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$\pi : X \to C$ sm lc3 fibration

$\gamma \in H_2(X, \mathbb{Z})^{\pi} = \ker \pi_*$

repr by curves

Sch theo on the fibers

$P_n(X, \gamma)$ mod sp of stp

$\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ coker s 0-dim'l $ch_2(\mathcal{F}) = \gamma$

pure 1 dim'l $\chi(\mathcal{F}) = n$

$\to \mathcal{F} = i_* \mathcal{G}$ $i : S \hookrightarrow \mathcal{F}$ fiber of π

$\mathcal{O}_X \to \mathcal{O}_S \xrightarrow{s} \mathcal{G}$ stp on S

$P_n(S, \beta)$ $\beta \in H_2(S, \mathbb{Z})$

$ch_1(\mathcal{G}) = \beta$

$\chi(\mathcal{G}) = n$

[PT] $P_n(S,\beta) \simeq \mathrm{Hilb}^{n+\beta^2/2}(\mathcal{C}/_M)$ ← Hilb sch of 1 dim'l subschemes,

univ curve

β irred then

Smooth of dim $n+\beta^2+1$

$P = P_n(X,\gamma)$ perf ob theory $E^\cdot$ of rk 0

$h^0(E^{\cdot\vee}) \simeq \mathcal{E}xt^1_{\pi_P}(\mathbb{I}^\cdot,\mathbb{I}^\cdot)_0$

$h^1(E^{\cdot\vee}) \simeq \mathcal{E}xt^2 \quad —— \quad)_0$

$P \times X \xrightarrow{\pi_P} P \qquad P_{n,\gamma} = \int_{[P]^{vir}} 1$

Thm γ irred $P_{n,\gamma} = \sum_{h=0}^{\infty} (-1)^{n+2h-1} \chi(\dot{P}_n(S,h))$.

S fiber in NL locus of γ $NL^{\pi}_{h,\gamma}$

$\exists$ c $C \subset S$ $i_*[C] = \gamma$

$[C]^2 = 2h-2$

$NL^{\pi}_{h,\gamma}$ = intersection # in the mod sp of quasi pol K3 surf # of fibers S cont'g

$\chi(P_n(S,h))$ known Kawai-Yoshioka 2000 idea of pf $\exists\ P_n(X,\gamma) \xrightarrow{\rho} C$

consists of typ I (non iso) type II (isol) comps

ρ_0 ρ_{iso}

$$h^1(E^{\cdot\vee})\big|_{\rho_0} \simeq \mathcal{E}xt^{\ni\ \text{smooth}}_{\pi_\varphi}(F, I^{\cdot})$$

$$\big|_{\rho_{iso}} \simeq \Omega_{\rho_{iso}}$$

$$[\rho_{iso}]^{vir} = [\rho_{iso}] \cap \mathrm{Top}(\Omega_{\varphi\,\rho_{iso}})$$

$$[\rho_0]^{vir} = [\rho_0] \cap (\mathrm{Top}(\Omega_{\rho_0/\mathcal{C}}) \cup C_i(\ldots)$$

S'pose that X has ft many nodal fibers $\rightsquigarrow$ Conifold trans $\widetilde{X}$

$X_{t,1} \subset \mathbb{P}^3 \times \mathbb{P}^1$

$X_{t,2}$

$$\begin{tikzcd} \widetilde{X} \arrow[d, "h"] \arrow[dr, "\varepsilon"] & \\ X_0 \arrow[r] & X \end{tikzcd}$$

$e_1, \dots, e_k$ excp curves

Degeneration formula

$$PT(X) = \sum_{n,\gamma} q^n t^\gamma P_{n,\gamma}$$

$$\varepsilon_* \frac{PT(\widetilde{X})}{PT^h(\widetilde{X})} = PT(X)^2$$

ε_*

$\varepsilon_* \quad \xi \to t^\gamma$

$$PT^h(X) = \sum_{\substack{n \\ h_* \gamma = 0}} \sim \; = \prod \text{ known}$$

Need to deal with $\gamma = h^! \gamma_0 + m e_i$.

