Lecture 1

Vladimir Sverak
University of Minnesota

Outline of the minicourse

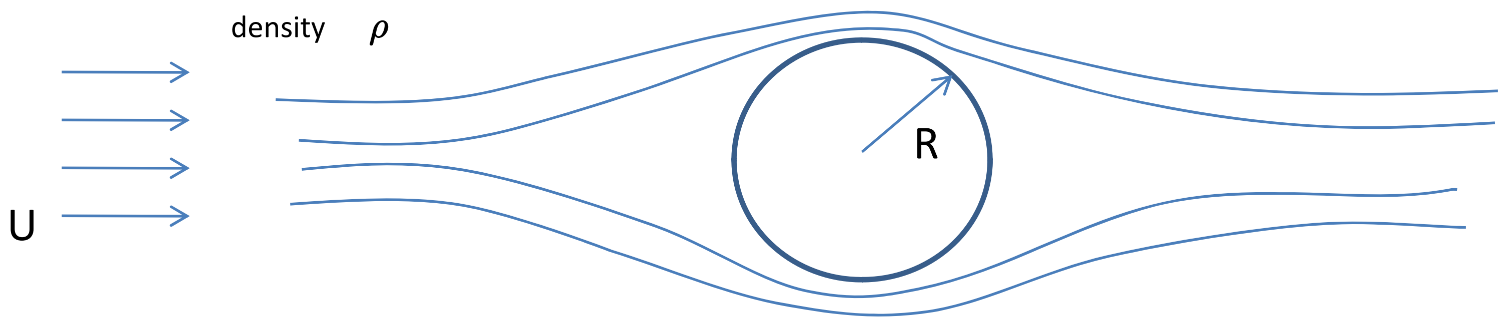
Lecture 1: Basic overview of Navier-Stokes/Euler

Lecture 2: Kolmogorov’s 1941 theory (Introduction)

Lecture 3: Geometry of Euler’s equation I

Lecture 4: Geometry of Euler’s equation II

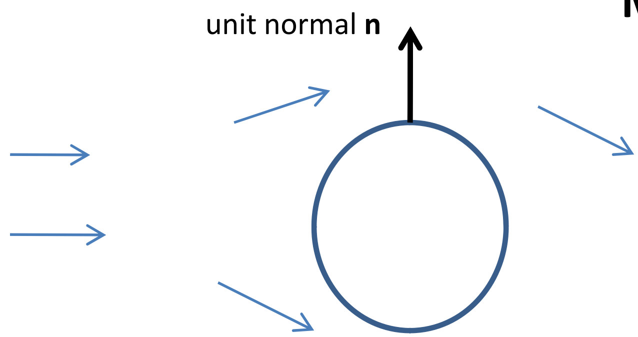
Flow around a sphere



Problem: Calculate the drag force F ("resistance of the medium")

History:

1687	Newton	$\mathbf{F} \sim \mathbf{c}\ \rho \mathbf{R}^2\mathbf{U}^2$	(Principia, Cor. 1 of Thm. 30, Book 2)
1752	d'Alembert	**F = 0**	in ideal fluids
1851	Stokes	$\mathbf{F} = \mathbf{6}\pi\rho\nu\ \mathbf{R}\ \mathbf{U}$	for low velocities



Mathematical Model

$u(x,t)$	velocity field
$p(x,t)$	pressure
ϱ	density (assumed to be constant)

$\Omega = R^3 \setminus$ ball domain of the flow

PDE for u and p, formula for F (Navier, Stokes, 1845-1851, for ν =0 Euler, 1757)

$$\left.\begin{aligned} \mathbf{u}_t + \mathbf{u}\,\nabla\mathbf{u} + \nabla \mathbf{p}/\varrho - \nu\,\Delta\mathbf{u} &= 0 \\ \operatorname{div}\mathbf{u} &= 0 \end{aligned}\right\} \quad \text{in} \quad \Omega\times(t_1, t_2)$$

u=0 at the boundary of Ω (**u n = 0** when ν=0)

u=U at ∞

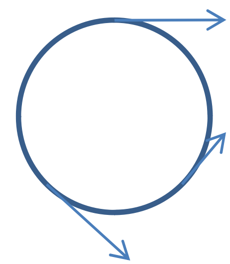
$$\mathbf{F} = \int_{\partial\Omega} [\mathbf{pn} - \sigma\mathbf{n}]\ \mathbf{dS}\,, \quad \text{with } \sigma_{ij} = \varrho\,\nu\,(u_{l,j} + u_{j,i}) \quad \text{(force at a given time)}$$

Attempt #1:

water: $\nu = 10^{-6}$, air: $\nu = 10^{-5}$ (in SI units)

Let's just take $\nu = 0$ and the Euler boundary condition **u n =0**

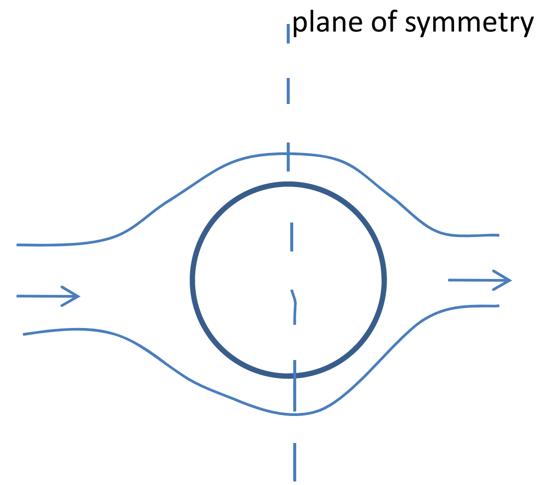
(rather than a very small $\nu>0$ and the natural Navier-Stokes boundary condition u=0)



We can find an explicit solution!

$$\mathbf{u}=\nabla \mathbf{h} \qquad \text{with} \qquad \mathbf{h} = \mathbf{Ux}\,(1+\mathbf{R}^3/2|\mathbf{x}|^3)$$

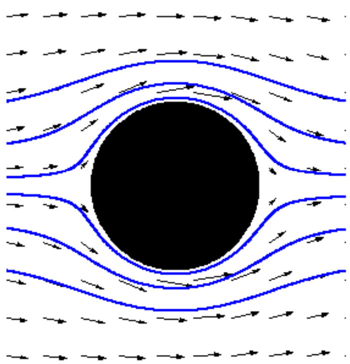
the drag force **F = 0**



"d'Alembert's paradox"; can be derived for general shape, mathematically there is no mistake, the source of the paradox is in the assumption ν=0

Stokes' Calculation (1851)

Calculate dF/dU at U=0 by solving the equation linearized about the trivial solution u=0



$$\left.\begin{array}{r} -\nu\, \Delta u + \nabla p = 0 \\ \operatorname{div} u = 0 \end{array}\right\} \quad \text{in } \Omega$$

u=0 at the boundary of the ball

u=U at ∞

Explicit solution, invariant w. resp. to rotations about the x_1 axis , in polar coordinates

