Filtering Sparse Regular Observed Linear and Nonlinear Turbulent System

John Harlim, Andrew J. Majda

Department of Mathematics and Center for Atmosphere Ocean Science
Courant Institute of Mathematical Sciences, New York University

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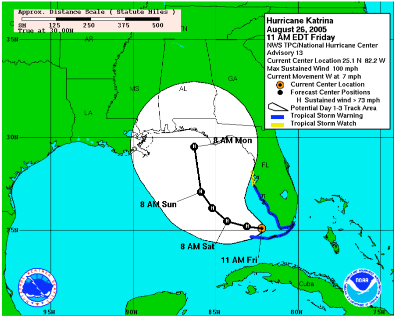
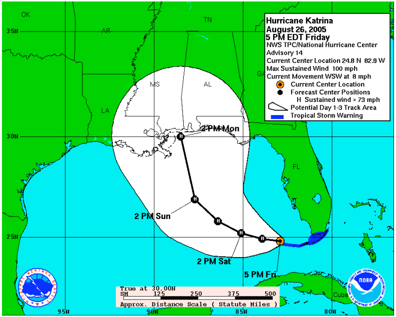
What is filtering (data assimilation)?





A predictor-corrector method that includes observations (via Bayesian update) to improve the real time prediction.

Difficulties in Real Time Filtering and Prediction of Turbulent Signals from Partial Observations

1. For turbulent signals from nature with many scales, even with mesh refinement the model has inaccuracies from parametrization, under-resolution, etc. How to overcome them? Can judicious model error help filtering?

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2. Computational efficiency: how big of ensemble is needed in representing the uncertainty of billions of variables?

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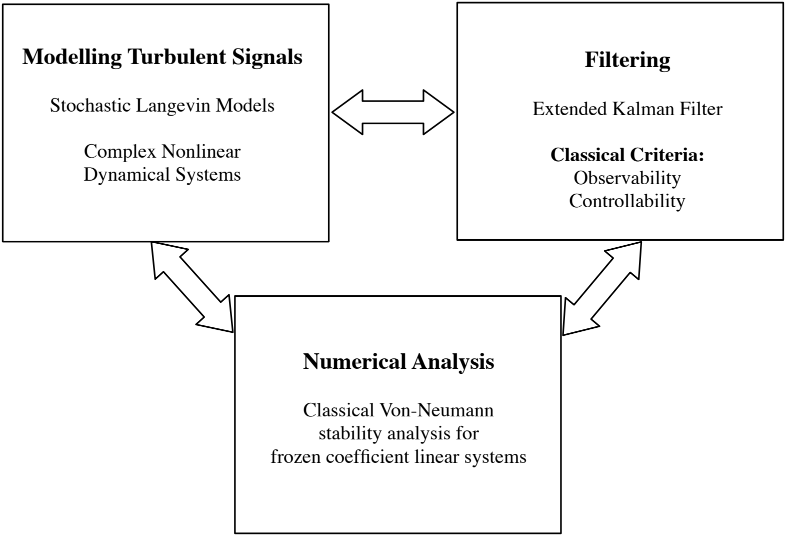
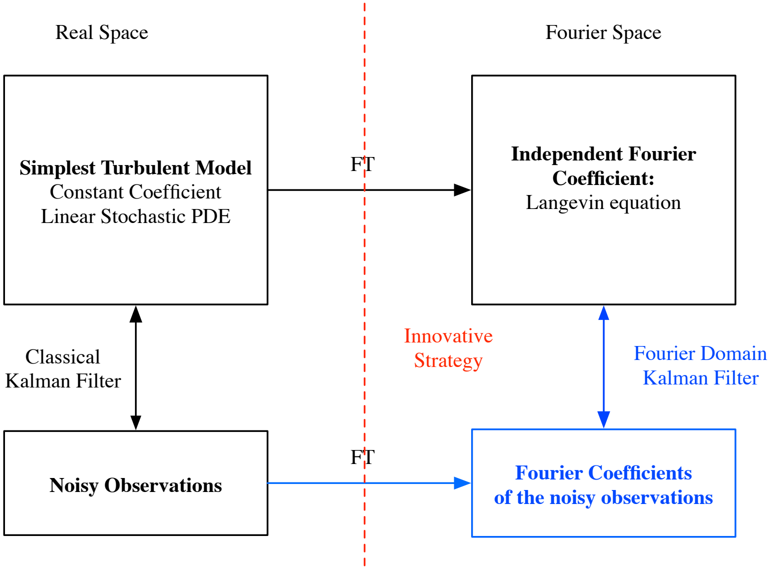
2. Computational efficiency: how big of ensemble is needed in representing the uncertainty of billions of variables?
3. The most accurate ensemble filters is not immune from "catastrophic filter divergence" (beyond machine infinity).

