Singular Diffusion Equations With Nonuniform Driving Force

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(Joint work with M.-H. Giga)

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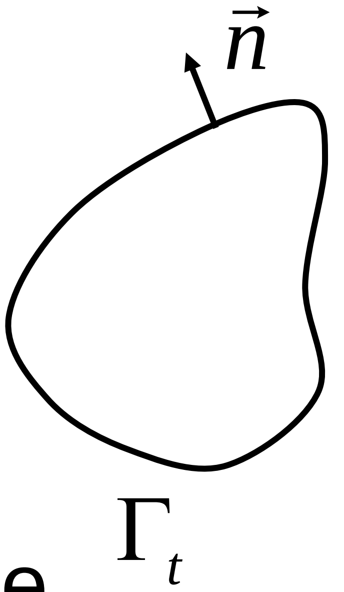
Contents

0. Introduction

curvature flow with a **singular** interfacial energy



$$V = M(\vec{n})\ (\kappa_\gamma + \sigma) \text{ on } \Gamma_t \subset \mathbf{R}^2$$

$\gamma = \gamma(\theta)$: interfacial energy density

$\vec{n} = (\cos\theta, \sin\theta)$: normal of Γ_t

$\kappa_\gamma = (\gamma''(\theta) + \gamma(\theta))\kappa$: weighted curvature

σ : driving force, $M(\vec{n}) > 0$: mobility

$$\gamma'' + \gamma = \Sigma c_k \delta(\theta - a_k)$$

crystalline energy: typical singular energy!

1. Typical Problems: (σ :given)

(a) $u_t = (\operatorname{sgn} u_x)_x + \sigma(x), \ x \in R, \ t > 0$

with I.C. $u(x,0) = u_0$

More generally,

(b) $u_t = a(u_x)[(W'(u_x))_x + \sigma(x)]$

W **: convex,** $a > 0 \ (W(p) = |p|, \ a \equiv 1)$

$a \neq \text{const} \ \Rightarrow$ **Non divergence form**

Feature

Energy density W has
jump discontinuities
so that diffusion is singular.

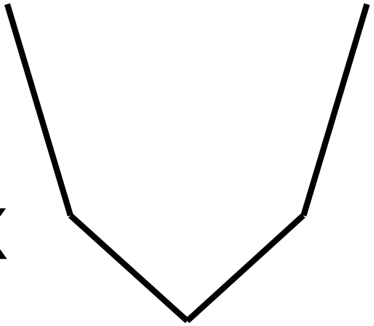
(a) is of the form

$$u_t = 2\delta(u_x)u_{xx} + \sigma(x).$$

Source of Problems

(i) Total variation flow: (a)

(ii) Crystalline curvature flow with driving force term: (b)



$W(p)$**: piecewise linear, convex**

$$a(p) = (1 + p^2)^{1/2}, \ \sigma = \text{constant}$$

Angenent-Gurtin ’89, Taylor ’91

Further Applications (Vertical Diffusion)

Burgers eq. $u_t + uu_x = 0$

$\Leftrightarrow\ V = y$ on Γ_t (*)

$\Gamma_t = \{(x,y);\ y = u(x,t)\}$ (graph of u)

A solution of (*) may overturn and cannot be viewed as the graph of entropy solution.

Consider

$$V = y - M\ \mathrm{div}\nabla_p \gamma(\vec{n}), \quad \gamma(p_1, p_2) = |p_2|$$

instead of (*), where $M > 0$.

‘Thm’ (M.-H. Giga – Y. G. ‘03)

If $M > 0$ is sufficiently large (with respect to jump size), then overturn is prevented and Γ_t becomes the graph of entropy solution at least for Riemann problem.

cf. Y. G. ’02, Y.-H. R. Tsai – Y.G. – Osher ’02,
Y. Brenier: ’09 formulation by an obstacle function

Notion of Solutions

(1) Subdifferential formulation: (a) (not (b))

$u_t \in -\partial E(u)$ **Fukui-Y.G. '96** $a = \mathrm{const}$

$\sigma : \mathrm{general}$

(2) Viscosity approach: (b)

Mi-Ho Giga-Y.G. '98, '99, '01

$\sigma : \mathrm{const},\ u_t + F(u_x,\ (W'(u_x))_x + \sigma) = 0$

Goal: Extend the theory for σ depending on X !

Nonlocal Quantity

Consider simplest eq $(W(p)=|p|,\ \sigma\equiv 0)$

$$u_t=(\operatorname{sgn} u_x)_x.$$

What is the speed of the facet (flat part)?

Assume 'facet stays as facet'

$$\int_{a-\delta}^{b+\delta} u_t = W'(-\bar{\delta}) - W'(+\tilde{\delta})$$

$$\Rightarrow u_t=\frac{-2}{b-a}\ (\delta\downarrow 0)$$

$$\tilde{\delta}=u_x(a-\delta), -\bar{\delta}=u_x(b+\delta)$$

▪▪▪▪▪▪► **Crystalline flow for admissible polygon.**

Non uniform σ

What is the quality

$$\Lambda_W^{\sigma}(u)\ (x) = (W'(u_x))_x + \sigma(x)?$$

In general Λ_W^{σ} is not constant on the facet so that facet may break.