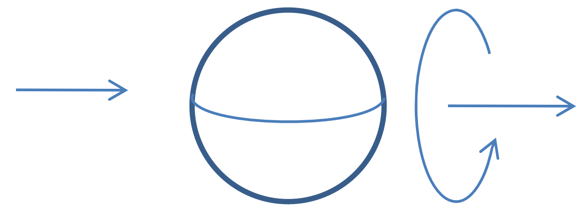
$$u_r = U \cos\theta\, [1-3R/2r+R^3/2r^3]$$

$$u_\theta = -U \sin\theta\, [1-3R/4r-R^3/4r^3]$$

gives **$F=6\pi\varrho\nu RU$** for infinitesimal U

Attempt #2:

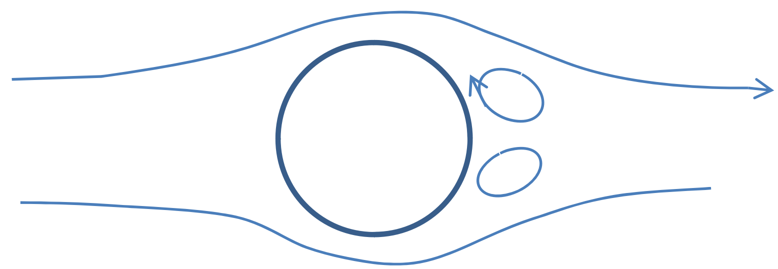
Continue Stoke's solution numerically to the non-linear regime (larger velocities), assuming the same symmetries:



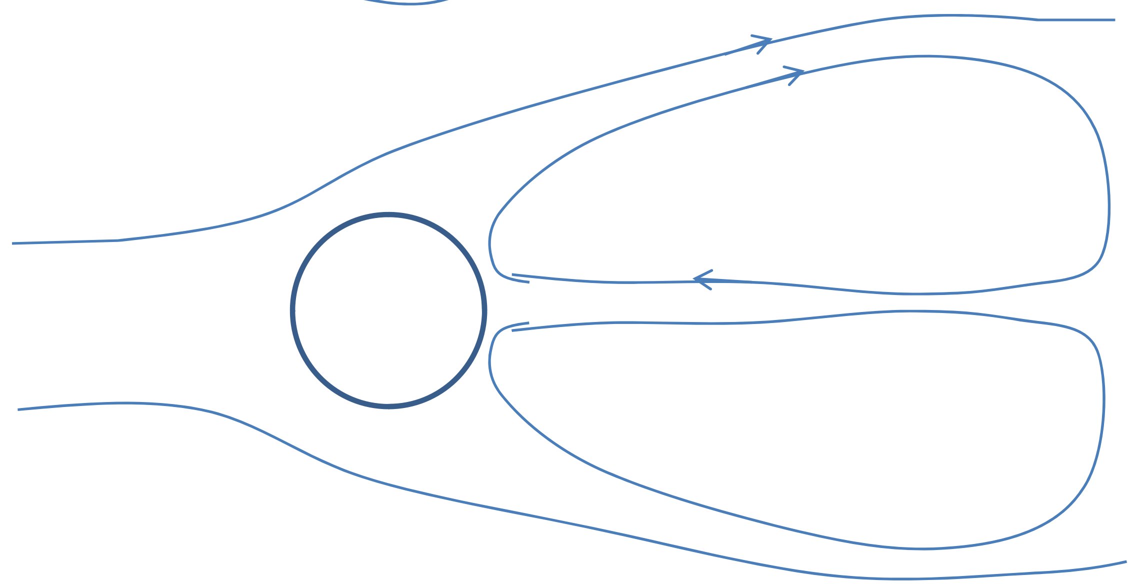
u=u(x), rotational symmetry about the x_1 axis

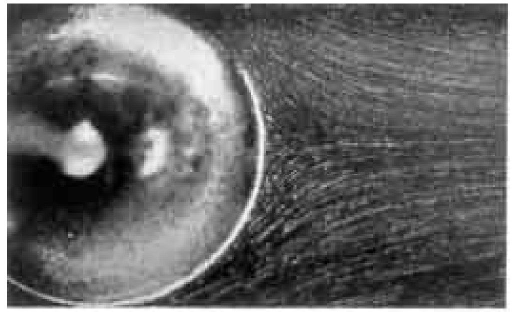
$$-\nu \Delta u + u\nabla u + \nabla p = 0$$
$$\text{div } u = 0$$
$$u=0 \text{ at } \partial\Omega$$
$$u=U \text{ at } \infty$$

Good results for small U
correct prediction of
drag force F

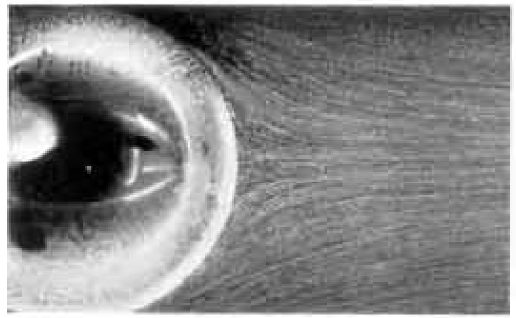


Unrealistic flows for larger U,
(re-circulation region too large)
drag force F too low

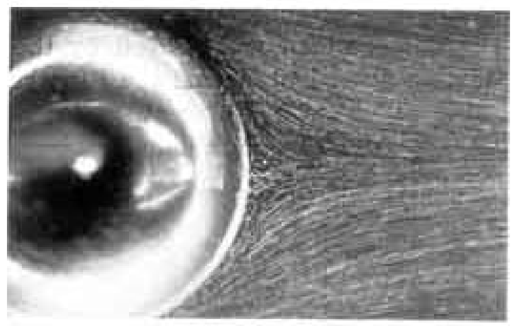




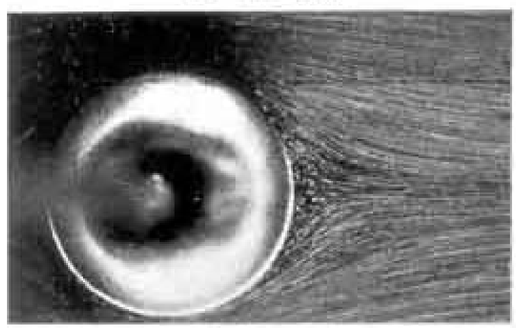
(a) $R = 9.15.$



(b) $R = 17.9.$

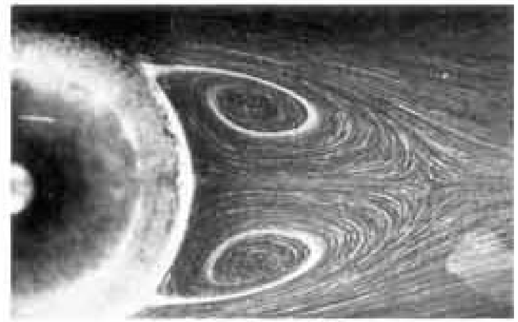


(c) $R = 25.5.$

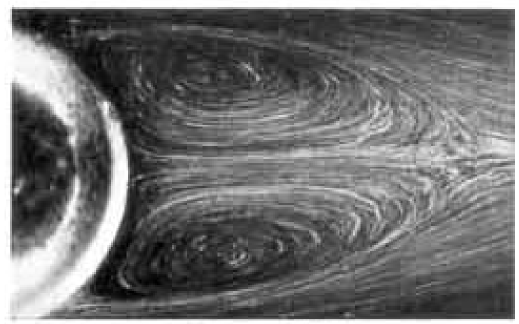


(d) $R = 26.8.$

(e) $R = 37.7.$



(f) $R = 73.6.$



(g) $R = 118.$

(h) $R = 133.$

What's wrong this time?