Goal: Provide math guidelines and new numerical strategies thru modern applied math paradigm

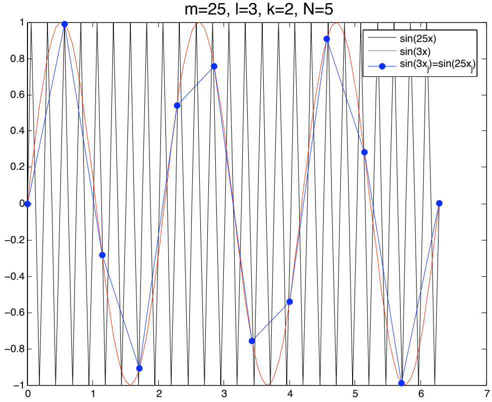


Filtering Linear Problem

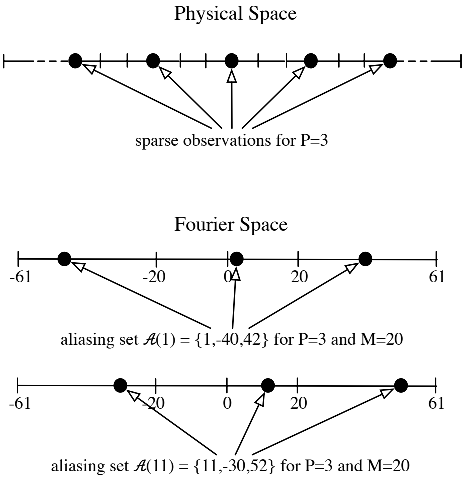


How to deal with Sparse Regularly Spaced Observations ?

ALIASING !!



Example: 123 grid pts (61 modes) but only 41 observations (20 modes) available



Example: Stochastically forced advection-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = -\frac{\partial}{\partial x}u(x,t) - \mu\frac{\partial^2}{\partial x^2}u(x,t) + \bar{F}(x,t) + \sigma(x)\dot{W}(t), 0 \le x \le 2\pi$$

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- **Fourier Domain Kalman Filter (FDKF)**

$$\begin{aligned} d\hat{u}_k(t) &= [(-\mu k^2 - \mathrm{i}k)\hat{u}_k(t) + \hat{F}_k(t)]dt + \sigma_k dW_k(t), \\ FDKF : v_\ell(t) &= \sum_{k_i \in \mathcal{A}(\ell)} u_{k_i}(t) + \eta_\ell^o(t), \end{aligned}$$

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- **Reduced Fourier Domain Kalman Filter (RFDKF)**

$$RFDKF : v_\ell(t) = u_\ell(t) + \eta^o_\ell(t),$$

where $\eta^o_\ell(t) \sim \mathcal{N}(0, r^o/2M+1), |k| \leq N, |\ell| \leq M.$

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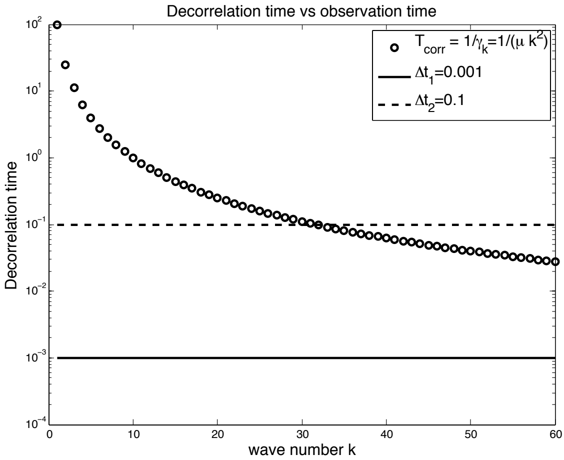
- **Strongly Damped Approximate Filter (SDAF, VSDAF)**: Observation is modeled as in FDKF but we implement it with dynamic-less unresolved modes.