Speed of Evolution (a)

Thm (Komura, Brezis - Pazy)

H: Hilbert space, E: convex, lower semicontinuous, $u_0 \in H$

⇒ There exists **unique** solution $u \in C([0,\infty),H) \cap AC([\delta,T],H)$

solving
$$\frac{du}{dt} \in -\partial E(u) \quad \text{a.e. } t>0,\ u(0)=u_0.$$

Moreover, u is right differentiable for all $t>0$ and

$$\frac{d^+u}{dt} = -\partial^0 E(u).$$

canonical restriction / minimal section:

$$\partial^0 E(u) = \operatorname{argmin}\{\|f\|_H \,;\, f \in \partial E(u)\}$$

Solution knows how to evolve !

How to calculate the Speed

$$E(u) = \int_{\mathbf{T}} \{W(u_x) - \sigma(x)u\}\, \mathrm{d}x, \quad \mathbf{T} = \mathbf{R} / \mathbf{Z}, \quad H = \mathrm{L}^2(\mathbf{T}),$$

$$f \in \partial E(u) \Leftrightarrow \quad f = -\eta_x - \sigma, \quad \eta(x) \in \partial W(u_x(x)),$$

$$f^0 = \partial^0 E(u)$$

$$\Leftrightarrow f^0 = \mathrm{argmin}\{\int_{\mathbf{T}} |f|^2\, \mathrm{d}x;\ f = -\eta_x - \sigma,$$

$$\eta(x) \in \partial W(u_x(x))\}.$$

Obstacle type condition at the place where the slope belongs to the jump of W.

Explicit Solutions

(1) σ : const **Angenent-Gurtin, Taylor**

'admissible polygon'

(2) $\sigma(x)=\sigma(-x),\ \sigma_x>0,\ \text{e.g. } a(p)=(1+p^2)^{1/2}$

Explicit solution starting with $u_0\equiv 0$ **.**

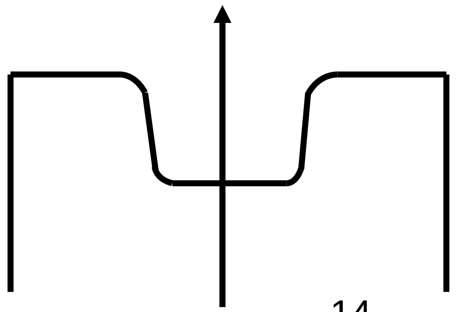
(bending solution)

Nonlocal Hamilton-Jacobi equations with unusual free boundary

Y.G.-Gorka-Rybka (2010?)

Y.G.-Rybka (2009)

[M.-H. Giga-Y.G. '98 for (a)]



2. Variational Characterization

From (a) we learn the speed Λ_W^σ is given by solving an obstacle problem.

$I=(a,b),\ W(p)=|p|$ **for simplicity.**

Admissible set

$$\tilde{K}_{\chi_-\chi_+}(I)=\{\eta\in H^1(I):\ |\eta|\le 1,\ \eta(a)=-\chi_-,\eta(b)=\chi_+\}$$

(χ_-,χ_+ takes either -1 or $+1$)

Energy and Curvature

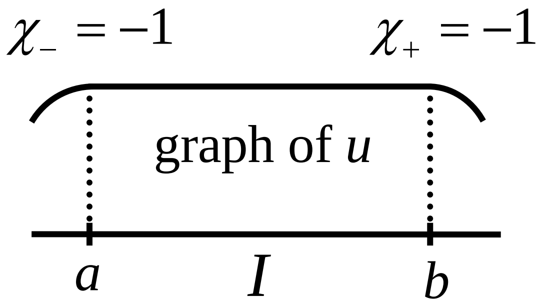
$$J_{\chi_-\chi_+}(w, I) = \int_I |w + \sigma|^2 \, dx$$

$$w_0^{\chi_-\chi_+} = \arg\min\{J_{\chi_-\chi_+}(w, I) \mid w = \eta_x, \ \ \eta \in \tilde{K}_{\chi_-\chi_+}(I)\}$$

u :faceted with slope zero on I with
transition number $\chi_-\chi_+$
Define

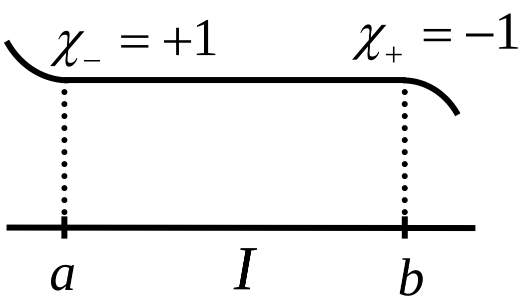
$$\Lambda_W^\sigma(u)\ (x) := w_0^{\chi_-\chi_+}(x) + \sigma(x), \ x \in I$$

Transition number of u near facet I



χ_-, χ_+ takes either $+1$ or -1

$+$ $-$



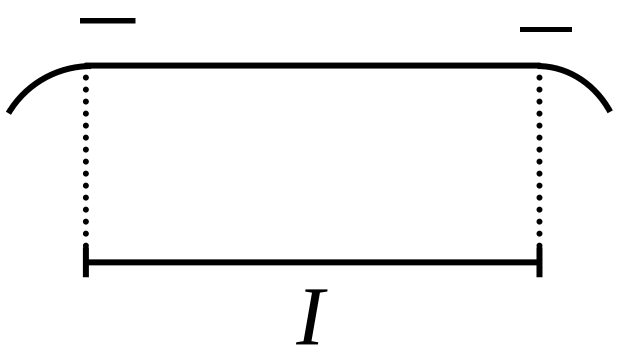
χ_- : left transition number

χ_+ : right transition number

$$\chi_- = \begin{cases} 1 \text{ if } u(x) \geq u(a) \text{ for } x \leq a \text{ near } a \\ -1 \text{ if } u(x) \leq u(a) \text{ for } x \leq a \text{ near } a \end{cases}$$