In the real world the unrealistic solutions we calculated are **unstable**.
They are stable in the computer, because of the extra symmetry assumptions we imposed (u=u(x), u symmetric under rotations about x_1 axis).

The symmetry assumptions restrict the degrees of freedom which real solutions can explore.

Solution:

- calculate the time-dependent equation
- do not impose the rotational symmetry
- make sure that the algorithm does not artificially impose extra symmetries
- possibly introduce small perturbations which break the symmetries

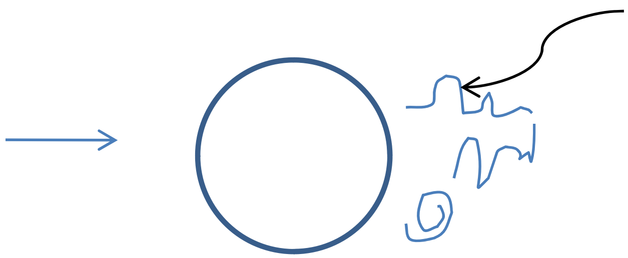
(There are non-trivial Numerical Analysis issues, some of them already present in the previous calculation, which could be a subject of a separate lecture...)

Attempt #3:

Full time-dependent equation

instanteneous drag force is F(t), resulting drag force is $F = \dfrac{1}{t_2 - t_1}\displaystyle\int_{t_1}^{t_2} F(t)\,dt$

At first everything looks very good, but another issue appears:



Fine-scale structures which put huge demands on the computing power

the computer is soon overwhelmed

Picture to be shown during the lecture

Example of a real flow which is beyond present-day computing power

Examples : (assuming the best available computers today)

tennis ball: cannot get beyond 5 to 10 m/s (not relevant for the game)

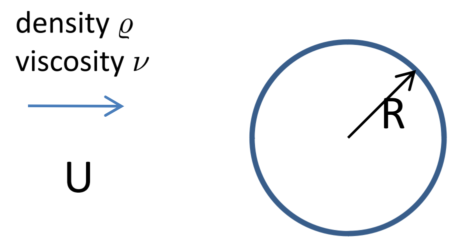
automobile: cannot do much better than 0.1 m/s

We also find out that doubling the speed needs about 8 to 10-fold increase in the computing power (once the oscillatory regime is reached).

<u>"Non-dimensionalization"</u>

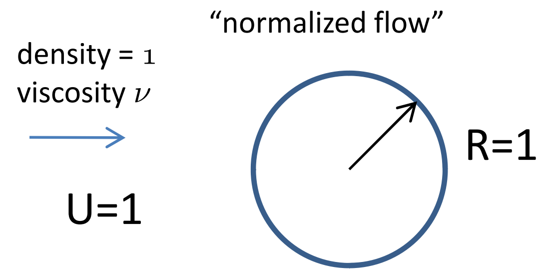
(reducing $F = F(\varrho, \nu, R, U)$ to a function of one variable)

- select the unit of length so that R=1
- select the unit of time so that U=1
- select unit of mass so that $\varrho=1$
- the one remaining given variable ν has changed to $\nu' = \nu/RU$.



- Reynolds number: **$\mathbf{Re = RU/\nu}$** $= 1/\nu'$ (the main parameter of the flow)

  (the equation becomes $u_t + u\nabla u + \nabla p - 1/Re\ \Delta u = 0$)



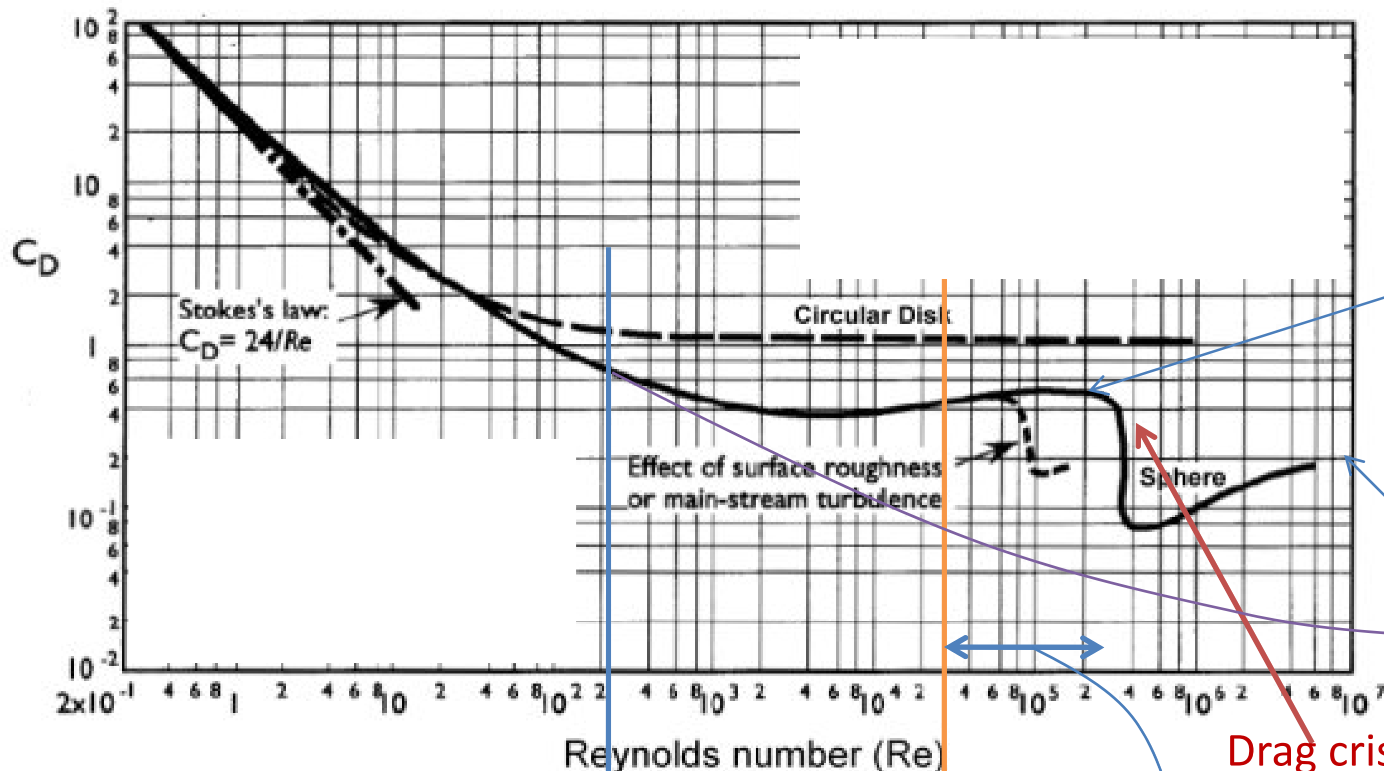
- the output variable F has changed to f=f(Re)

