u: displacement

$v = \partial_t u$: particle velocity

σ: stress tensor

Inverse problem: least squares formulation

1. Data: $d_{r,s,t}$
2. source: $f_s(x,t) = \delta(x-s)w(t)$, ω: source wavelet (signature)

Def: $J[c] = \frac{1}{2}\|d_{r,s,t} - u_s(r,t)\|_2^2$, where u_s comes from solving the corrsponding wave equaiton. Problem:

$$\min_{c(\mathbf{x})} J[c] \text{ s.t. } \begin{cases} \frac{1}{c^2(\mathbf{x})}\partial_t^2 u_s - \Delta u_s = f_s, \\ u_s(x,0) = \partial_t u_s(x,0) = 0 \\ \text{ propper boudary conditions} \end{cases}$$

Write $u = F[c]$; $F[c] = F[c_0] + \frac{\delta F}{\delta(c^2)}[c_0](c^2 - c_0^2) + ...$ Note, $\frac{\delta F}{\delta(c^2)}$ is an operator, Frechet derivative. Assume know background wave speed $c_0(\mathbf{x})$. Want to call $F[c_0]$ the incident wave, often assume c_0 is smooth. This approach works well when $c^{-2} - c_0^{-2}$ is oscillatory, (has zero mean, etc). Want to call $\frac{\delta F}{\delta(c^2)}[c_0]$ primary reflected waves; further terms would correspond to multiples. If these conditions are not met, then further terms in the Taylor expansion are significant, otherwise not.

$$\begin{cases} \frac{1}{c_0^2(\mathbf{x})}\partial_t^2 u_{\text{inc}} - \Delta u_{\text{inc}} = f_s \\ \frac{1}{c^2(\mathbf{x})}\partial_t^2 u_s - \Delta u_s = f_s \\ \to \text{(subtract)} \begin{cases} \frac{1}{c_0^2(\mathbf{x})}\partial_t^2 u_{\text{scat}} - \Delta u_{\text{scat}} = -V(\mathbf{x})\partial_t^2 u \\ \text{zero initial conditions, propper boundary conditions.} \end{cases} \end{cases}$$

$$\to u_{\text{inc}}(x,t) = \int\int G(x,y;t\tau) f_s(y,\tau)\,dy\,d\tau = Gf$$

if $C_0 = 1$, $G(x,y,y) = \frac{\delta(t - \|x-y\|/c)}{ct = \|x-y\|}$.

$$\to u_{\text{scat}} = -G(V\partial_t^2 u), u = u_{\text{inc}} - GV\partial_t^2 u$$

$$(I + GV\partial_t^2)u = u_{\text{inc}},$$

Lippman-Schwinger equations.

Born sereis:

$$u = u_{\text{inc}} - GV\partial_t^2 u_{\text{inc}} + GV\partial_t^2 GV u_{\text{inc}} + ...$$

Write $\mathbf{F} = \frac{\delta F}{\delta c^2}[c_0]$, which is the forward / linearized modeling operator, $\mathbf{F}V = u_x^B$ (superscript stands for Born). Weak scattering: require that $\|G(V\partial_t^2)\|_2 < 1$, which implies the series converges; usually this is too strong a contidion, and is not practical.

Linearized Inverse Problem: minimize over V

$$J_L[V] = \frac{1}{2}\|d_{r,s,t} - u_{\text{inc},s}(r,t) - u_{x,s}^B(r,t)\|_2^2 = \frac{1}{2}\|\tilde{d} - \mathbf{F}V\|_2^2 \text{ s.t. } \begin{cases} (\frac{1}{c^2(\mathbf{x})}\partial_t^2 - \Delta)u_{x,s}^B = -V(x)\partial_t^2 u_{\text{inc},s}, \\ \text{zero initial conditions} \\ \text{propper boudary conditions} \end{cases}$$

(removes multiples, compares to primaries).

Solution of Linearized Inverse Problem: is $V = \mathbf{F}^{-1} u^B_{\text{scat}} = (\mathbf{F}^\star \mathbf{F})^{-1} \mathbf{F}^\star u^B_{\text{scat}}$. $\frac{\delta J_L}{\delta V} = \mathbf{F}^\star(\mathbf{F}V - \tilde{d})$, $\delta V = -\alpha \frac{\delta J_L}{\delta V}$. Gradient descent works for full non-linear problem: $\frac{\delta J}{\delta c^{-2}} = \mathbf{F}^\star(\mathbf{F}[c] - d)$, Landweber iteration. Generally, we're motivating the use of the adjoint state operator $\mathbf{F}^\star$, there are many reasons why this is usefull.