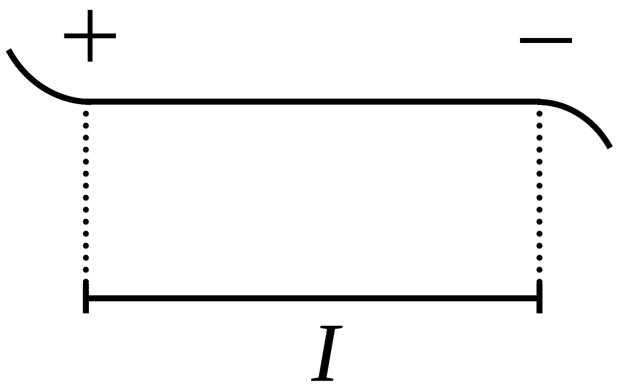
Example

（1） $\Lambda_W^{\sigma}(u)\ (x) = w_0^{--}(x) + \sigma(x)$



graph of u

（2） $\Lambda_W^{\sigma}(u)\ (x) = w_0^{+-}(x) + \sigma(x)$

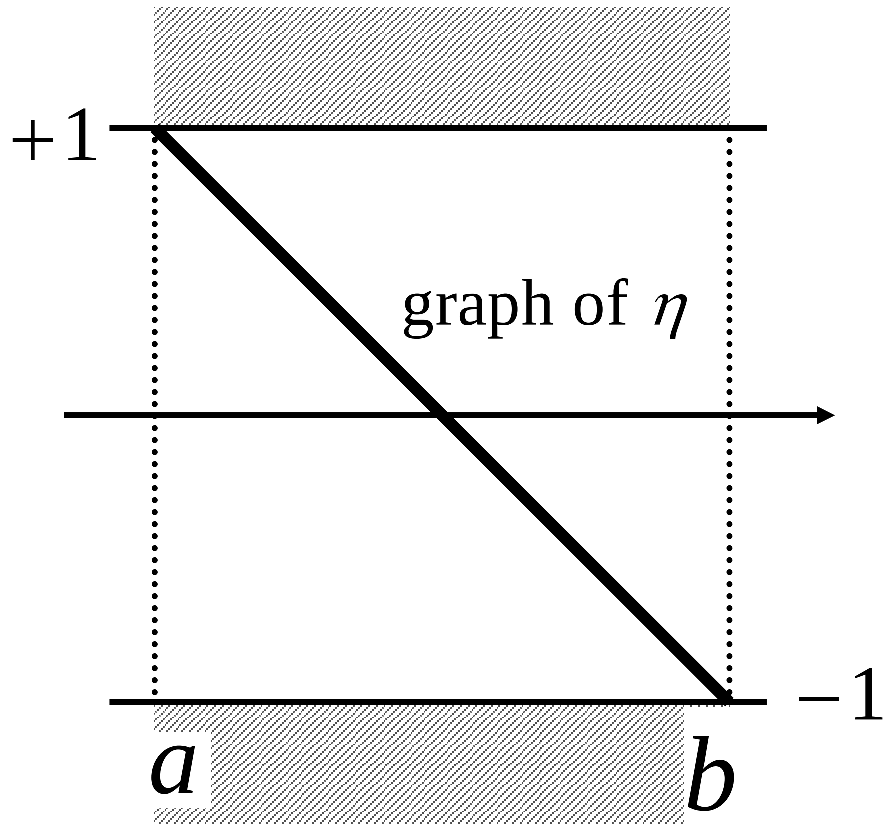


graph of u

More on Values of Λ_W^σ

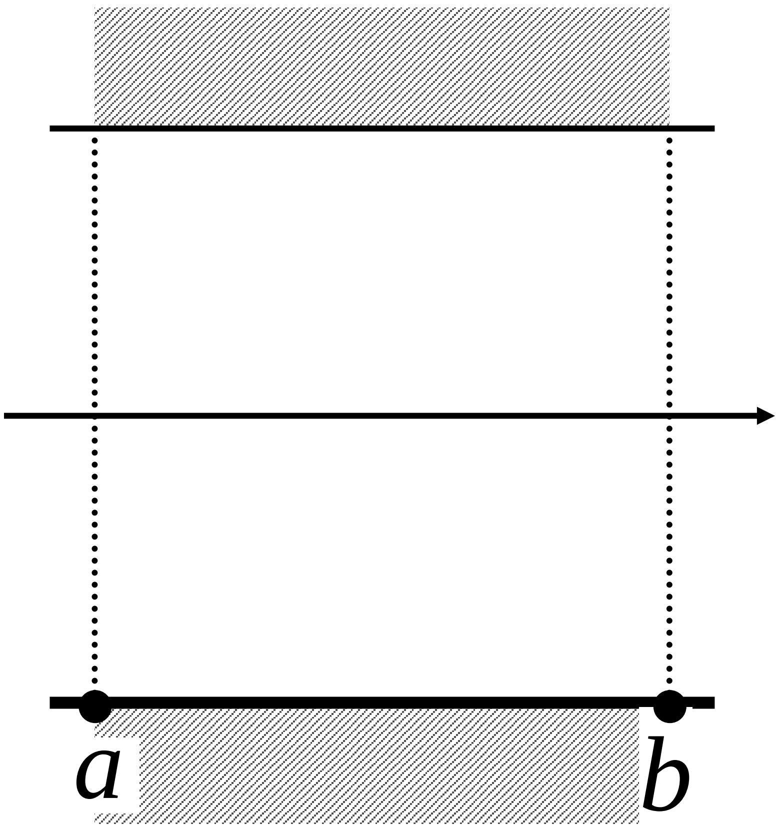
The case $\sigma \equiv \text{const}$

(1)



$$\Lambda_W^\sigma = \frac{-2}{b-a} + \sigma$$

(2)



$$\Lambda_W^\sigma = 0 + \sigma$$

Nonconstant Driving Force: Reformulation

Better to rewrite problems: Formally set $\xi_x = w + \sigma$

