Onsager's Conjecture and Turbulent dissipation.

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Euler (Navier-Stokes)

$$\frac{\partial u}{\partial t} = -(u \cdot \nabla)u - \nabla P + \nu \Delta u + F$$

$$\nabla \cdot u = 0$$

$u \in L^2(\mathbb{R}^3)$ or $u \in L^2(\mathbb{T}^3)$

Energy Balance

$$\frac{1}{2}\frac{d}{dt}\int |u|^2 dx = -\int (\mathbb{P}(u \cdot \nabla u), u)\, dx - \nu \int |\operatorname{grad} u|^2 dx + \int F \cdot u \, dx$$

When u is "sufficiently" smooth

$$\int (\mathbb{P}(u \cdot \nabla u), u)\, dx = 0$$

$$\Rightarrow \quad \frac{d}{dt} E = 0 \qquad \text{(Euler)}$$

$$\frac{dE}{dt} = -\nu \int |\operatorname{grad} u|^2\, dx + \int F \cdot u\, dx \qquad \text{(NSE)}$$

Weak Solutions - Onsager's Conjec

Onsager Conjectured that in 3 D

1) Weak Solutions of Euler with Holder Continuity $h > 1/3$ conserve energy

2) Solutions with $h \leq 1/3$ could dissipate energy

i.e For 3-D turbulent flow dissipation of energy persists without viscosity "turbulent" or "anomolous dissipation" (1949)

(NSE) Kolmogorov's Law (1941)

The energy spectrum $E(|k|)$ for turbulent flow in the "inertial" range i

$$E(|k|) = c\,\bar{\varepsilon}^{2/3}\,|k|^{-5/3}$$

$\bar{\varepsilon}$ is the average dissipation rate

k is the wave number vector

Stated in physical space: Holder exponent h is $1/3$ in a statistically averaged sense.

Note: $\bar{\varepsilon}$ does not vanish as $r \to 0$.

Conjecture underlying turbulence supported by experiments & numerics

average dissipation rate for NSE converges to the average anomolous dissipation rate for Euler in the limit $\nu \to 0$.

For turbulent flows this rate is positive.

So far there is <u>no</u> <u>proof</u>.

"Scale Locality"

Euler equ^n's in Fourier Space $\mathbb{T}^3$

$$u = \sum_{k} e^{2\pi i\, k\cdot x} a(k) \quad , \quad a\cdot k = 0$$

$$\frac{d}{dt} a(k) = -2\pi i \sum_{k'} [a(k-k')\cdot k'] \left[a(k') - \frac{k}{|k|^2}(a(k')\cdot k)\right]$$

$$\frac{1}{2}\frac{d}{dt} a^2(k) = -2\pi i \sum_{k'} [a(k-k')\cdot k'][a(k')\cdot a(k)]$$

Essential interactions in the energy cascade are between wave numbers of similar magnitude.

i.e. the "flux" is controlled by local interactions.

Onsager's Conjecture (one direction)

Eyink (1994): Conservation of energy
Stronger assumption $h > 1/3$

Constantin, E, Titi (1994).
Conservation of energy in $B^{\alpha}_{3,\infty}$, $\alpha > 1/3$.

Duchon + Robert (2000)
slightly weaker condition.

Cheskidov, Constantin, Friedlander, Shvydkoy.
Conservation of energy in $B^{1/3}_{3,\infty\,\mathrm{van}}$. (2008).

Euler Equations

Weak solution

$u_0 \in L^2(\mathbb{R}^3)$: test fn $\varphi \in C^\infty([0,T]\times\mathbb{R}^3)$

$$\nabla\cdot\varphi = 0$$

$$u(t)\,\varphi(t) - u(0)\,\varphi(0) - \int_0^t u(s)\frac{\partial\varphi}{\partial s}\,ds = \int_0^t b(u,\varphi,u)(s)\,ds$$

nd $\nabla\cdot u = 0$ in the sense of distributions $t\in[0,T]$

$$b(u,v,w) = \int_{\mathbb{R}^3} (u\cdot\nabla v)\cdot w\,dx$$

Existence of a weak Solution to Euler with positive smoothness that does not conserve energy remains open.

recall examples of Scheffer, Shnirelman of weak solutions in L^2

non uniqueness

non conservation of energy

but no smoothness.

De Lellis & Szekelyhidi.

Th (Cheskidov, Constantin, F., Shvydkoy 07)

Every weak solution to the Euler equations conserves energy for

$$u \in L^3([0,T];\ B^{1/3}_{3,\infty\,\text{van}})$$

This theorem strengthens the earlier results of Constantin et al and Duchon-Robert

We conjecture the result is sharp. i.e. the precise borderline space for Onsagers conjecture.