Adjoint-State Method:

$$\langle d, \mathbf{F}V \rangle_{r,t,s} = \langle \mathbf{F}^\star d, V \rangle_x \Rightarrow \sum_r \int d_r(t) \mathbf{F}V(r,t) dt = \int \mathbf{F}^\star d(x) V(x) dx \ \ (\text{fix } s)$$

one construction:

$$d_{\text{ext}}(x,t) = \sum_r d(r,t) \delta(x-r)$$

then,

$$\sum_r \int d_r(t) \mathbf{F}V(r,t) dt = \int \int d_{\text{ext}}(x,t) \mathbf{F}V(x,t) dt dx$$

now, $u^B_x(x,t) = \mathbf{F}V(x,t)$, which solves the wave equation, so let

$$d_{\text{ext}}(x,t) = (\frac{1}{c_0^2}\partial_t^2 - \Delta) q(x,t)$$

called the adjoint wavefield. Now integrate by parts. Boundary terms:

$$\int \frac{1}{c_0^2} \partial_t q u |_0^T dx, \int \frac{1}{c_0^2} q \partial_t u |_0^T dx, \int_0^T \int_S \frac{\partial q}{\partial u} dS_x dt, \int_0^T \int_S q \frac{\partial u}{u} dS_x dt$$

The last two terms are zero, due to finite speed of propagation. The lower boundary terms in the frist two terms are zero due to initial conditions. Require $q(x,T) = \frac{\partial q}{\partial t}(x,T) = 0$, then the upper boundary terms are zero aswell. Final value problem (backward in time). This is the adjoint state problem:

$$\begin{cases} d_{\text{ext}}(x,t) = (\frac{1}{c_0^2}\partial_t^2 - \Delta) q(x,t) \\ q(x,T) = \frac{\partial q}{\partial t}(x,T) = 0 \end{cases}$$

LHS:

$$\int \int q(x,t) \underbrace{(\frac{1}{c_0^2}\partial_t^2 - \Delta) u^B_{\text{scatt}}(x,t)}_{-V(x)\partial_t^2 u_{\text{inc}}(x,t)} dx dt = \int V(x) \underbrace{\left[(-1) \int q(x,t) \partial_t^2 u_{\text{inc}}(x,t) dt \right]}_{\mathbf{F}^\star d(x) \text{ Imaging Operator}} dx$$

Fix s.

$$\hat{u}_{\text{inc}}(x,\omega) = \int \hat{G}(x,y,\omega) \hat{f}(y,\omega) dy$$

$$u^B_{\text{scat}}(x,\omega) = \int \hat{G}(x,y,\omega)(-V(y))(-\omega^2) \hat{u}_{\text{inc}}(y,\omega) dy =$$

$$\omega^2 \int \int \hat{G}(x,y,\omega) V(y) \hat{G}(y,z,\omega) \hat{f}(z,\omega) dy dz$$

Parseval:

$$\langle d, u^B_x \rangle_{r,t} = \langle \hat{d}, \hat{d}^B_x \rangle$$

$$= \sum_r \int \hat{d}(r,\omega) \overline{\hat{u}^B_{\text{scat}}(r,\omega)} d\omega$$

$$= \sum_r \int \hat{d}(r,\omega)\omega^2 \int\int \overline{\hat{g}(r,y,\omega)} V(y) \overline{\hat{G}(y,z,\omega)\hat{f}(z,\omega)} dydzd\omega$$

$$= \int dyV(y) \int d\omega\omega^2 \sum_r \overline{\hat{G}(r,y,\omega)} \hat{d}(r,\omega) \int dz\overline{\hat{G}(y,z,\omega)\hat{f}(z,\omega)}$$

$$\mathbf{F}^\star d(x) = \int d\omega(\omega^2) \underbrace{\sum_r \overline{\hat{G}(r,x,\omega)}\hat{d}(r,\omega)}_{\hat{q}(x,\omega)} \underbrace{\overline{\int dz\hat{G}(x,z,\omega)\hat{f}(z,\omega)}}_{\overline{\hat{u}_{\text{inc}}(x,\omega)}} \quad \text{(RTM)}$$

$$= \int d\omega\omega^2 \hat{q}\overline{\hat{u}_{\text{inc}}} = -\langle \hat{q}, (-\omega^2)\overline{\hat{u}_{\text{inc}}}\rangle_\omega$$