γ_0 irrel

Can extend thm in some cases

Another approach: perverse str theory

let $\mathcal{C} = \{ E \in D^b(\mathrm{Coh}\,\tilde{X}) \mid Rh_* E = 0 \}$

$\mathrm{Per} = \mathrm{Per}^{\tilde{X}/X_0} = \{ E \mid Rh_* E \in \mathrm{Coh}\, X_0,\ \mathrm{Hom}(\mathcal{C}^{<0}, E) = \mathrm{Hom}(E, \mathcal{C}^{>0}) = 0 \}$

heart of bd t-str $\mathcal{O}_{\tilde{X}} \in \mathrm{Per}$

$\mathrm{Per}_{\leq 1} = \{ E \mid \dim(\mathrm{supp}\, E) \leq 1 \}$ $\mathrm{Per}_0 = \{ \}$

Perv stp $\mathcal{O}_X \xrightarrow{s} \mathcal{F}$ $\quad \mathcal{F} \in \mathrm{Per}_{\leq 1}$

$\mathrm{Hom}(\mathrm{Per}_0, \mathcal{F}) = 0$

$P_n^{\mathrm{Per}}(\tilde{X}, \gamma)$ mod sp $\quad$ coker $s \in \mathrm{Per}_0$

Perf ob thry

Conjecture $\dfrac{PT_n(\tilde{X})}{PT(\tilde{X})} = PT^{\mathrm{Per}}(\tilde{X})$

holds EG version

* If conj true: $\mathcal{E}_* PT^{\mathrm{Per}}(\tilde{X}) = PT(X)^2$

* How to find $PT^{Per}(\tilde{X})$

Key lemma: $\mathcal{O}_{\tilde{X}} \to \mathcal{F}$ perv stp with $h_*[\mathcal{F}]$ irred then

$$\mathrm{Hom}(\mathcal{F}, \mathcal{F}) \simeq \mathbb{C}$$

X CY3 sm proj. $X \xrightarrow{\pi} \mathbb{P}^1$ s.t all fibers integral
gen fiber sm K3

$\rightsquigarrow$ $J(r,\gamma,n)$: generalized DT invs of Gieseker
SS sheaves with $ch = (0, rF, \gamma, n)$

$\gamma \in H_2(X,\mathbb{Z})^\pi$

Thm

$$PT(X) = \prod_{\substack{r>0\ \gamma>0 \\ n>0}} \exp((-1)^{n-1} J(r,\gamma,r+n)\, q^n t^\gamma)^{n+2r}$$

$$\cdot \prod_{\substack{r>0\ \gamma\geq 0 \\ n>0}} \exp(J(\ )\, q^{-n} t^\gamma)^{n+2r}$$

$$D_0 = D^b \mathrm{coh}_\pi(X)$$

$$E \in D_0 \rightsquigarrow ch(E) = (0, rF, \gamma, n)$$

ω ample div $\quad \mu_\omega(E) = \int \frac{\gamma \cdot \omega}{r} \quad \infty \; r=0$

$$\hookrightarrow \quad T_\omega = \left\langle {}_{,}\, \overset{E\ \mu_\omega\ ss}{\mu_\omega(E) > 0} \right\rangle_{ex} \quad F_\omega = \left\langle \overset{E\ \mu_\omega\ ss}{\mu_\omega(E) \le 0} \right\rangle_{ex}$$

(T_ω, F_ω) torsion pair $\rightsquigarrow \underset{=\mathcal{B}_\omega}{\langle F_\omega, T_\omega[-1] \rangle} \overset{\subset \mathrm{coh}\pi}{} \wr$

$$v \in \Pi_0 = \mathbb{Z} \oplus H_2(X, \mathbb{Z})^{\Pi} \oplus \mathbb{Z}$$

$$\bar{Z}_{\omega,0}(v) = \int_X e^{i\omega} v$$

$$\forall t > 0 \qquad \sigma_{t\omega} = (\bar{Z}_{\omega,0}, B_\omega)$$

Bridgeland st

$$D = \langle \pi^* \mathrm{Pic}\, \mathbb{P}^1, \mathrm{coh}_\pi \rangle_{tr}$$

$$A_\omega = \langle \pi^* \mathrm{Pic}, B_\omega \rangle \subset D$$

$E \in \mathcal{A}_\omega \quad ch E = (R, rF - \gamma, n)$

$\sigma_{t\omega} = (Z_{t\omega}, \mathcal{A}_\omega) \quad t > 0$

weak st on D

$Z_{\omega,1}(R) = R\sqrt{-1}$

$Z_{\omega,0}(v) = \int_X e^{-i\omega} v$

$\mathcal{M}_{tw}(r,\gamma,n)$ mod stk of σ_{tw}

ss $E \in \mathcal{A}_w$

$\operatorname{ch}(E) = (1-r, -\gamma, -n)$

$$\widehat{DT}(r,\gamma) \xleftarrow{\;0^+ \leftarrow t\;} DT_{tw}(r,\gamma,n) \xrightarrow{\;t\to\infty\;} L(\beta,n)$$

to Coh

θ

$\pi^* \mathrm{Pic}\,\mathbb{P}^1$ $\quad J(r,\gamma,r)$

wallcrossing

$J(r,\gamma,n)$

PT

$J(\gamma,\beta,n)$