PROBLEM SESSION: HOMOGENOUS DYNAMICS

Notes by Lior Silberman (lior@math.ubc.ca), who is responsible for all errors.

1. Introduction: J. Athreya

Homogenous dynamics: $H \curvearrowright G/\Gamma$ where G ois locally compact group, $H < G$ a closed subgroup, $\Gamma < G$ a lattice.

Important base case is the action of $\mathrm{SL}_2(\mathbb{R})$ on $\mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$. Interpreting this as the space X_2 of unimodular lattices we can generalize this to $\mathrm{SL}_n(\mathbb{R})/\mathrm{SL}_n(\mathbb{Z})$ ("increase the dimension") but also ("increase genus") to an $\mathrm{SL}_2(\mathbb{R})$-action on the space of metrics on a high-genus surface, that is Teichmüller dynamics.

2. U. Hamenstädt

2.1. **Weyl chamber flow.** This is $A \curvearrowright G/\Gamma$ where G is simple Lie group (non-compact type), $\Gamma < G$ a uniform lattice, A a maximal split Cartan subgroup, $M = Z_K(A)$. Inside have a 1d Anosov flow using the "barycenter direction" in the Weyl chamber of A; this is part of the natural decomposition of the geodesic flow.

Problem 1. Is there a symbolic coding for this flow?

Problem 2. What about specification and orbit closures?

Fact 3. *Periodic orbits are compact tori of dimension equal to the rank.*

Problem 4. Can you count conjugacy classes in Γ according to length? What about classes near the "barycenter" direction?

- Can count periodic orbits *weighted* by the volume of the corresponding torus (Bowen), but these weights might be *exponentially large* in the length of the orbit (this is a problem in classfield theory)

2.2. **Higher Teichmüller theory.** Related to paper of Bowen IHES volume 60: let $\Gamma = \pi_1(S)$ be a surface group and consider an injective representation $\rho\colon \Gamma \to \mathrm{PSL}_2(\mathbb{C})$ with convex cocompact image. Bowen shows there is a hyperbolic metric on S such that the length of each closed geodesic γ is at least the length of $\rho(\gamma)$ with equality iff $\rho(\Gamma)$ is Fuchsian.

Problem 5. Replace $\mathrm{PSL}_2(\mathbb{C})$ with by G, ρ an Anosov representation. Entropy goes down when moving off Fuchsian locus. Can we choose the hyperbolic structure such that the lengths will go up?

Date: August 1, 2017.

3. I. Gekhtman

Let $M = \tilde{M}/\Gamma$ be a compact negatively curved manifold (say $\tilde{M} = \mathbb{H}^2$). Let μ be either a finite-range random walk on Γ or a discretization of Brownian motion. Let ν be a μ-stationary measure on $\partial\tilde{M} = \partial\Gamma$ (e.g. if $\tilde{M} = \mathbb{H}^2$ and μ discretizes Brownian motion then ν is ac wrt Lebesgue measure).

Let m be a geodesic-flow-invariant measure on T^1M which is the projection of a measure on $T^1\tilde{M}$ in the class of $\nu \times \nu \times dt$ where $T^1\tilde{M} \simeq \left(\partial\tilde{M} \times \partial\tilde{M} \setminus \Delta\right) \times \mathbb{R}$.

Let $\omega_n = g_1 g_2 \cdots g_n$ where g_i chosen according to μ be the mu-RW. It's known that they are hyperbolic for large n, say with axis γ_n.

Conjecture 6. *Let $A \subset T^1M$ have boundary of measure zero. Then for almost every trajectory of the μ-RW, the projections $\bar{\gamma}_n$ of the axes to T^1M (where they are closed geodesics) we have*

$$\frac{\ell\left(\bar{\gamma}_n \cap A\right)}{\ell\left(\bar{\gamma}_n\right)} \xrightarrow[n\to\infty]{} m(A).$$

4. F. Ledrappier

Problem 7. Give an example of a measure-preserving action of $\mathbb{R}^n$ that is mixing but not mixing of all orders.

Remark 8. No such example for action of semisimple G with finite centre [5]

Problem 9. Let M be a complete open manifold with pinched negative curvature (i.e., all sectional curvatures bounded above and below by some negative numbers). Let Ω be the non-wandering set of the geodesic flow and assume Ω is compact. Is the geodesic flow mixing for the measure of maximal entropy?

Remark 10. This is true for some special cases: (a) CAT(0)-space admitting a rank one axis [7]. (b) The limit set is the whole boundary.

- For the next two problems let S be a hyperbolic surface (isometric to $\mathbb{H}/\Gamma$ for some discrete Γ). On T^1S we have the *horocycle flow* $N = (n_x)_{x\in\mathbb{R}}$ and the *geodesic flow* $A = (g_t)_{t\in\mathbb{R}}$.

Problem 11. Description of closed invariant sets for the horocycle flow $N = (n_x)_{x\in\mathbb{R}}$ on a surface with constant negative curvature.

Remark 12. Let $A^+ = (g_t)_{t\geq 0}$ be the positive geodesic flow. Alexandre Bellis gave an example [1] of a geometrically infinite hyperbolic surface $S = \mathbb{H}/\Gamma$ such that the horocycle $\overline{Nu}$ meets the geodesic ray A^+u at an unbounded sequence of times, then $A^+u \subset \overline{Nu}$.

Problem 13. Classification of all Radon measures which are invariant and ergodic under the horocycle flow on T^1S.

Remark 14. The geometrically finite case is well understood thanks to h[4, 2, 3, 6]. In the geometrically infinite case, Sarig [8] shows that if S is weakly tame then any non-trivial ergodic horocycle-flow-invariant Radon measure on T^1S is quasi-invariant under the geodesic flow.

