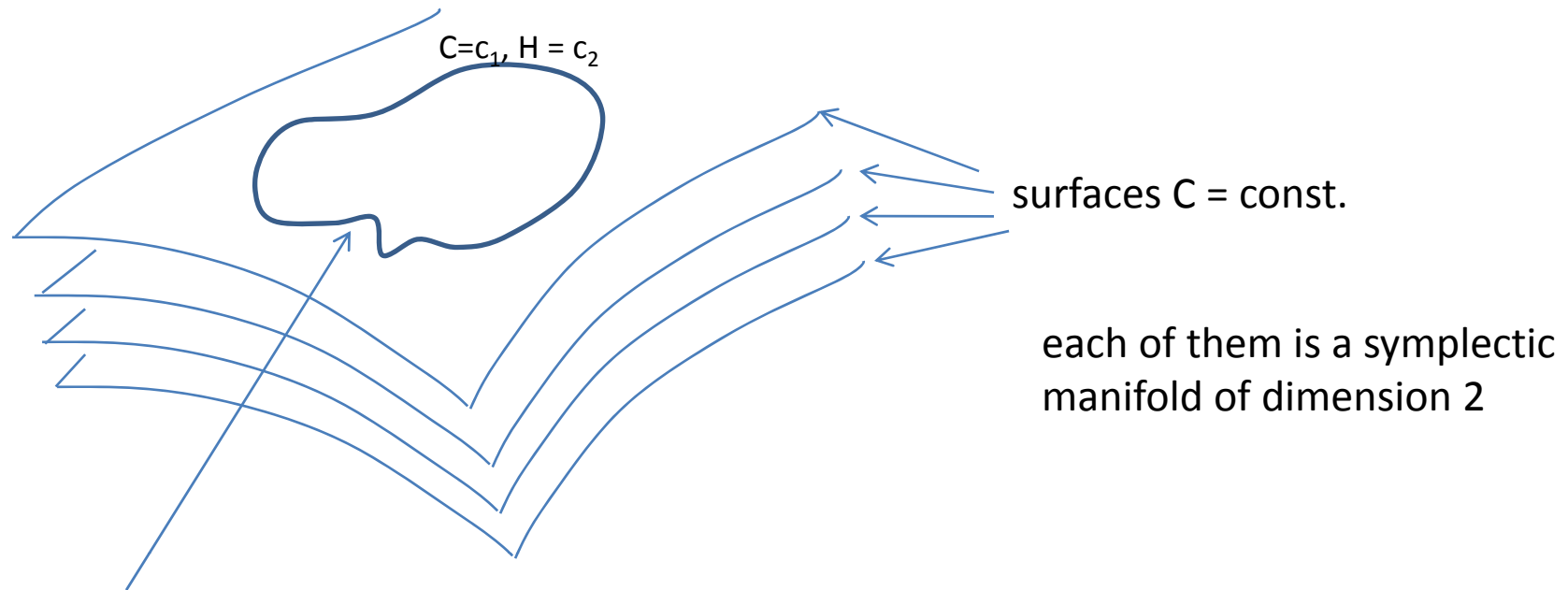
Solution of d/dt y_j = {H, y_j} moving along C=const. H=const.

In general, the structure of the manifolds Y = G \ X is similar:
Y is foliated into "symplectic leaves", the leaves "do not interact".

In general, Y is not a manifold, the foliation can have singularities, etc.

The conservation law C=const. looks first unexpected:
naively we expect to reduce the dimension of the system by the dimension
of the group G, but the symplectic structure gives often more!

Example: general ODE system   with a 1d symmetry group G on a Manifold X

$$\dot{x} = f(x)$$

symmetry $$f(g \cdot x) = g \cdot f(x)$$

the reduced system is on Y=G \ X,   dim Y = dim X – dim G

Symplectic situation: we still have the same reduction as in the general case.

In addition: the equations on Y are again $dy_i/dt = \{ H, y_i \}$
          and (given dim G=1), we have an additional reduction:
          There is at least one Casimir function C   ←   Typically C is given by
           and the evolution in Y takes place on C=const.   the function generating
                                                                the conservation law

Example: we can see without calculation that a rigid body rotating about a given point in the absence of external forces should be integrable:

Configuration space: M= SO(3)
Phase space: X=T*M
Symmetry groups: G=SO(3) (acting on M by left multiplication)
Reduced space: Y = G \ X, dim Y = 3 (it turns out Y= so(3) $\sim$ R^3)

But symplectic leaves must have even dimension,
so there should be at least one Casimir function

2d symplectic leaves $\Rightarrow$ integrable