<u>Def^n</u> $B^{1/3}_{3,\infty_{van}}$ is the space of tempered distributions $u \in \mathbb{R}^3$

$$\lim_{j\to\infty} \lambda_j^{1/3} \| \Delta_j u \|_3 = 0 , \quad j \geq -1$$

Here $\lambda_j = 2^j$ and $\Delta_j u$ is the j-th piece of the Littlewood-Paley decomposition.

$$H^{5/6} \hookrightarrow \underset{p\geq 2}{B^{1/3}_{3,p}} \hookrightarrow B^{1/3}_{3,\infty_{van}} \hookrightarrow B^{1/3}_{3,\infty}$$

We define the energy flux through a sphere of radius 2^j as follows:

Let $S_j u = F^{-1}(\chi(2^{-j}\xi)\, Fu)$

where $\chi(\xi)$ is a cutoff function supported in the ball radius 1 and $\chi(\xi)=1$ for $0 \le \xi \le 1/2$.

The Littlewood - Paley piece of u is

$$\Delta_j u = S_{j+1} u - S_j u$$

The energy flux is defined as

$$\Pi_j = -\int_{\mathbb{R}^3} (u \cdot \nabla S_j^2 u, u)\,dx$$

Using the test function $\varphi = S_j^2 u$ in the weak formulation of Euler $\Rightarrow$

$$\frac{1}{2}\frac{d}{dt}\|S_j u\|_2^2 = -\Pi_j(t)$$

The total flux $\Pi = \lim_{j \to \infty} \Pi_j$

Note when $\Pi = 0$ we have

$$\lim_{j \to \infty} \frac{d}{dt}\|S_j u\|_2^2 = \frac{d}{dt} E(t) = 0.$$

Th (C-C-F-S)

$$|\Pi_j| \le C\left(\sum_{i=-1}^{\infty} 2^{-\frac{2}{3}|j-i|}\, 2^{\frac{2i}{3}} \|\Delta_i u\|_3^2\right)^{3/2}$$

Conclusions

i) $\Pi = 0$ when $\limsup_{i\to\infty} 2^{i/3} \|\Delta_i u\|_3 \to 0$

i.e. $u \in B^{1/3}_{3,\infty\,\mathrm{van}}$, $\nabla\cdot u = 0$

ii) The theorem gives a precise description of locality.
It demonstrates the decay of non local interactions postulated by Kraichnan.

<u>Th</u>: The total energy flux of any div. free vectorfield in $B^{1/3}_{3,\infty_{van}} \cap L^2$ vanishes.

In particular, every weak sol^n to Euler that belongs to $L^3([0,T]; B^{1/3}_{3,\infty_{van}}) \cap C_w([0,T]; L^2)$ Conserves energy.

<u>Example</u> (following Eyink) is presented of a div free vector field in $B^{1/3}_{3,\infty}$ that gives a positive transfer of energy in frequency space as $J \to \infty$

i.e. $\liminf_{J \to \infty} \Pi_J >$ +ve constant.

<u>Note</u>: existence of a weak solution to Euler in $B^{1/3}_{3,\infty}$ that exhibits anomolous dissipation is still open.

1/ Conservation of helicity $\int (u \cdot \nabla \times u)\, dv$
for $u \in B^{2/3}_{3,\infty\, van}$

2/ In 2D conservation of enstrophy $\int |curl\, u|^2 dv$
for $curl\, u \in L^3([0,T]: L^3)$

3/ For 3D Navier-Stokes weak solutions
$u \in L^3([0,T);\ B^{1/3}_{3,\infty})$ satisfy the
energy balance eqn.

Dyadic Model (F-Cheskidov-Pavlovic)

Decompose Fourier Space into shells

$$2^j \le |k| < 2^{j+1}$$

Define $a_j^2 \simeq \|\Delta_j u\|_2^2$, energy in j-th shell

Recall $|\Pi_j| \le C\left(\sum_{i=-1}^{\infty} 2^{-\frac{2}{3}|j-i|} 2^{2i/3} \|\Delta_i u\|_3^2\right)^{3/2}$

Define model flux $\Pi_j = 2^{cj} a_j^2 a_{j+1}$

where the scaling parameter $c \in [1, 5/2]$

Definition of norms for $\{a_j\}$:

$$\|a\|_{L^2}^2 = \sum_{j=0}^{\infty} a_j^2.$$

$$\|a\|_{H^s}^2 = \sum_{j=0}^{\infty} 2^{2sj} a_j^2$$

Note: 3D $\|u\cdot\nabla u + \nabla P\|_{L^2} \leq \|u\|_{L^2} \|u\|_{H^s}$

$s > 5/2$

note $\lim_{j\to\infty} \Pi_j = \lim_{j\to\infty} 2^{cj} a_j^2 a_{j+1}$

$\to 0$ when $a \in H^s$, $s \geqslant \frac{c}{3}$

Regular solutions have bounded $H^{c/3}$ norm.