$$K_{\chi_-\chi_+}(I) = \{\xi \in H^1(I) : Z - 1 \le \xi \le Z + 1 \text{ (obstract condition)}$$

$$\xi(a) = Z(a) - \chi_-, \ \xi(b) = Z(b) + \chi_+ \text{ (B.C.)}\}$$

Here $Z(x) = \int^x \sigma\, dx$: primitive of σ.

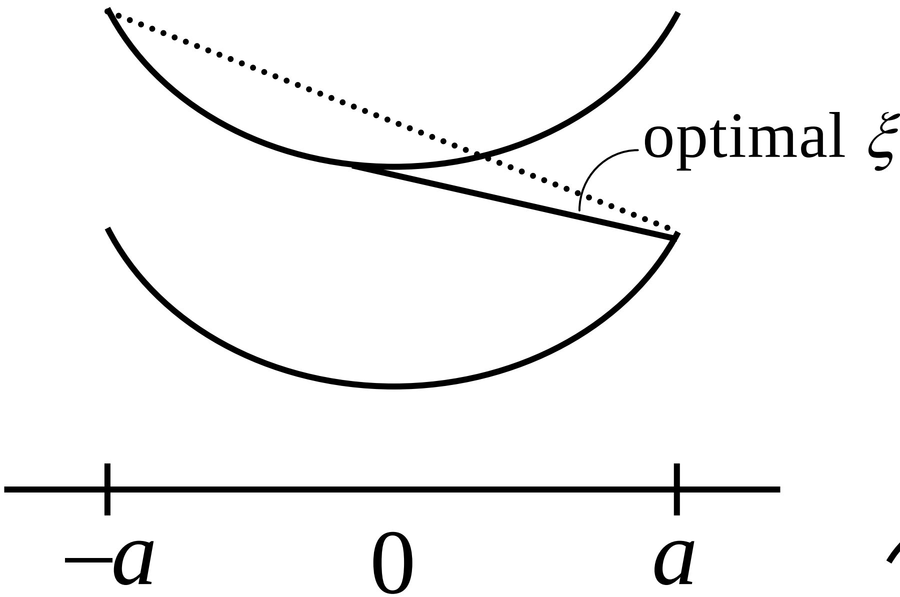
$$\boxed{w_0^{\chi_-\chi_+} + \sigma := \arg\min \left\{ \int_I |\xi'|^2 dx \,|\, \xi \in K_{\chi_-\chi_+} \right\}}$$

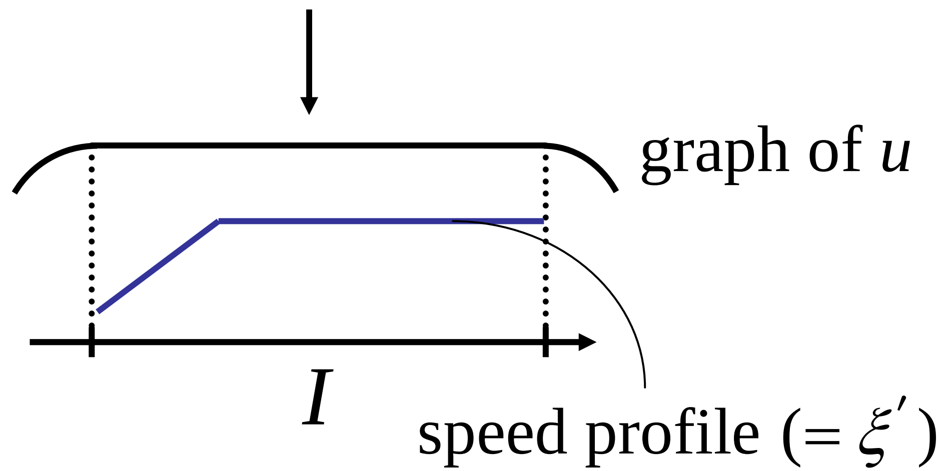
Obstacle Problem

Ex. $\sigma(x) = 2x, \quad I = (-a, a)$

$\chi_- = -1, \quad \chi_+ = -1$

a : not small





$\xi' =$ const. on I

$\Leftrightarrow I$ is calibrable (Cheeger set)

