Simulation of random processes

Random variables.

Problem 1. Use `hist` to plot a histogram of the empirical distribution of a sample of 1000 numbers obtained with `randn`. Plot on the same picture the theoretical pdf $p(x) = 1/\sqrt{2\pi}\exp(-x^2/2)$.
Hint: If Z is the vector containing your sample, then the following matlab instructions give a normalized histogram of it:

```
[ns,xs]=hist(Z,20);
ns=ns/length(Z)/(xs(2)-xs(1));
```

`bar(xs,nx);` *or* `plot(xs,nx);`
The histogram is here normalized so that the total area is equal to one.

Gaussian vectors.

Algorithm to generate a realization of a Gaussian vector $\boldsymbol{Y}$ with the distribution $\mathcal{N}(\boldsymbol{m}, \mathbf{R})$:
a) compute a square root $\mathbf{S}$ of $\mathbf{R}$.
b) draw a random vector $\boldsymbol{n}$ with the distribution $\mathcal{N}(\mathbf{0}, \mathbf{I})$ (use `randn`) and compute:

$$\boldsymbol{y} = \boldsymbol{m} + \mathbf{S}\boldsymbol{n}$$

The computationally expensive part is the computation of the square root of the covariance matrix (do it only once if you want to generate several realizations !). Use a Cholesky method for that. Here, use `sqrtm`.

Gaussian processes.

In order to generate realizations of a Gaussian process $\mu(\boldsymbol{x})$ with mean zero and covariance function $R(\boldsymbol{x}, \boldsymbol{x}') = \mathbb{E}\big[\mu(\boldsymbol{x})\mu(\boldsymbol{x}')\big]$, the Cholesky method is suitable for an arbitrary grid $\{\boldsymbol{x}, \boldsymbol{x} \in \mathcal{D}\}$:
a) evaluate the covariance matrix $\mathbf{R} = \{R(\boldsymbol{x}, \boldsymbol{x}'), \boldsymbol{x}, \boldsymbol{x}' \in \mathcal{D}\}$.
b) compute a square root $\mathbf{S}$ of $\mathbf{R}$.
c) draw a random vector $\boldsymbol{n}$ with the distribution $\mathcal{N}(\mathbf{0}, \mathbf{I})$ and compute:

$$\boldsymbol{y} = \mathbf{S}\boldsymbol{n}$$

Problem 2. Plot several realizations of a Brownian motion $(W_z)_{z\geq 0}$ over the regular grid $\{z_j = (j-1)h,\ j = 1, \ldots, n\}$ with $n = 2^{10}$ and $h = 0.05$ (it is a Gaussian process with mean zero and covariance function $\mathbb{E}[W_z W_{z'}] = \inf(z, z')$). Use both the Cholesky method and the partial sum method (use `cumsum`).

Stationary Gaussian processes.

In order to generate realizations of a stationary Gaussian process $\mu(\boldsymbol{x})$ with mean zero and covariance function $C(\boldsymbol{x} - \boldsymbol{x}') = \mathbb{E}\big[\mu(\boldsymbol{x})\mu(\boldsymbol{x}')\big]$, we can make use of the spectral representation of the random process $\mu(\boldsymbol{x})$:

$$\mu(\boldsymbol{x}) = \int_{-\infty}^{\infty} e^{i\boldsymbol{k}\cdot\boldsymbol{x}} \hat{C}(\boldsymbol{k})^{1/2} \hat{n}_{\boldsymbol{k}} d\boldsymbol{k}$$

where $\hat{n}_{\boldsymbol{k}}$ is a complex Gaussian white noise.

The Fourier algorithm makes use of the FFT and is suitable for a regular grid, say in 1D: $\{x_j = (j-1)h,\ j = 1, \ldots, n\}$.

Problem 3. Plot several realizations of a zero-mean Gaussian process $\mu(x)$ with Gaussian covariance function $C(x) = \exp(-x^2)$ over the grid $x_j = (j-1)h$ with $n = 2^{10}$ and $h = 0.05$. Use both the Cholesky method and the Fourier method.
Plot the histogram of the empirical distribution $(\mu(x_j))_{j=1,\ldots,n}$ and compare with the theoretical pdf $p(x) = 1/\sqrt{2\pi}\exp(-x^2/2)$.
Plot the empirical covariance function

$$C_e(x_j) = \frac{1}{n-j+1}\sum_{i=1}^{n-j+1}\mu(x_i)\mu(x_{i+j-1})$$

and compare with the theoretical function $C(x)$.
Repeat with a larger n, say $n = 2^{20}$, with the Fourier method.

Problem 4. Plot (with `imagesc` or `pcolor`) a realization of a zero-mean Gaussian process $\mu(\boldsymbol{x})$ with Gaussian covariance function $C(x) = \exp(-|\boldsymbol{x}|^2)$ over a 512×512 grid with step 0.05. Use the Fourier method with `fft2`.

Problem 5. Plot several realizations of a zero-mean Gaussian process $\mu(x)$ with Gaussian covariance function $C(x) = \exp(-x^2)$ over a grid covering $[0, 6]$, given the observations: $\mu(2) = 1$ and $\mu(4) = -1$. Compute the posterior mean and covariance matrix using the formula given below and use the Cholesky method (note: there exists a more efficient method called conditional kriging).

Remember (for problem 5): If

$$\begin{pmatrix}\mathbf{X}_1\\ \mathbf{X}_2\end{pmatrix} \quad\sim\quad \mathcal{N}\left(\begin{pmatrix}\mathbf{0}\\ \mathbf{0}\end{pmatrix}, \begin{pmatrix}\mathbf{R}_{11} & \mathbf{R}_{12}\\ \mathbf{R}_{21} & \mathbf{R}_{22}\end{pmatrix}\right)$$

then

$$\mathcal{L}(\mathbf{X}_1|\mathbf{X}_2 = \boldsymbol{x}_2) \quad=\quad \mathcal{N}(\mathbf{R}_{12}\mathbf{R}_{22}^{-1}\boldsymbol{x}_2, \mathbf{R}_{11} - \mathbf{R}_{12}\mathbf{R}_{22}^{-1}\mathbf{R}_{21})$$