Model Energy Flux Equation:

$$\frac{1}{2}\frac{d}{dt}\Big(\sum_{i=0}^{j} a_i\Big)^2 = -\Pi_j - \nu\sum_{i=0}^{j} 2^{2i}a_i^2 + \sum_{i=0}^{j} f_i a_i$$

$$\Rightarrow \frac{1}{2}\frac{d}{dt}a_j^2 = 2^{c(j-1)}a_{j-1}^2 a_j - 2^{cj}a_j^2 a_{j+1}$$
$$- \nu 2^{2j}a_j^2 + f_j a_j$$

The ODE system corresponding to this

$$\frac{d}{dt}a_j = 2^{c(j-1)}a_{j-1}^2 - 2^{cj}a_j a_{j+1} + f_j$$
$$- \nu 2^{2j}a_j$$

Oceanographic turbulence – Desnyanski Novikov, 1974.

Results for the Inviscid Model ($f_0 \neq 0$, $f_j = 0$, $j > 0$)

Th 1 There exists a unique fixed point

$$\alpha_j = 2^{-cj/3}\, 2^{c/6}\, f_0^{1/2}$$

Th 2 Every solution blows up in finite time T in the $H^{c/3}$ norm.

Th 3 The H^s norms for $s < c/3$ are locally square integrable in time.

Note $\|a\|_{H^s} = \left(\sum_{j=0}^{\infty} 2^{2sj} a_j^2 \right)^{1/2}$

Th 4 "Anomalous" dissipation occurs for $t > T$

Th 5 The fixed point is a global attractor.

The inviscid model satisfies Onsager's conjecture on both sides of the "critical" regularity

"Turbulent" Energy dissipation:

Since the support of any time averaged measure belongs to the global attractor, the average dissipation rate of the system equals the dissipation rate of the fixed point

at the fixed point $\varepsilon_d = f_0 \alpha_0$

(i.e. dissipation rate = energy input rate]

hence $\varepsilon_d = f_0^{3/2}\, 2^{c/6}$

and the energy density $E(k) = 2^{c/3} f_0\, k^{-\frac{2c}{3}-1}$

$$\Rightarrow \quad E(k) = C\, \varepsilon_d^{2/3}\, k^{-\frac{2c}{3}-1}$$

case $c=1$: $E(k) \sim \varepsilon_d^{2/3}\, k^{-5/3}$

"Navier-Stokes Model"

Th 1 There exists a unique fixed point $\{\alpha_j\}$ where

i) for $\nu > 0$, $\alpha \in H^s$ for all s

ii) $\lim_{\nu \to 0} \alpha_j = 2^{-cj/3} 2^{c/6} f_0^{1/2}$. (i.e. inviscid fixed point

Th 2 The fixed point is a global attractor.

Dissipation anomaly (Kolmogorov's Law)

Th 3: Let a^ν be a solution to the viscous model then

$$\lim_{\nu \to 0} \lim_{T \to \infty} \frac{1}{T} \int_0^T \nu \, \|a^\nu(t)\|_{H^1}^2$$

$\longrightarrow \bar{\varepsilon}_d$ average inviscid dissipation rate.

$= 2^{c/6} f_0^{3/2} > 0.$

Energy density $E(k) \sim \bar{\varepsilon}_d^{2/3} k^{-\frac{2c}{3}-1}.$

as $\nu \to 0$

Inertial range for the viscous model

The dissipation wave number k_d for the fixed point

$$k_d = \left(\frac{f_0^{3/2}}{\nu^3}\right)^{\frac{1}{4}\frac{2}{3-c}} \sim \left(\frac{\varepsilon_d}{\nu^3}\right)^{\frac{1}{4}\frac{2}{3-c}}$$

Thus for $c=1$ we recover "Kolmogorov"

$$E(k) \sim \varepsilon_d^{2/3} k^{-5/3} , \quad k_d \sim \left(\frac{\varepsilon_d}{\nu^3}\right)^{1/4}$$

for $c = 5/2$

$$E(k) \sim \varepsilon_d^{2/3} k^{-8/3} , \quad k_d \sim \frac{\varepsilon_d}{\nu^3}$$