Comparison of curvatures

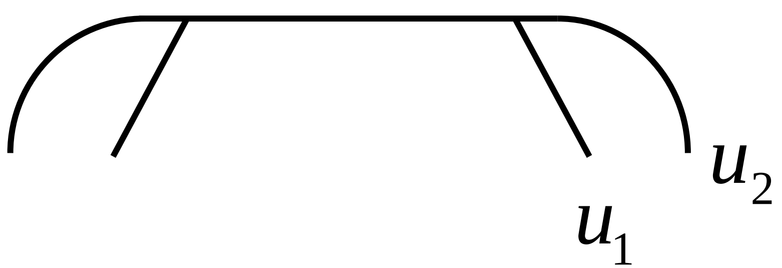
Assume: σ **is locally Lipschitz on** R**.**

Lemma $u_1 \le u_2,\ u_i:$ faceted on $I_i \quad (i = 1, 2)$

$u_1 = u_2$ on $I_1 \cap I_2$

$\Rightarrow \Lambda_W^\sigma(u_1)\ (x) \le \Lambda_W^\sigma(u_2)\ (x)$

for $x \in I_1 \cap I_2$



Stability

Lemma

$$\Lambda(\chi_-,\chi_+,I,x) = w_0^{\chi_-\chi_+}(x) + \sigma(x),\ x \in I$$

For each $r > 0$

$$\sup\{|\Lambda(\chi_-,\chi_+,I,x) - \Lambda(\chi_-,\chi_+,I-\mu,x-\mu)|:$$

$$|I| < r,\ x \in I\}$$

$$\to 0 \text{ as } \mu \to 0.$$

3. Definition of Solutions

We consider 'energy density' W **s.t.**

(i) $W : \mathrm{convex}$

(ii) $\exists P$ **finite set** $W \in C^2(R \setminus P)$

(iii) $\sup \{W''(p) : K \cap (R \setminus P)\} < \infty$

for any compact set K **in** R**.**

((iv) W' **jumps on** P **)**

Ex. W : piecewise linear convex (crystalline)

What is a suitable class of test functions to define viscosity like solution?
(c.f. M.-H. Giga – Y.G. ’98 $\sigma = \mathrm{const}$)

Faceted function

$$f \in C^2(\Omega),\ \ \Omega\text{: interval}$$

We say that f **is** P**-faceted if for** $x_0 \in \Omega$ **with** $f'(x_0) \in P$
$^{\exists} I$ **closed interval s.t.**