Connection with RADAR:

$$\hat{d} = \mathbf{F}V(s,\omega) = \int \exp(2i\omega\|\gamma(s) - y\|/c)A(\omega,s,y)V(y)dy$$

$$\mathbf{F}^\star \hat{d}(x) = \int \exp(2i\omega\|\gamma(s) - y\|/c)\overline{A(\omega,s,y)}\hat{d}(s,\omega)dy$$

$$\mathbf{F}^{-1}\hat{d}(x) = \int \exp(2i\omega\|\gamma(s) - y\|/c)B(\omega,s,y)\hat{d}(s,\omega)dy$$

RADAR:

$$d(s,t) \approx \int \delta(t - \frac{2}{c}\|\gamma(s) - x\|)V(x)dx$$

$$I_{KM}(x) = \int \delta(t - \frac{2}{c}\|\gamma(s) - x\|)d(s,t)dsdt$$

CT:

$$g(\theta,s) = \int \delta(s - x\cdot\theta)f(x)dx$$

$$I_{KM} = \int \delta(s - x\cdot\theta)g(\theta,s)d\theta ds = \int g(\theta, x\cdot\theta)d\theta$$

(unfiltered back projection)
Seismology:

$$d(r,t) \approx \int \delta(t - \tau(r,x) - \tau(s,x))V(x)dx$$

$$I_{KM}(x) = \sum_r \int \delta(t - \tau(r,x) - \tau(s,x))d(r,t)dt = \sum_r d(r, \tau(r,x) + \tau(s,x))$$

Assume $c(x)$ is smooth (no reflection, refraction). y: take off point. (Eikonal, Transport Equation)
Examples, with $c(x) = c_0$: $\tau(x,y) = \|x - y\|/c_0$, $\tau(x,y) = x_1/c_0$, distance function to any curve.
Iso-phase lines of $e^{(i\omega\tau)}$ = iso-level lines of τ = wavefronts. See viscocity solution of the eikonal equation, progressive wave expansion.

Eikonal:

$$\|\nabla_x \tau(x,y)\| = \frac{1}{c(x)}$$

Rays: characterstic curves. Wavefronts: level curves of τ, rays: perpendicular to wavefronts. So

$$\dot{X}(t) = c(X(t))\frac{\nabla\tau(X(t),y)}{\|\tau(X(t),y)\|}$$

chain rule:

$$\frac{d}{dt}\tau(X(t),y) = \dot{X}(t)\nabla_x\tau(X(t),y) = c(X(t))\frac{\nabla_x\tau(X(t),y)}{\|\nabla_x\tau(X(t),y)\|}\cdot\nabla_x\tau(X(t),y) = c(X(t),y)\|\nabla_x\tau(X(t),y)\| = 1$$

τ: travel time; $\tau(X(t),y) - \tau(X(0),y) = t$, $\tau(X(0),y) = 0$. Example: for $X(t)$ still depens on τ.

$$p(t) = \nabla_x\tau(X(t),y), \dot{p}(t) = \nabla\nabla\tau\cdot\dot{X}$$

$$= \nabla\nabla\tau\cdot\nabla\tau c^2 = \frac{1}{2}\nabla\|\nabla\tau\nabla^2 c^2 = -\frac{1}{2}\nabla(c^2)\|p(t)\|^2$$

get

$$\begin{cases}\dot{X}(t) = c^2(X(t))p(t), \quad X(0) = x_0\\ \dot{p}(t) = -\frac{1}{2}(\nabla c^2)(X(t))\|p(t)\|^2 \quad p(0) = p_0\end{cases}$$

and $\tau(X(t),y) = t$.

WAVE:

$$(\frac{1}{c^2}\partial_t^2 - \Delta)G = \delta(x-y)$$

$$G(x,y,t) \approx a_0(x,y)\delta(\underbrace{t-\tau(x,y)}_{\phi(t,x,y)})$$

Hamilton-Jacobi
FINISH

$$\tau(x,y) = \inf\{T : \exists X \text{ s.t. } X(0) = 0, X(T) = y, \|X(t)\| = c(X(t))\forall 0 < t < T\}$$

arguement is $X(t)$ that solves hamilton system for osme p_0.

$$\tau(x,y) = \inf\left\{\int_0^s \mathcal{L}(X(s),\dot{X}(s))ds : X(0) = x, X(s) = y\right\}$$

(any parameterization), $\mathcal{L}(x,\dot{x}) = \frac{1}{x(c)}\|\cdot x\|$, $\int_p \frac{1}{c(x)}d\ell$ and any min is $X(t)$ for some p_0. Euler-Lagrange: $\frac{\delta\mathcal{L}}{\delta X} - \frac{d}{ds}\frac{\delta\mathcal{L}}{\delta\dot{X}} = 0$.