- back to the original units:

$$\mathbf{F = \varrho R^2 U^2 f(Re) = \tfrac{1}{2}\, c_D(Re)\ \varrho\ R^2\ U^2} \quad \text{(conventional normalization)}$$

the drag coefficient

Experimental curves $c_D = c_D(Re)$ (Newton's formula equivalent to c_D = const.)



tennis serve
(recreational player)

automobile
at typical speeds

axi-symmetric,
time independent sol

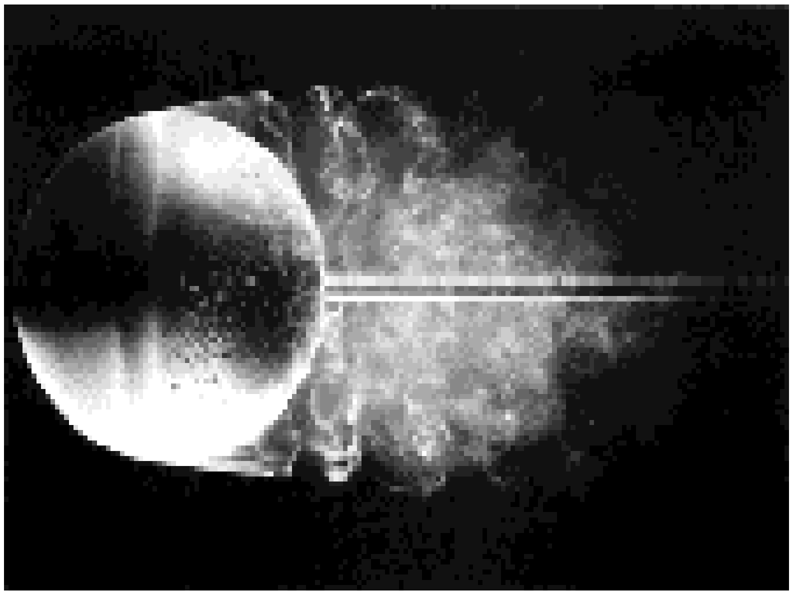
Limit for
stable steady
state solutions

Limit for 3d DNS
(present day computers)

how much further
we would get by
a 1000-fold increase
of computing power

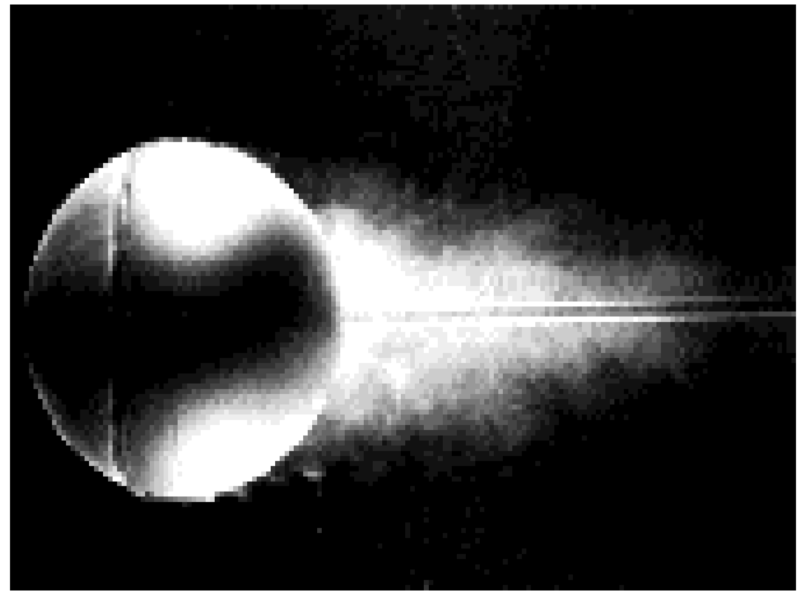
<u>Drag crisis:</u>
not only c_D but the drag
force itself drops as U
is increased

It seems to be unknown if this
also happens in 2d flows (since
it is hard experimentally to keep
to flows two dimensional for
large Re). The conclusion from
what is known numerically is unclear.



a

b



Change of the flow geometry accompanying the drag crisis

Challenges for PDE theory:

- Are the turbulent flows still described by the Navier-Stokes equations? In particular, do the Navier-Stoked equations have (smooth) solutions which would correspond to the experimental flows?

- If we were able to calculate/solve the Navier-Stokes solutions would we observe some of the remarkable experimental effects, such as

  a) The fact that for large Reynolds numbers the drag force is quite independent of the viscosity. (From the PDE point of view, the viscosity term $\nu \Delta u$ is the dominant term in the equations!)

  b) The drag crisis and its sensitivity to the roughness of the boundary

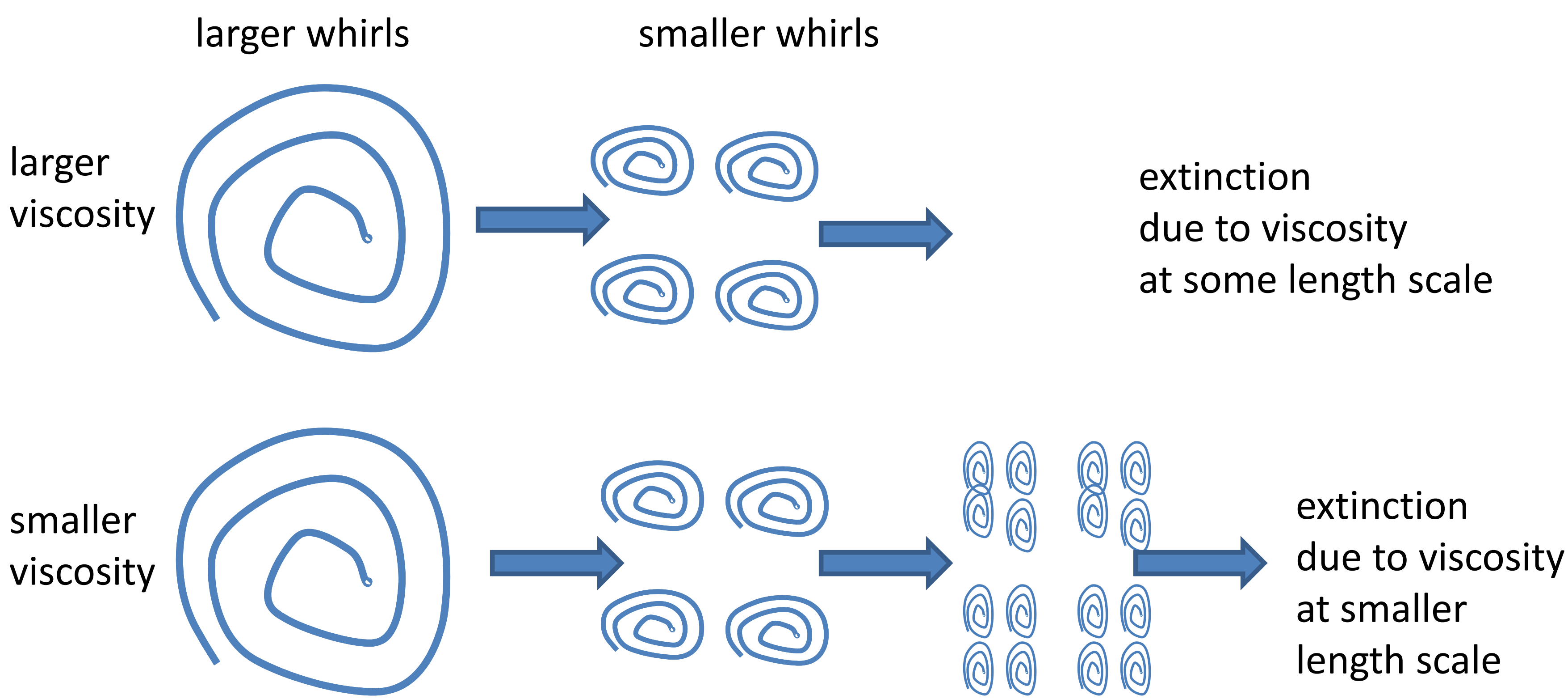
- Understanding the nature of the turbulent oscillations so that they could be treated "statistically", without calculating all the microscopic details