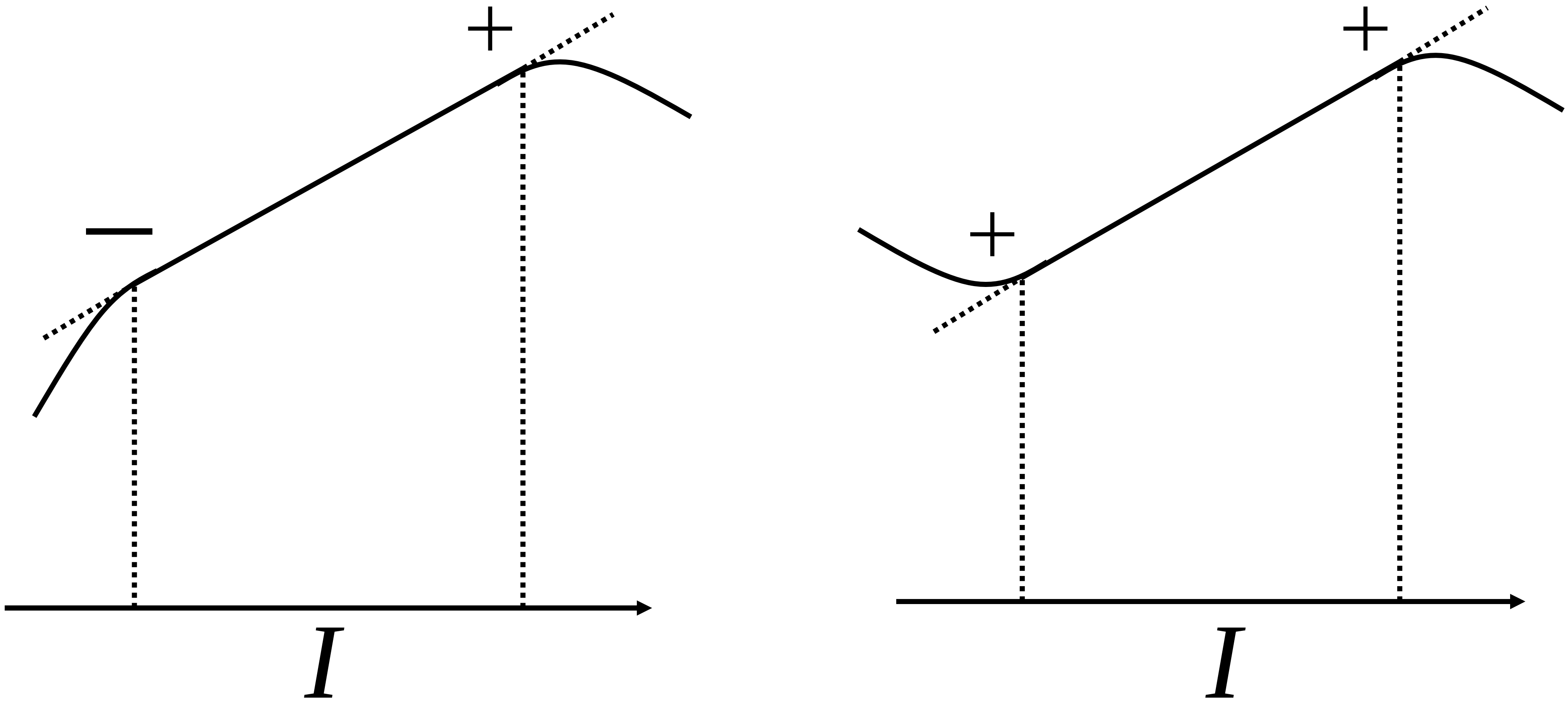
(i) $x_0 \in I \subset \Omega$

(ii) $f'(x) \in P$ **for** $x \in I$

(iii) $f'(x) \notin P$ **for** $x \in^{\exists} \text{nbd} \setminus I$

(iv) 'transition number' is defined.

graph of f



I : **faceted region**

Admissible functions

$$C_P^2(\Omega) = \{ f \in C^2(\Omega) \mid f : P - \text{faceted} \}$$

$$\varphi \in C^{1,2}(Q),\ Q = \Omega \times (0,T)$$

φ is admissible if φ is of the form

$$\varphi(x,t) = f(x) + g(t)$$

$$f \in C_P^2(\Omega),\ g \in C^1(0,T).$$

$A_P(Q)$ **: the set of admissible functions**

Definition (Global version)

$$u_t + F(u_x, (W'(u_x))_x + \sigma(x)) = 0$$

$u \in C(Q)$ **: subsolution in** Q **if**

$$\varphi_t + F(\varphi_x, \Lambda_W^\sigma(\varphi)) \le 0 \text{ at } (\hat{x}, \hat{t})$$

whenever $\max(u - \varphi) = (u - \varphi)\ (\hat{x}, \hat{t})$

for $\varphi \in A_P(Q)$.

($\Lambda_W^\sigma(\varphi) = (W'(\varphi_x))_x + \sigma$ **if** $\hat{x} \notin$ **faceted region)**

Jet in faceted region

$$\mathcal{J}^+_{\mathcal{P}}\varphi(\hat{x},\hat{t}) = \{\tau \in R \,|^{\exists}\, \omega : \text{ **modulus**}$$

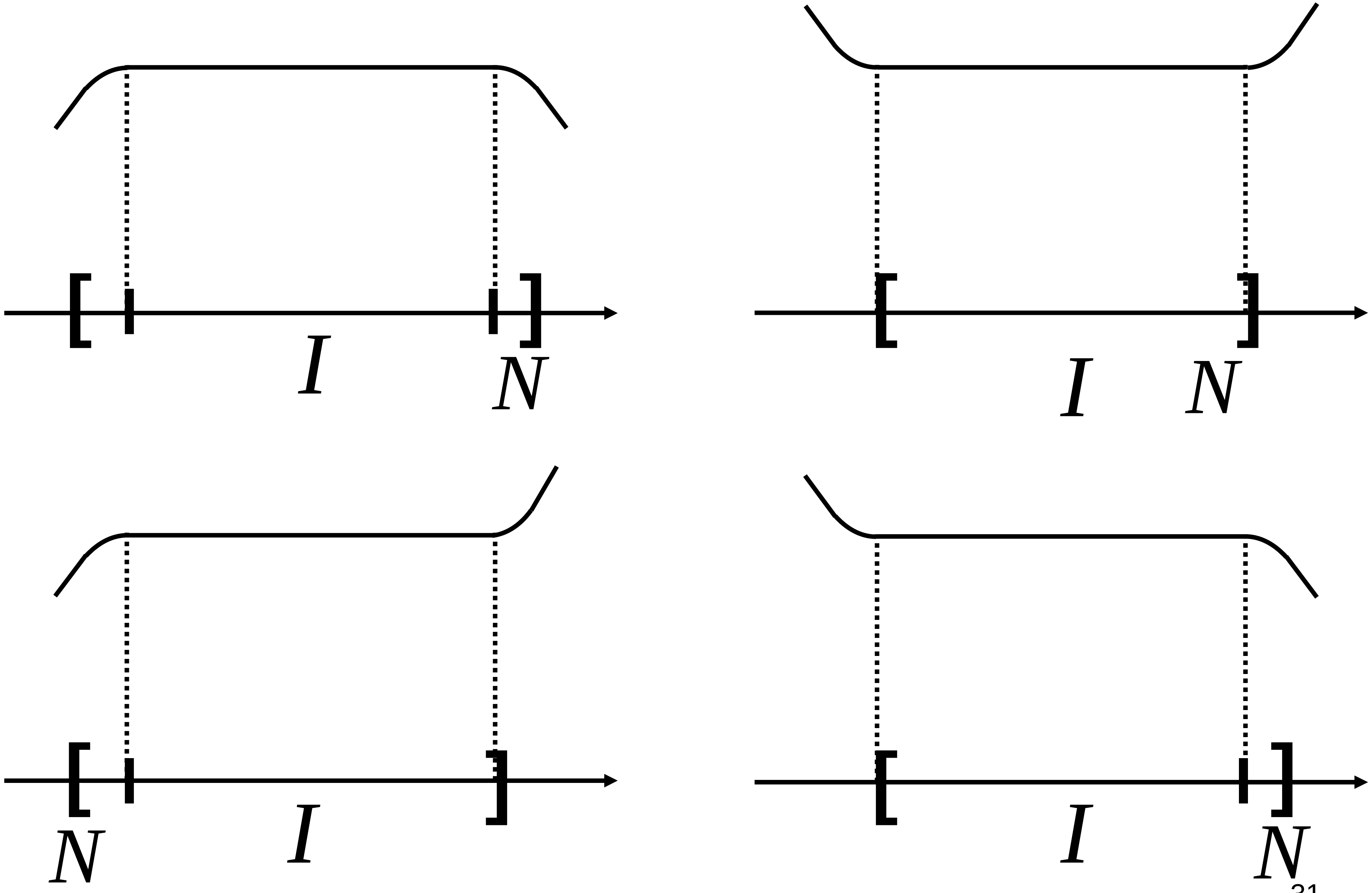
$$\varphi(x,t) \le \varphi(\hat{x},\hat{t}) + p\cdot(x-\hat{x}) + \tau(t-\hat{t})$$