Amplitude:

$$2\nabla\tau\cdot\nabla a_0 + a_0\Delta\tau = 0$$

$$\frac{d}{dt}a_0(X(t),y) = \dot{X}(t)\nabla_x a_0 = c^2\nabla\tau\nabla a_0 = -\frac{c^2}{2}a_0\Delta\tau$$

$$\nabla\cdot a^2\nabla\tau = 0$$

Model Velocity Estimator:

$$J[c] = \frac{1}{2}\|d - F[c]\|_2^2$$

$$\frac{1}{c^2} = \frac{1}{c_0^2} + V$$

$$u_s(r,t) = F[c] \Leftrightarrow (\frac{1}{c^2}\partial_t^2 - \Delta)u_s = f_s$$

$$u^B_{\text{scat},s}(r,t) = \frac{\delta F}{\delta c^{-2}}[c_0]V = \mathbf{F}V \Leftrightarrow (\frac{1}{c^2}\partial_t^2 - \Delta)u^B_{\text{scat},s} = -V\partial_t^2 u^B_{\text{scat},s}$$

$$-\frac{\delta J}{\delta c^{-2}}[c_0] = \mathbf{F}^\star(d - \mathbf{F}[c])$$

$$\mathbf{F}_s^\star d_s(x) = -\int_0^T q_s(x,t)\partial_t^2 u_{\text{inc},s}(x,t)dt$$

(1) Landweber iterations:

$$\begin{cases} \delta V_{k+1} = -\alpha \frac{\delta J}{\delta c^{-2}}[c_k] \\ \frac{1}{c_{k+1}^2} \end{cases}$$

???????

$$\max_{c_0} |\langle d, f(r - c_0 \cdot)\rangle|$$

Full Waveform Inversion: new data as $\hat{d}_{r,s}(\omega)$ add freq ω from low to high (Karzmarz), λ from large to small. Requires knowledge of $\hat{d}_{r,s}(\omega)$ for small ω.

(2) Extension Principles (symes): $d_{r,s,t}$, $\mathbf{F}^\star d(x) = \sum_s \mathbf{F}_s^\star d_s(x)$ (stack). $\mathbf{F} = (\mathbf{F}_1, ..., \mathbf{F}_{N_s})$, $\mathbf{F}^\star = \sum_s \mathbf{F}_s^\star$ (adjoint and operator-sum commute). Idea: let $V_s(x) = \mathbf{F}_s^\star d_s(x)$, look at images before summing. If c_0 is good, then $V_{s_1} \approx V_{s_2}$, else not. Example: Differential semblance optimization (Symes).

$$J_{\text{DSO}}[c_0, (V_s, ..., V_{N_s})] = \frac{1}{2}\sum_s |V_{s+1} - V_s| + \frac{1}{2}\sum_s \|d_s - \mathbf{F}_s V_s\|_2^2, \ \ \min J_{\text{DSO}}$$

$\Omega \subseteq \mathbb{R}^3$, Extension: manifold $\overline{\Omega}$, operator χ, such that (see notes).
Example: (Standard extension), $\overline{\Omega} = \Omega \times S$, $(x \in \Omega, s \in S)$. $\overline{V} \in \mathcal{D}'(\overline{\Omega})$ in $\overline{V}(x,s)$.

$$(\chi V)(x,s) = V(c)$$

.

$$(\mathbf{F})V_{r,s,t} = \int dx V(x) \int d\tau G(x, x_r, \tau)\partial_t^2(x_s, x, t-\tau)$$

$$(\chi V)(x,s) = V(c)$$

.

$$(\overline{\mathbf{F}})\overline{V}_{r,s,t} = \int dx \overline{V}(x) \int d\tau G(x, x_r, \tau)\partial_t^2(x_s, x, t-\tau)$$

(insert)

(3) Travel Time Tomography:

$$J_{TT}[c] = \frac{1}{2}\|\tau_{r,s} - \tau(x_r, x_s)\|, \quad \min_c J_{TT}[c] \text{ s.t. } \|\nabla_x \tau(x, x_r)\| = \frac{1}{c(x)}$$

Solution in layered media(Hergoltz-Wiechert (1910)). Data: $\tau(r)$, fix s. Horizontal slowness $T'(x) = p(x) = \frac{1}{c_0}\cos(\theta_0)$.