Reverse-time (and Kirchhoff) migration

Preliminaries.

The homogeneous two-dimensional Green's function $\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{y})$ is solution of

$$\Delta_{\boldsymbol{x}} \hat{G} + \omega^2 \hat{G} = -\delta(\boldsymbol{x} - \boldsymbol{y}), \qquad \boldsymbol{x} \in \mathbb{R}^2$$

with the Sommerfeld radiation condition. It is given by

$$\hat{G}(\omega, \boldsymbol{x}, \boldsymbol{y}) = \frac{i}{4} H_0^{(1)}(\omega |\boldsymbol{x} - \boldsymbol{y}|)$$

where $H_0^{(1)}$ is the Hankel function

$$H_0^{(1)}(s) = J_0(s) + i Y_0(s)$$

and J_0 is the Bessel function of the first kind and Y_0 is the Bessel function of the second kind (use `besselj` and `bessely`).

Time-harmonic localization - full aperture.

Consider N_a transducers on a circular array centered at $\mathbf{0}$ with radius R_a.

Consider a point reflector at $\boldsymbol{x}_{\mathrm{ref}}$.

Generate the data set, i.e. the matrix of the time-harmonic amplitudes $\hat{d}_{rs}(\omega)$ recorded by the rth receiver when the sth source emits a time-harmonic signal with unit amplitude and frequency ω. Use a Born approximation for the reflector.

Plot the (two-dimensional) RT and KM imaging functional $\mathcal{I}_{\mathrm{RT}}(\boldsymbol{x})$ and $\mathcal{I}_{\mathrm{KM}}(\boldsymbol{x})$ using the data set.

Compare the focal spot with the theoretical function $J_0^2(\omega |\boldsymbol{x} - \boldsymbol{x}_{\mathrm{ref}}|)$.

Use $\omega = 2\pi$, $R_a = 100$, $N_a = 100$, and $\boldsymbol{x}_{\mathrm{ref}} = (10, 20)$, and play with the numbers (move the reflector).

Time-harmonic localization - partial aperture.

We use the convention $\boldsymbol{x} = (x, z)$.

Consider N_a receivers on a linear array (along the x_1-direction) centered at $\mathbf{0}$ with length R_a.

Generate the data set, i.e. the matrix of the time-harmonic amplitudes $\hat{d}_{rs}(\omega)$ recorded by the rth receiver when the sth source emits a time-harmonic signal with unit amplitude and frequency ω. Use a Born approximation for the reflector.

Plot the (two-dimensional) RT and KM imaging functional $\mathcal{I}_{\mathrm{RT}}(\boldsymbol{x})$ and $\mathcal{I}_{\mathrm{KM}}(\boldsymbol{x})$ using the data set. Plot also the MUSIC-type imaging functional:

$$\mathcal{I}_{\mathrm{MU}}(\boldsymbol{x}) = \left| \left\langle \hat{\boldsymbol{g}}(\omega, \boldsymbol{x}), \boldsymbol{v}_1 \right\rangle \right|^2$$

where $\hat{\boldsymbol{g}}(\omega, \boldsymbol{x})$ is the vector of Green's functions from the array to the search point $\boldsymbol{x}$ and $\boldsymbol{v}_1$ is the first singular vector of the response matrix $\hat{\mathbf{d}}$.

Use $\omega = 2\pi$, $R_a = 50$, $N_a = 100$, and $\boldsymbol{x}_{\rm ref} = (0, 100)$, and play with the numbers (reduce R_a, move the reflector).

Look at the focal spot in the (cross-range) x-direction and compare with the theoretical function $\mathrm{sinc}^2(\pi|x - x_{\rm ref}|/r_c)$, with $r_c = \lambda|\boldsymbol{x}_{\rm ref}|/R_a$ and $\lambda = 2\pi/\omega$.

Look at the focal spot in the (range) z-direction and compare with the theoretical function $\big|\int_0^1 \exp(-i\frac{\pi}{2}s^2\frac{|z-z_{\rm ref}|}{r_l})ds\big|^2$, with $r_l = 2\lambda|\boldsymbol{x}_{\rm ref}|^2/R_a^2$.

Add noise and increase the noise until the functionals become blurred. Note the robustness of RT compared to MUSIC.

Implement the Hadamard acquisition scheme to generate the data set.

Time-dependent localization - partial aperture.

Consider N_r receivers on a linear array (along the x-direction) centered at $\mathbf{0}$ with length R_a.

We now assume that the sources emit a broadband signal with $\hat{f}(\omega) = \mathbf{1}_{[\omega_0-B,\omega_0+B]}(\omega)$.

Generate the data set, i.e. the matrices of the time-harmonic amplitudes $\hat{d}_{rs}(\omega)$ recorded by the rth receiver when the sth source emits a time-harmonic signal with unit amplitude and frequency ω, for ω sampled over $[\omega_0 - B, \omega_0 + B]$.

Plot the (two-dimensional) RT and KM imaging functional $\mathcal{I}_{\rm RT}(\boldsymbol{x})$ and $\mathcal{I}_{\rm KM}(\boldsymbol{x})$ using the data set.

Use $\omega_0 = 2\pi$, $B = 0.05\omega_0$, $R_a = 20$, $N_a = 40$, and $\boldsymbol{x}_{\rm ref} = (0, 100)$, sample 20 frequencies over $[\omega_0 - B, \omega_0 + B]$, and play with the numbers (reduce the bandwidth).

Look at the focal spot in the (cross-range) x-direction and compare with the theoretical function $\mathrm{sinc}^2(\pi|x - x_{\rm ref}|/r_c)$, with $r_c = \lambda_0|\boldsymbol{x}_{\rm ref}|/R_a$ and $\lambda_0 = 2\pi/\omega_0$.

Look at the focal spot in the (range) z-direction and compare with the theoretical function $\mathrm{sinc}(2B|z - z_{\rm ref}|)$.