Remark: An extra puzzle – significant drag reduction by minute amounts of polymer additives

Out of reach of rigorous analysis for forseable future?

The heuristics of the ν independence (L. F. Richardson, 1920s)

cartoon picture (in reality it is important that the flow is 3D)



The net macroscopic effect remains (approximately) the same

Picture to be shown during the lecture

Real situation illustrating the previous cartoon picture

This “Richardson effect” is what makes it possible to do practical calculation of flows which cannot be resolved by DNS (and also explains why scale models work better than one would expect based on comparison of Reynolds numbers)

We are not interested in the microscopic details of the velocity field u(x,t) , we are only interested in the macroscopic quantity F

Cartoon picture:

Do the calculation with $\boldsymbol{\nu=\nu(x)}$ where in some (well chosen) areas ν(x) is much larger than the original viscosity. This terminates the cascade sooner - we save many degrees of freedom. At the same time we will find that for

$$\mathbf{F} = \mathbf{F}[\nu(\mathbf{x})]$$

$$\delta \mathbf{F}/\delta\nu(\mathbf{x}) \sim \mathbf{0} \quad \text{in some large regions}$$

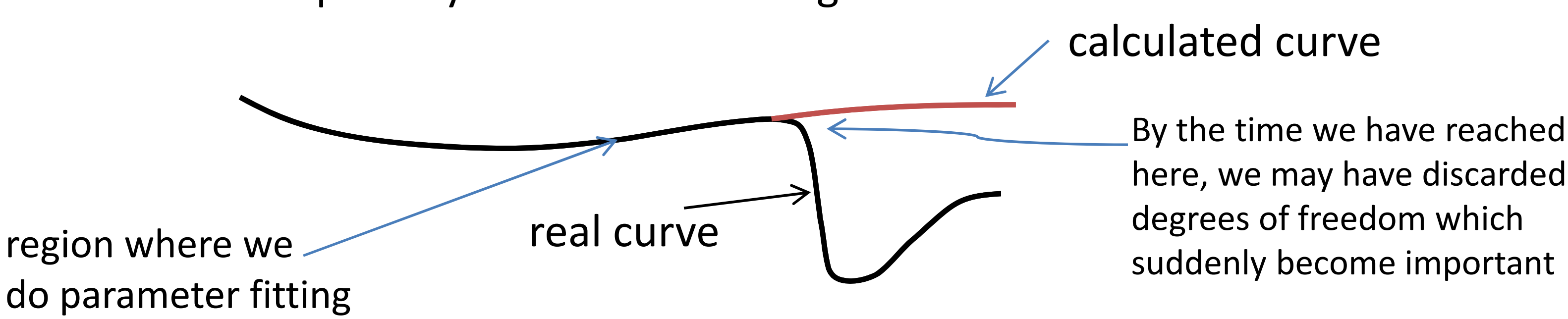
Manipulating ν(x) is only one of many ways to regularize the flow.

Other possibilities include:

- Regularizing u in $u\nabla u$ by a "high frequency filter" ("LES")
- Add terms modeling fine-scale structures ("Reynolds stress")
- Assess the importance of various degrees of freedom by "sensitivity tests" (analogues of $\delta \mathbf{F}/\delta\nu(\mathbf{x})$)
- many other ideas…

All this is a *huge* research area by itself, with obvious practical consequences for engineering and weather prediction.

Remarkable successes, but many problems remain. A typical example: try to "catch" the drag crisis:

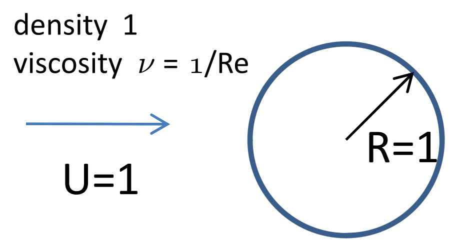


Examples of flows where we cannot directly solve the Navier-Stokes equations and must instead solve some modified model equations.

Pictures to be shown during the lecture

Challenges for PDE theory:

Are the turbulent flows described by smooth solutions of Navier-Stokes? What if the smooth solutions cease to exist as the viscosity gets very small and we get into the highly oscillatory regimes?



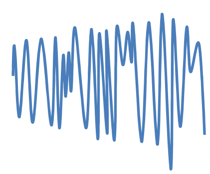
$$u_t + u\,\nabla u + \nabla p - \nu\,\Delta u = 0$$
$$\operatorname{div} u = 0$$

Possible scenario: the smooth solutions exist only if the Reynolds number stays below some critical value.