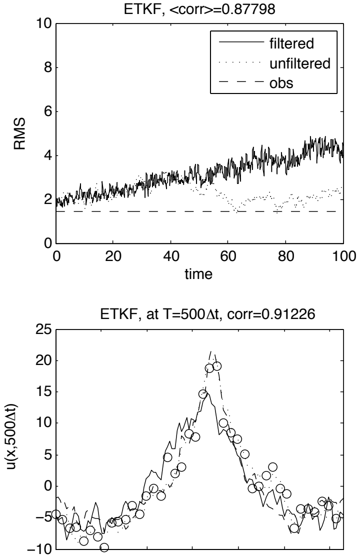
$$\begin{aligned} \mathrm{e}^{-\mu k_1^2 \Delta t} &= \mathcal{O}(1), \\ \mathrm{e}^{-\mu k_i^2 \Delta t} &= \mathcal{O}(\epsilon) \ll 1, \quad 2 \le i \le P. \end{aligned}$$

Skill of cheaper approximate filters with model error depends on observation time at given wave number and how rough spectrum is for truth signal.

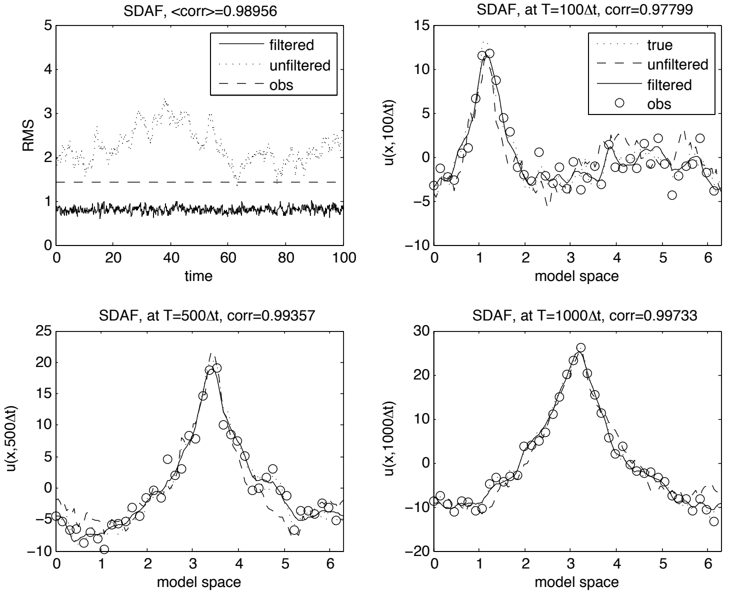


ETKF Filter Divergence ($K = 150, r = 40\%$), observability is violated

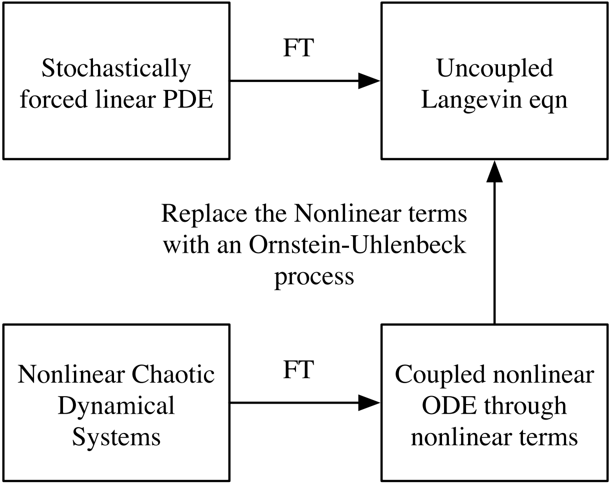
Extreme event, $\Delta t_2 = 0.1, E_k = k^{-5/3}$



SDAF high skill (observability is satisfied) Spontaneous development of extreme event for $\Delta t_2 = 0.1$ and $E_k = k^{-5/3}$