$$+\omega(|t-\hat{t}\,|)\,|t-\hat{t}\,| \text{ in } N \times (\hat{t}-\delta, \hat{t}+\delta)\},$$

where N **is a 'semi nbd' of faceted region.**

Here $\varphi \in A_P(Q)$

Set N



Admissible superfunction

$$\varphi \in USC(Q),\ \varphi(\ \ ,\hat{t}) \in C(\Omega)$$

φ **is admissible superfunction at** $(\hat{x},\hat{t})$

if one of following hold.

(A) $\varphi(\ ,\hat{t}) : P-$**faceted at** $\hat{x}$ **and**

$\mathcal{J}_p^+\varphi(\hat{x},\hat{t})$ **: non empty**

(B) $^{\exists}(\tau, p, X) \in \mathcal{P}^{+}\varphi(\hat{x}, \hat{t})$

(C) $\varphi(\cdot, x) : P-$ **faceted at** $\hat{x}$ **and** $\hat{x}$ **is on bdry of the faceted region**

$^{\exists}(\tau, p, 0) \in \mathcal{P}^{+}\varphi(\hat{x}, \hat{t})$

Equivalent Definition (infinitesimal version)

$u \in C(Q)$ **subsolution in** Q

if φ **: admissible superfunction testing** u **at** $(\hat{x}, \hat{t})$ **from above satisfies**

(i) $\tau + F(\varphi_x(\hat{x}, \hat{t}),\ \Lambda_W^\sigma(\varphi)(\hat{x}, \hat{t})) \leq 0$

for all $\tau \in \mathcal{T}_P^+ \varphi(\hat{x}, \hat{t})$ **if (A) holds;**

(ii) $\tau + F(p, W''(p)X + \sigma(\hat{x})) \leq 0$ **for all** $(\tau, p, X) \in \mathcal{P}^{+}\varphi(\hat{x}, \hat{t})$ **if (B) holds;**

(iii) $\tau + F(p, \sigma(\hat{x})) \leq 0$ **for all** $(\tau, p, 0) \in \mathcal{P}^{+}\varphi(\hat{x}, \hat{t})$ **if (C) holds and** $\hat{x}$ **is a strict max of** $u - \varphi$ **near** $\hat{x}$ **in** x **–direction.**

(cf. M.-H-.Giga－Y.G. 2001)

4. Comparison Principle

$$u_t + F(u_x, (W'(u_x))_x + \sigma(x)) = 0$$

Assumption

(F1) $F \in C(R \times R)$

(F2) $F(p, X) \leq F(p, Y)$ **for** $X \geq Y$

(F3) $|F(p, X) - F(p, Y)| \leq C(1 + |p|)\,|X - Y|$

(Lipschitz continuity)

Example

$$u_t - a(u_x)[W'(u_x))_x + \sigma(x)] = 0$$

a **:continuous and nonnegative**

Assumption (iii) is standard if we consider the case that $W'' \equiv 0$ **; eq is**
$u_t + H(u_x, x) = 0.$

(x-depending HJ equations)

Comparison Principle

Thm

Assume that F satisfies (F1)-(F3).

Assume that σ is Lipschitz on $\overline{\Omega}$ where Ω is a bdd interval.

u **: sub** v **: super**

$u^* \le v_*$ **on** $\partial_p Q \Rightarrow u^* \le v_*$ **in** Q

Idea: $u, -v \in USC(\overline{Q})$

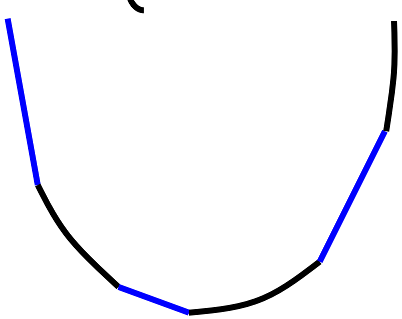
Argue by contradiction.

$$\Phi(x,y,t,s) := u(x,t) - v(y,s) - \alpha(t-s)^2$$

$$-\frac{1}{\beta}B((x-y)\beta) - \frac{\gamma}{T-t} - \frac{\gamma}{T-s}$$

α, β **: large** γ **: small**

$$\begin{cases} B: \ C_P^2 \text{ convex function, } B(0)=0 \\ B(x) \sim x^2 \text{ for large } x, \ B(x) \geq 0 \end{cases}$$



graph of B

γ : small, α : large fixed.