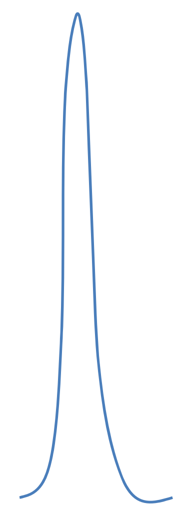
$\mathbf{u}^{(\nu)}$ … solutions for a given ν=1/Re

imagine a critical $\nu_0 > 0$ such that $\mathbf{u}^{(\nu)}$ "break down" as $\nu \to \nu_0$

various possibilities:



frequencies increase beyond any limit



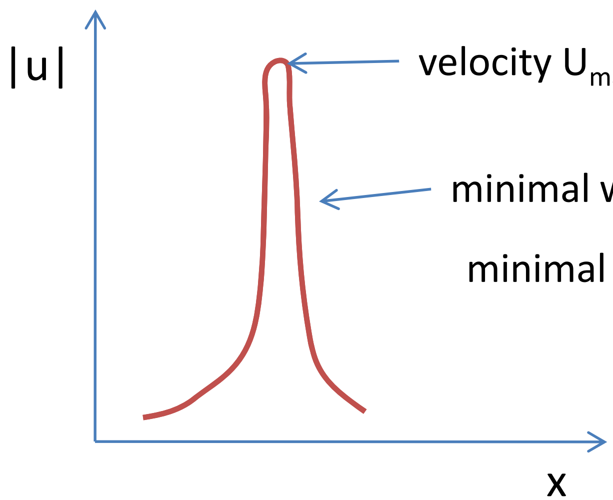
"bursts" with maximal velocities increasing beyond any limit

Regularity criteria
(non-technical description)

(Leray, Ladyzhenskaya, Prodi, Serrin, Scheffer, Caffarelli-Kohn-Nirenberg, Struwe, Constantin-Fefferman, Seregin - V.S., and many others;)

The only possibility for the breakdown of the smooth solutions is through the bursts of high velocity (growing beyond any limits). Moreover, the bursts must have certain minimal dimensions.

Dimensions of the bursts



velocity U_{max}

minimal width $L \sim \nu / U_{max}$

minimal duration $T \sim \nu / U_{max}^2$

Heuristics:

$u_t + u \nabla u + \nabla p - \nu \Delta u = 0$

naively one expects

$u_t \sim U_{max}/T$

$u \nabla u \sim U_{max}^2 / L$

$\nu \Delta u \sim U_{max} / L^2$

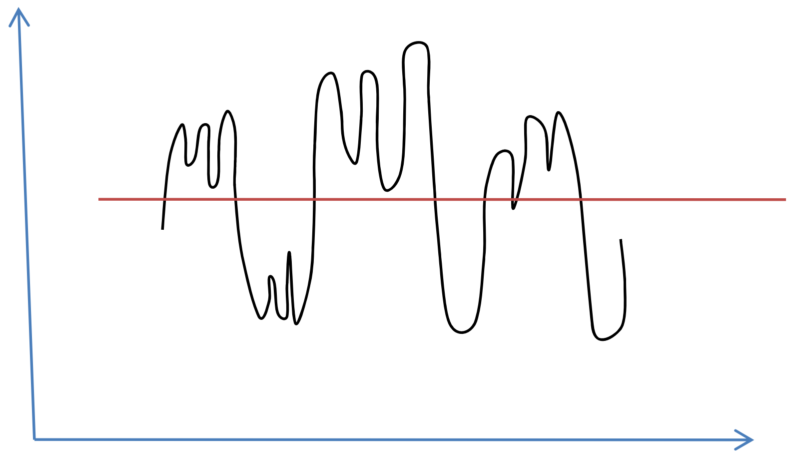
and the terms in the equation should be of the same order of magnitude:

$U_{max}/T \sim U_{max}^2/L \sim \nu U_{max}/L^2$

Kinetic energy in the burst $\sim \varrho L^3 U_{max}^2 \sim \varrho \nu^3 / U_{max}$ – no contradiction!

However, a (much more sophisticated) version of this calculation can be used to show that the singularity set must be mall (CKN).

Typical data from numerical calculations and experiments:



The quantities do oscillate,
but seem to stay within reasonable
distance from the averages –
no extreme bursts

Can we be sure that bursts with exceedingly large velocities do not exists, or do the measurements / numerics just miss them because the volumes involved are too small?
It is likely the former, but we do not really know with 100% certainty.

Example: The complex Ginzburg – Landau equation

Consider complex-valued functions $u\colon R^3 \times (t_1, t_2) \to \mathbf{C}$ and the equation

$$iu_t + (1 - i\varepsilon)\Delta u + |u|^2 u = 0$$

with initial condition $u(x, 0) = u_0$.

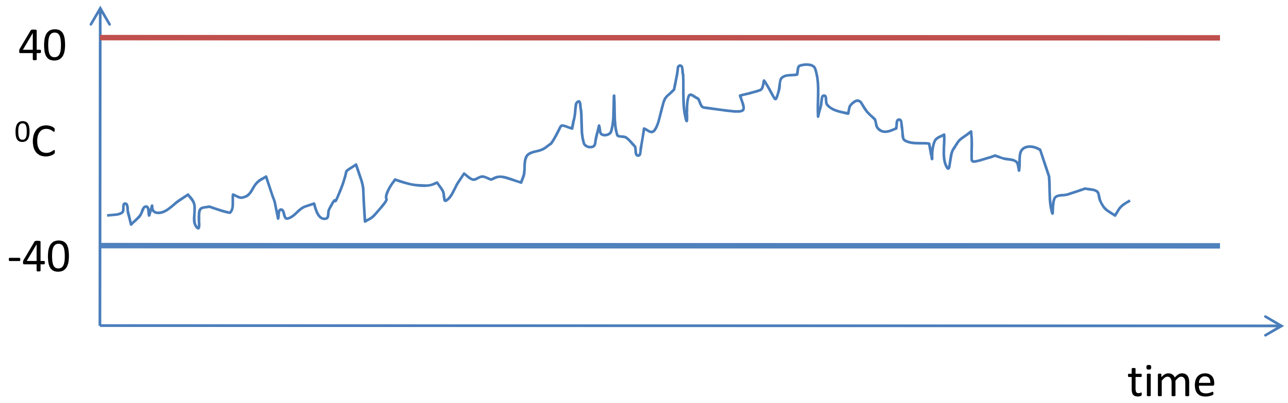
This equations shares with Navier-Stokes

- the energy estimate $\int_{R^3} |u(x,t_2)|^2\,dx + \varepsilon \int_{t_1}^{t_2} \int_{R^3} 2|\nabla u|^2\,dx\,dt = \int_{R^3} |u(x,t_1)|^2\,dx$
- the scaling symmetry $u(x,t) \to \lambda u(\lambda x, \lambda^2 t)$ (so that the Reynolds number considerations are the same).
- regularity criteria analogous to Navier-Stokes: possible breakdown of solutions can happen only through "bursts" of u with the same estimates for the dimension for the bursts.

Solutions of this equation can blow-up in finite time (from smooth initial data), with self-similar bursts of the same dimensions as the estimates for Navier-Stokes. (P. Plechac, V. S.)

The regularity problem is somewhat related to the following issue:

daily temperatures in Minneapolis in 2008

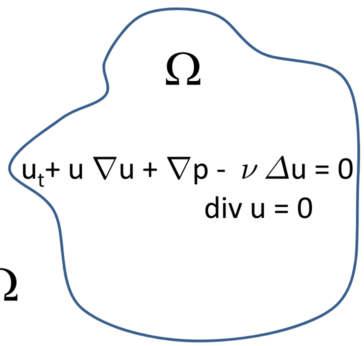


What keeps the temperatures quite firmly between -40 and 40 ^{0}C ?
Energy conservation by itself would be insufficient, there must be other mechanisms.

Do we have similar mechanisms for Navier-Stokes solutions, so that the energy is kept “dispersed” and the high velocity bursts do not happen?