5. T. Meyerovich

Problem 15. Find an "algebraic" characterization of the pseudo-orbit tracing property for algebraic actions.

Definition 16. An action of a countable group Γ on a compact metric space (X, d) has the *pseudo-orbit tracing property* if for all $\epsilon > 0$ there is $\delta > 0$ and a finite set $S \subset \Gamma$ such that every (S, δ)-pseudo-orbit is ϵ-traced by a genuine orbit.

Definition 17. An (S, δ)-pseudo orbit is a function $x_\cdot \colon \Gamma \to X$ such that for all $S \in S$ and $\gamma \in \Gamma$, $d\left(s \cdot x_\gamma, x_{s\gamma}\right) < \delta$. This is *$\epsilon$-traced* by the orbit of $x \in X$ if $d\left(\gamma x, x_\gamma\right) < \epsilon$ for all $\gamma \in \Gamma$.

Example 18 (Shadowing Lemma; Bowen)**.** Anosov diffeomorphisms.

Example 19. In the expansive totally disconnected situation this is equivalent to being a subshift of finite type.

Definition 20. An action of Γ is *algebraic* if X is a compact abelian group and and Γ acts by continuous automorphisms.

Prototypical algebraic actions are automorphisms of $\mathbb{R}^d/\mathbb{Z}^d$. Algebraic actions are determined by their *dual module* in the group algebra $\mathbb{Z}[\Gamma]$.

Example 21. Known cases

- Every expansive principal algebraic action has property.
- Expansive algebraic $\mathbb{Z}$-actions have the property.
- If an expansive action has the property, the dual module is *finitely presented.*
- For expansive algebraic Γ-actions on totally disconnected group (Γ-subshifts) the property is equivalent to having a finitely presented dual.
- There are examples of expansive algebraic actions with a finitely presented dual but without the pseudo-orbit tracing property.
- S. Bhattacharya recently gave an example of an expansive action of a polycyclic group that does not have the property.

6. R. Tanaka

Let Γ be a discrete group acting by isometries on the Poincaré disc $\mathbb{D}$. Suppose Γ has a compact fundamental domain (can be generalized further). Let μ be a measure on Γ supported on a generating set and consider the associated random walk on Γ. The exit measure on $\partial\Gamma = \partial\mathbb{D}$ is a μ-stationary measure ν.

Problem 22. For which μ is the stationary measure ν doubling? Specifically, is an exponential moment enough?

Definition 23. A measure ν on a metric space (X, d) is *doubling* if there is $C > 0$ such that for all $x \in X$, $r > 0$ one has

$$\nu\left(B(x, 2r)\right) \leq C\nu\left(B(x, r)\right) .$$

Note that the definition of doubling depends on the metric on the boundary; we take the usual distance on the circle as $\partial\mathbb{D}$ but note that hyperbolic isometries of $\mathbb{D}$ act on $\partial\mathbb{D}$ by Lipschitz maps so the doubling property is unchanged.

Example 24. When μ has finite range, it has a nice Green's function and doubling for the exit function is known. Similarly for strong enough moment conditions.

So, an equivalent version of the problem is whether the harmonic measure can be understood without the Green's function.

Example 25. Take the SRW on the free group $\Gamma(2)$. The exit measure is then well-behaved from the point of view of the cantor-set structure of the leaves of the tree, but not doubling for the Euclidean structure of the boundary.

7. S. Hurder

Plante: connection between growth of a leaf and dynamics of a leaf. E.g.: in an Anosov system the strong stable leaf has polynomial growth. Can prove hyperfiniteness from that (as opposed to the proof based on symbolic coding).

Say M is compact, $L \subset M$ is an immersed leaf of our foliation. Can then restrict the metric to L and consider the volume growth of L in that metric, that is the g

View the moving along leaves as a multiparameter dynamical system; the volume growth of the leaf describes the number of states accessible at "time" r.

Can ask what the results of Bowen et al mean for the dynamics-of-foliations problems stemming from the developments of Hirsch, Thurston, Plante.

One way to get a foliation is to let the group $\Gamma = \pi_1(B)$ act on a compact manifold T ("transversal"). Then the *suspension* $M = \tilde{B} \times T/\Gamma$ (diagonal action) has a foliation with leaves T. In general for a foliation $\mathcal{F}$ on a manifold M we have local transversals T to the foliation, and Ha.e. fliger constructed the "foliation peudogroup" acting on T.

More generally can investigate the dynamics of pseudogroup actions.

Ghys, Langevin, Walczak introduced the *geometric entropy* of a foliation:

$$h(\mathcal{F}) = \lim_{\epsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log \#S(n,\epsilon)$$

where $S(n,\epsilon)$ is defined as follows: for two points x, y in a transversal T_α apply a word of length n in the pseudogroup (that is, move along the leaf). If both reach the same transversal T_β see if the distance is at least ϵ (if they don't reach the same transversal they are definitely separated). Then $S(n,\epsilon)$ counts distinct histories in this sense.

Example 26. Consider $F_2 \simeq \mathbb{Z} \star \mathbb{Z}$ acting by isometries on S^2. Then there is entropy but no growth (action by isometries).

More generally needs to look at the growth function $\#S(n,\epsilon)$.

Problem 27. Find slow-entropy invariants when $\#S(n,\epsilon)$ grows subexponentially.

Problem 28. How to relate the Ghys–Langevin–Walchak entropy of a foliation with the Lyapunov structure of the foliation.

For the Lyapunov structure, take x, y in the leaf and flow by geodesic flow on the leaf from x to y. See what this does to the transverse volume.

Problem 29. Find a notion of measure (metric) entropy for foliations and prove a maximum principle.

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