maximizer $(x_\beta, y_\beta, t_\beta, s_\beta)$

- **May assume that x_β, y_β are not on the bdry of faceted region. It is in a interior of faceted region for large β.**

- **Max Principle and (faceted) sup volution**

  $^{\exists}U_1$ **admissible superfunction**

  $^{\exists}U_2$ **admissible subfunction**

s.t. lengths of faceted region agree

Sup Convolution with Faceted Functions

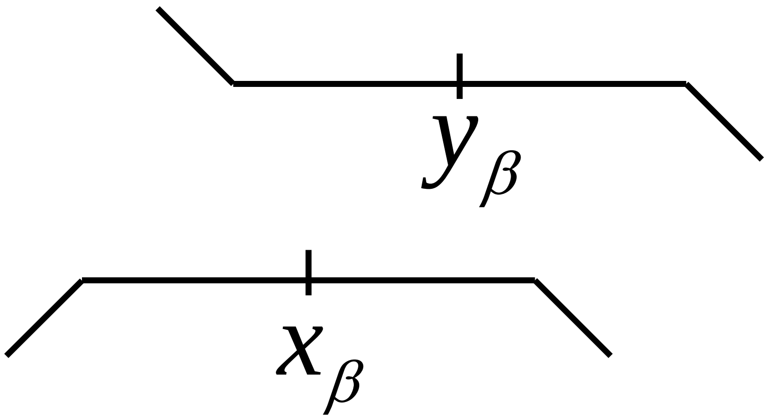
M.-H. Giga – Y. G. '98

$$\vartheta(x,\ \rho) = \begin{cases} (x-\rho)^2/\rho, & x > \rho, \\ 0, & |x| \le \rho, \\ (x+\rho)^2/\rho, & x < -\rho, \end{cases}$$

$$u^\alpha(x) = \sup_\xi \{u(\xi) - \vartheta(x-\xi, \alpha)\}.$$

If u attains a local maximum at x_0 ,

then u^α is 0 – slope faceted at x_0 .



Use definition of infinitesimal version

$$
\begin{cases}
2\alpha(t_\beta - s_\beta) + \dfrac{\gamma}{T^2} + F(p, \Lambda_W^\sigma(U_1)(x_\beta, t_\beta)) \le 0 & \cdots(1) \\
2\alpha(t_\beta - s_\beta) - \dfrac{\gamma}{T^2} + F(p, \Lambda_W^\sigma(U_2)(y_\beta, s_\beta)) \ge 0 & \cdots(2)
\end{cases}
$$

(cf. M.-H. Giga－Y.G. 2001)

$$\Lambda_W^\sigma(U_1)(x_\beta, t_\beta) - \Lambda_W^\sigma(U_2)(y_\beta, s_\beta) \le o(1)$$

as $x_\beta - y_\beta \to 0 \ (\beta \to \infty)$.

Proved by

- **Comparison of curvature**

  and

- **Stability**

Recall P **: finite set**

$F(p,X) - F(p,Y) \to 0$ **as** $X - Y \to 0$

$F(p,X) \le F(p,Y)$ **if** $X \ge Y$

Then (1)-(2) yields

$$\frac{2\gamma}{T^2} \le 0$$

as $\beta \to \infty$ **so that** $x_\beta - y_\beta \to 0.$

Contradiction !

Related Problems

3-dimensional problem is widely open.

only studied when $\sigma \equiv 0.$

Even in this case facet may break.

Bellettini, Paolini, Novaga…

Further possible development

Problems

- Stability: approximation smooth energy
  (σ:const; M.-H. Giga – Y. Giga 1999)

- Level set approach:
  [σ: const; M.-H. Giga – Y. Giga 2001]

- Convergence of Allen-Cahn type
  [$\sigma = 0$, Ohtsuka – Schätzle – Y. G. 2006]
  [Bellettini – Goglione – Novaga 2000]
  (Convergence to crystalline flow)

Special Project
A minisemester on evolution of interfaces, Sapporo 2010
July 12, 2010 - August 13, 2010

Organizers: Y. Giga (Tokyo), H. Ishii (Tokyo), T. Funaki (Tokyo), Y. Tonegawa (Sapporo), R. V. Kohn (New York), P. Rybka (Warsaw)

Venue: Department of Mathematics, Hokkaido University

Intensive Activities
(1) Tutorial Lectures and Interdisciplinary Conference
Mathematical Aspects of Crystal Growth
In cooperation with SIAM
July 26 -July 30, 2010
Organizers: Y. Giga (Tokyo), H. Ishii (Tokyo), R. Kohn (New York), P. Rybka (Warsaw), E. Yokoyama (Tokyo)
Tutorial Lecturers: R. Caflisch (Los Angeles), D. Margetis (College Park), R. Monneau (Paris)

(2) Tutorial Lectures and International Workshop
Singular Diffusion and Evolving Interfaces
August 2 -August 6, 2010
Organizers: T. Funaki (Tokyo), Y. Giga (Tokyo), P. Rybka (Warsaw), Y. Tonegawa (Sapporo)
Tutorial Lecturers: G. Bellettini (Rome), B. Kawohl (Cologne)

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