Can something beyond energy conservation be said about the energy and its transportation ?

Navier-Stokes in a bounded domain
with zero boundary condition:



$u=0$
at $\partial\Omega$

In the absence of forcing terms the solution must decay to zero as $t \rightarrow \infty$, just based on energy conservation. Steady state solutions must be trivial.

Replace Ω by all space with $u=0$ at ∞

Can the solution sustain itself by drawing energy from ∞ and transporting it to sustain itself (despite the dissipation)?
The total energy may not be finite, but the equation may put constraints on how it can be transported.

Relevance to possible singularities:
watch the singularity by a microscope in slow motion.
At the microscopic scale, events at a finite distance from the singularity are effectively at ∞.

Ancient solutions and Liouville theorems

Classical Liouville Theorem:

$\Delta u = 0$ in R^n and u is bounded, then u is constant

Parabolic version:

$u_t - \Delta u = 0$ in $R^n \times (-\infty, 0)$ and u is bounded, then u is constant.

Conjectured steady-state Navier-Stokes version:

$u \nabla u + \nabla p - \nu \Delta u = 0$, $\operatorname{div} u = 0$ in R^3 and u is bounded, then u is constant.

Conjectured time-dependent Navier-Stokes version:

$u_t + u \nabla u + \nabla p - \nu \Delta u = 0$, $\operatorname{div} u=0$ in $R^3 \times (-\infty, 0)$ and u,p are bounded, then u,p are constant.

Status of the conjectures:

true in 2 dimensions, open in 3d, except for some partial results for 3d axi-symmetric flows (Koch, Nadirashvili, Seregin, V.S.), and the case of Leray's self-similar singularities (Necas-Ruzicka-V.S., Tsai).

Another related topic: behavior of solutions with $u_0 \in L^\infty$ (Y. Giga); do they stay bounded?

Both the regularity of solutions, and the validity of the Liouville conjecture would require extra properties of the equation beyond energy conservation and the usual consequences of diffusion.

Where can the extra properties come from?

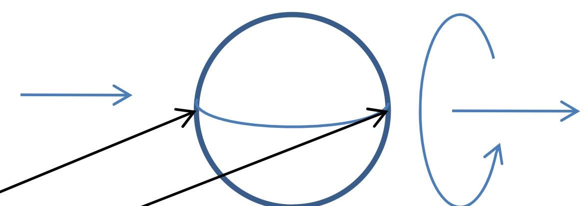
One possibility: hidden monotone quantities, maximum principles, etc.

<u>Simple classical example</u>: axisymmetric flows without swirl

$u_t + u \nabla u + \nabla p - \nu \Delta u = 0$, div $u = 0$, $u(Rx,t)=Ru(x,t)$ for reflections R about planes containing the x_1 axis

Hidden monotone quantity (cylindrical coord.):

$(\mathrm{curl}\ u)_\phi / r$



gives full regularity of solutions, except possibly at and at
also gives the Liouville conjecture for this special case

A non-classical monotone quantity to rule out self-similar singularities

(Necas, Ruzicka, V.S.; Tsai)

Leray's self-similar singularities

$$u(x,t) = \frac{1}{\sqrt{2\kappa(T-t)}}\, U\Big(\frac{x}{\sqrt{2\kappa(T-t)}}\Big)\ .$$

Monotone quantity ruling out these singularities:

$$(T-t)\ \Big(\frac{|u|^2}{2} + p + \kappa x u\Big)\ ,$$

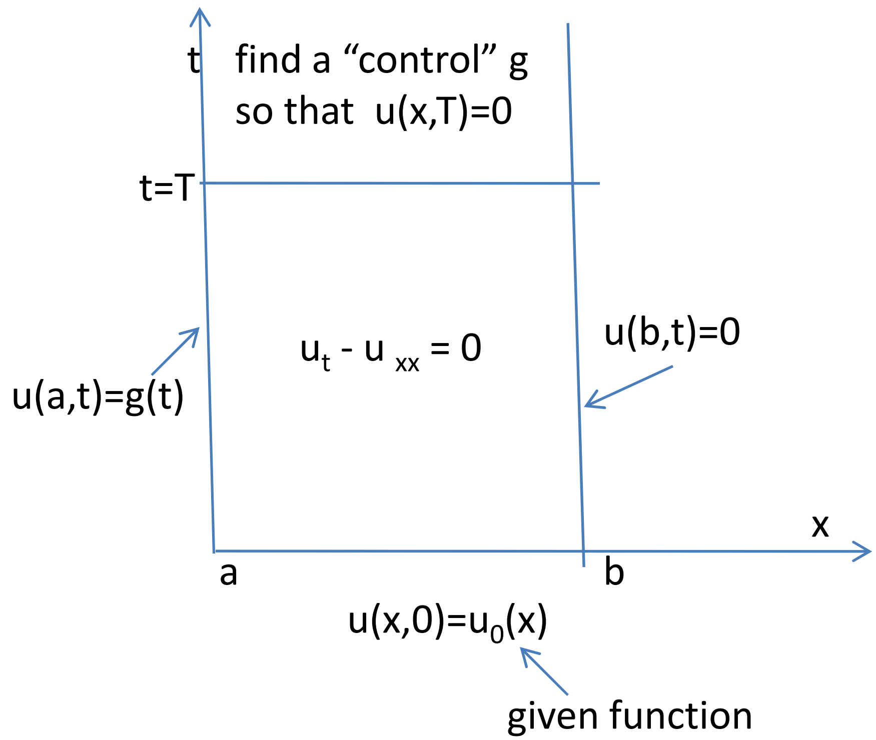
a modification of the Bernouli quantity

$$\frac{|u|^2}{2} + p\ .$$

Link between the regularity theory and the control theory

(Escauriaza-Seregin-V.S.)