Radical Filtering Strategy for Nonlinear System

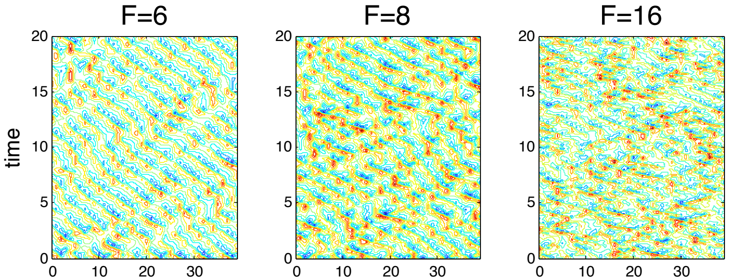


Filtering turbulent nonlinear dynamical systems

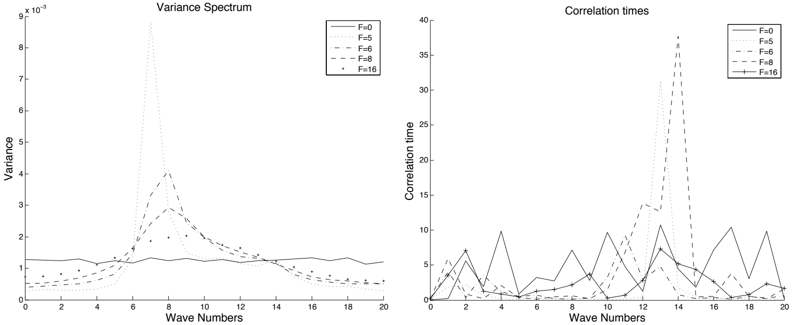
L-96 model (Lorenz 1996), 40 modes.
(absorbing ball property)

$$\frac{du_j}{dt} = (u_{j+1} - u_{j-2})u_{j-1} - u_j + F, \quad j = 0, \ldots, J-1$$

Energy Rescaled Variables:
F=6 weakly chaotic
F=8 strongly chaotic
F=16 fully turbulent

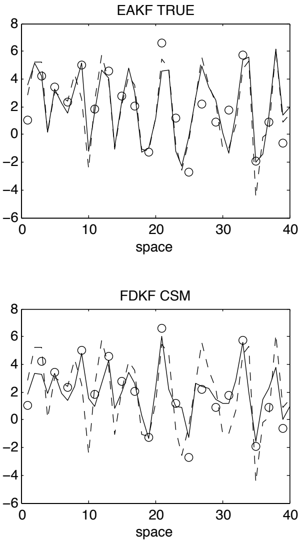


Climatological Variance and Correlation time



Climatological Stochastic Model (CSM): fit the damping coefficient and stochastic noise strength to these climatological statistical quantities.

Regularly spaced sparse observations: weakly chaotic regime $F = 6, P = 2, r^o = 1.96, \Delta t = 0.234$

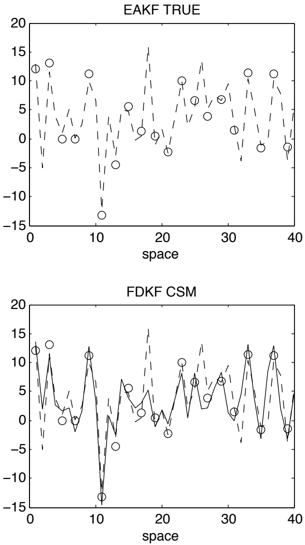


Regularly spaced sparse observations: weakly chaotic regime
$F = 6, P = 2, r^o = 1.96, \Delta t = 0.234$ hrs

Table: This is a regime where EAKF true is superior.

scheme	RMS	corr.
EAKF true	0.82	0.95
EAKF CSM	2.20	0.64
ETKF true	∞	-
ETKF CSM	2.50	0.55
FDKF CSM	2.07	0.69
RFDKF CSM	2.39	0.60
No Filter	2.8	-

Regularly spaced sparse observations: fully turbulent regime
$F = 16, P = 2, r^o = 0.81, \Delta t = 0.078$ hrs



Regularly spaced sparse observations: fully turbulent regime
$F = 16, P = 2, r^o = 0.81, \Delta t = 0.078$ hrs

Table: This is a regime where FDKF is superior.

Scheme	RMS	corr.
EAKF true	∞	-
EAKF CSM	5.15	0.61
ETKF true	∞	-
ETKF CSM	5.80	0.54
FDKF CSM	4.80	0.66
No Filter	6.3	-

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- The "poor-man's" CSM model degrades the filtering skill in the weakly chaotic regime but suggests encouraging results in the strongly chaotic and fully turbulent regimes.
- Practically, our radical strategy is independent of tunable parameters and ensemble size.
- Catastrophic filter divergence in a chaotic DS with absorbing ball property needs further mathematical theory.

References:

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HM2 Harlim and Majda, "Mathematical test criteria for filtering complex systems: Regularly sparse observations", J. Comp. Phys., 227(10), 5304-5341, 2008.

HM3 Harlim and Majda, "Catastrophic filter divergence in filtering nonlinear dissipative systems", submitted to Comm. Math. Sci., 2008.