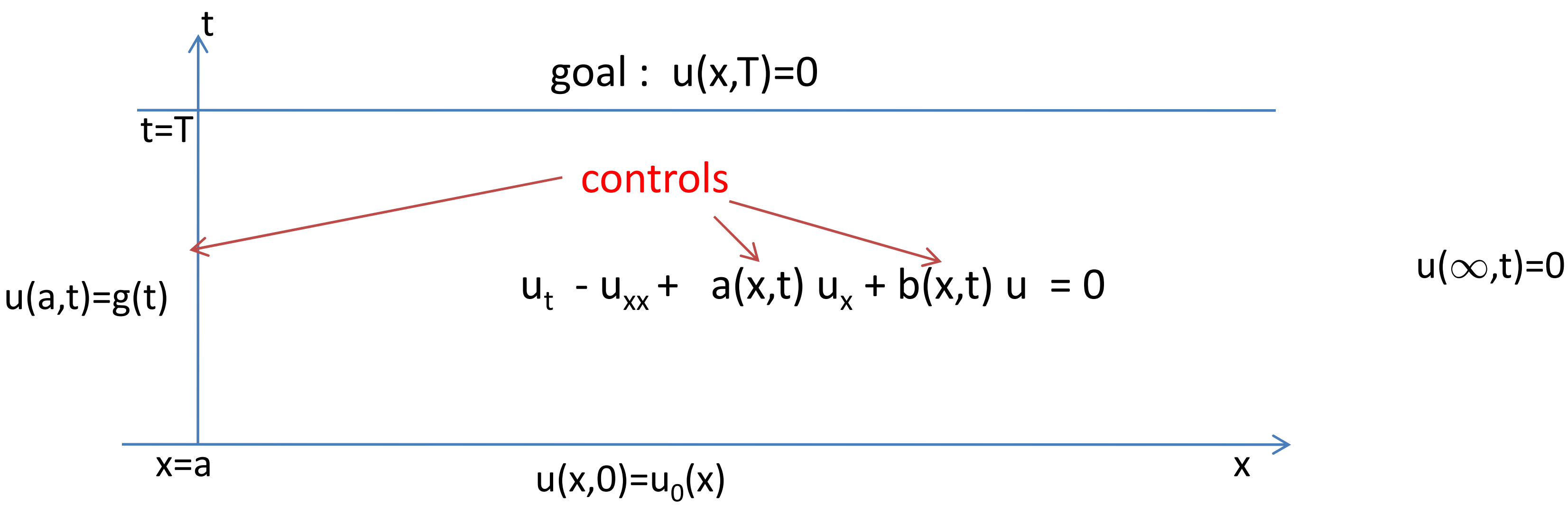
A typical control theory problem



Classical result:

For each u_0 a suitable g can be found

A situation arising in NSE regularity theory (simplified picture)



Interpretation: $g(t)$, $a(x,t)$, $b(x,t)$ represent unknown effects of the non-linearity

achieving $u(x,T) = 0$ $\longrightarrow$ the solution can concentrate, the non-linear effects can prevent the expected diffusion of the energy

via some rescaling and limiting procedures

Result (ESS): such controls never exists, unless $u_0 = 0$

Implication for regularity: $\int |u(x,t)|^3\, dx$ stays bounded $\Rightarrow$ no singularity.

Remark: $||u(t)||_{L^3} / \nu$ is dimensionless ....analogue of Reynolds number

The world of special solutions

So far we have talked about “general solutions” at relatively high Reynolds number. But there is another world of “special solutions”, which are not chaotic and resemble more the world of celestial mechanics, rather then the fluctuating ensembles of statistical mechanics. The study of such solutions goes back to the classics of the 19th century (Kelvin, Poincare, …)

Movie to be shown during the lecture

Euler's equation and geometry

$$u_t + u\nabla u + \nabla p = 0$$
$$\mathrm{div}\, u = 0$$

Particles which interact through the incompressibility constraint

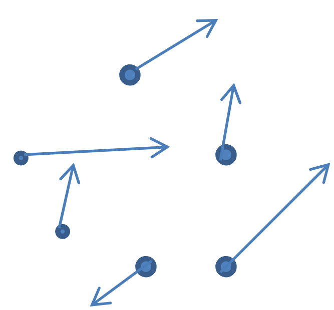
Compare with free particles

$$u_t + u\nabla u = 0$$

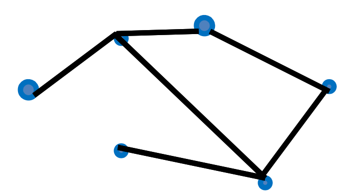
or free particles with (not very physical) friction

$$u_t + u\,\nabla u - \nu\,\Delta u = 0$$

Burger's equation
- it is much better understood than Euler/Navier-Stokes;
$\nu>0$ – no singularities, due to not-so-hidden max principles,
$\nu=0$ – well understood singularities

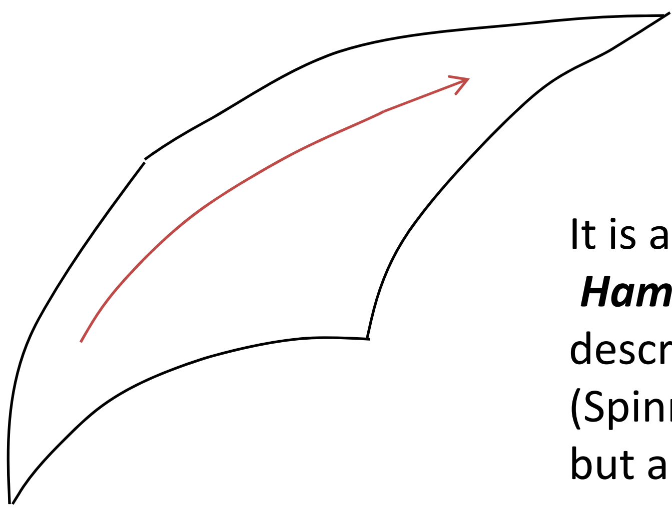


Free particles – easy to understand



A constraint system – details of the motion can be counter-intuitive and difficult to predict without calculation - think of gyroscopes

Euler's equations is completely geometric – it describes geodesics on the non-linear manifold of volume preserving diffeomorphisms (V.I.Arnold). It has no "free parameters" - the whole structure is canonical and quite "rigid" – it is hard to deform it meaningfully.



It is an (infinite-dimensional and complicated) ***Hamiltonian system*** which can be roughly described as **an infinite dimensional spinning top**. (Spinning tops have only 3 degrees of freedom, but are already complicated enough!)

boundary of the
"manifold of solutions"

Examples of a few limiting regimes of Navier-Stokes / Euler

(there are many more)

conjectural KAM
tori of "lifted solutions"

1d non-linear Schroedinger
equation (Hasimoto)

$$i\psi_t + \psi_{xx} - |\psi|^2\psi = 0$$

Thin vortex filaments
(dolphin movie)

1d water wave equations:
Benjamin-Ono equation, KdV , …

$$u_t + uu_x + Hu_{xx} = 0$$

$$u_t + uu_x - u_{xxx} = 0$$

Euler solutions

Navier-Stokes
solutions

2d Euler solutions

Finite-dim Hamiltonian systems
(point vortices),
Statistical Mechanics with
negative temperatures (Onsager)

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \ , \ \dot{p}_i = -\frac{\partial H}{\partial q_i}$$

"finite-dimension-like"
picture of steady-states
(Choffrut, V.S.)

Steady 3d
solutions

behaviour at ∞,
regularity of very
weak solutions

Conformal geometry of
the sphere S^2, via Landau
solutions (V.S.)

$$-\Delta\varphi + 2 = 2e^{\varphi}$$

open problems "dual" to
the regularity problem
regularity – high frequencies
behaviour ay ∞ – low frequencies

conjectural applications
to behaviour at ∞

Classical Mechanics
(planets, spinning tops…):
relatively few degrees of freedom,
follow all of them, use geometry…

Statistical Mechanics:
extremely many degrees of freedoms,
“average out” all but a few, use conservation
laws to deal with those which remain

Fluid Mechanics:
too many degrees of freedom to follow, but not enough
“disorder” for the known averaging methods to work reliably.

Future Solutions:
fusion of Geometry + Statistical Physics + PDE theory + Numerical Analysis
+ Computer Science + “brute force”

A lesson from computer chess:
do not underestimate brute force!