Random paraxial wave equation

Propagation of time-harmonic waves in a homogeneous medium.

We consider a time-harmonic initial condition (with frequency ω) in the plane $z = 0$ whose transverse profile is a Gaussian with radius r_0:

$$\phi_0(x) = \exp\Big(-\frac{x^2}{r_0^2}\Big).$$

Solve the Schrödinger equation from the plane $z = 0$ to the plane $z = L$

$$\partial_z \phi = \frac{i}{2k}\partial_x^2 \phi, \qquad \phi(z=0,x) = \phi_0(x),$$

using the Fourier method (use `fft`). Compute the transmitted wave profile $\phi_t(x) = \phi(L, x)$ for an initial Gaussian beam with radius $r_0 = 4$, a grid of 2^{10} points with a transverse grid step 0.1, $k = \omega = 1$, and $L = 10$.

Check that the transmitted wave profile $\phi_t(x)$ is:

$$\phi_t(x) = \frac{r_0}{r_t}\exp\Big(-\frac{x^2}{r_t^2}\Big), \qquad r_t = r_0\Big(1 + 2i\frac{L}{kr_0^2}\Big)^{1/2}$$

by comparing the square modulus of the numerical transmitted wave with

$$|\phi_t(x)|^2 = \frac{r_0}{R_t}\exp\Big(-\frac{2x^2}{R_t^2}\Big), \qquad R_t = r_0\Big(1 + \frac{4L^2}{k^2 r_0^4}\Big)^{1/2}.$$

Time reversal for time-harmonic waves in a homogeneous medium.

Consider the (compactly supported) time-reversal mirror in the plane $z = L$:

$$\chi_M(x) = \Big(1 - \big(\frac{x}{2r_M}\big)^2\Big)^2 \mathbf{1}_{[-2r_M, 2r_M]}(x).$$

Perform a time-reversal experiment by backpropagating the time-reversed wave ϕ_t (i.e. its complex conjugate) from the time-reversal mirror support at $z = L$ to the plane $z = 0$. Compute the refocused wave $\phi_{tr}(x)$ in the plane $z = 0$ for different values for the radius r_M (from 2 to 20) of the mirror. Note how refocusing becomes poor when r_M becomes small.

Note: if one considers a "Gaussian time-reversal mirror" in the plane $z = L$, with a truncation function of the form

$$\chi_M(x) = \exp\Big(-\frac{x^2}{r_M^2}\Big),$$

then one can check analytically that the refocused wave profile $\phi_{tr}(x)$ is given by

$$\begin{aligned}
\phi_{tr}(x) &= \frac{1}{a_{tr}}\exp\Big(-\frac{x^2}{r_{tr}^2}\Big), \\
r_{tr}^2 &= \Big(\frac{1}{r_M^2} + \frac{1}{r_0^2 - 2i\frac{L}{k}}\Big)^{-1} + 2i\frac{L}{k}, \qquad a_{tr} = \Big(1 + \frac{4L^2}{k^2 r_0^2 r_M^2} + 2i\frac{L}{kr_M^2}\Big)^{1/2}.
\end{aligned}$$

Check this result numerically.

Propagation of time-harmonic waves in a random medium.

Implement a split-step Fourier method with longitudinal step h to solve the Schrödinger equation

$$\partial_z \phi = \frac{i}{2k}\partial_x^2 \phi + \frac{ik}{2}\mu(z,x)\phi, \qquad \phi(z=0,x) = \phi_0(x) = \exp\Big(-\frac{x^2}{r_0^2}\Big)$$

Consider a potential μ of the form:

$$\mu(z,x) = \mu_n(x), \qquad \text{if } z \in [nz_c, (n+1)z_c)$$

where $\mu_0(x)$, $\mu_1(x)$, …, $\mu_{[L/z_c]}(x)$ are independent realizations of a Gaussian process with mean zero and covariance function $\mathbb{E}[\mu_n(x)\mu_n(x')] = \sigma^2 \exp(-(x-x')^2/x_c^2)$.

Take $h = 1$, $z_c = 1$, $x_c = 4$, $\sigma = 1$.

Check that the mean transmitted wave profile $\phi_t(x) = \phi(L,x)$ is:

$$\mathbb{E}[\phi_t(x)] = \frac{r_0}{r_t} \exp\Big(-\frac{x^2}{r_t^2}\Big) \exp\Big(-\frac{\gamma_0 \omega^2 L}{8}\Big)$$

with $\gamma_0 = \sigma^2 z_c$. For the evaluation of the mean take the average over 100 runs with 100 independent realizations of the random medium.

Time reversal for time-harmonic waves in a random medium.

Consider a Gaussian time-reversal mirror in the plane $z = L$: $\chi_M(x) = \exp(-x^2/r_M^2)$.

Perform a time-reversal experiment by backpropagating the time-reversed wave ϕ_t in the same random medium and compute the refocused wave $\phi^{tr}(z,x)$ in the plane $z = 0$. Check that the mean refocused wave profile $\phi_r^{tr}(x) = \phi^{tr}(0,x)$ is given by

$$\mathbb{E}[\phi_r^{tr}(x)] = \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big) \exp\Big(-\frac{x^2}{r_a^2}\Big)$$

with $r_a^{-2} = \gamma_2 \omega^2 L/48$ and $\gamma_2 = 2\sigma^2 z_c/x_c^2$.

Try different values for the radius r_M of the mirror. Note that refocusing becomes significantly better in the random medium case than in the homogeneous medium case when the mirror is relatively small (say $r_M = 2$).

Perform a time-reversal experiment by backpropagating the time-reversed wave ϕ_t in a homogeneous medium and compute the refocused wave $\phi^{tr}(z,x)$ in the plane $z = 0$. Check that the mean refocused wave profile $\phi_r^{tr}(x) = \phi^{tr}(0,x)$ is given by

$$\mathbb{E}[\phi_r^{tr}(x)] = \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big) \exp\Big(-\frac{\gamma_0 \omega^2 L}{8}\Big)$$

Time reversal for time-dependent waves in a random medium.

Here we consider a time-dependent initial condition, whose spectrum is flat over $[\omega_0 - B, \omega_0 + B]$, with $\omega_0 = 1$ and $B = 0.75$, and whose transverse profile is a Gaussian with radius r_0.

Perform a time-reversal experiment for this wave: simply sum the frequency components computed in the previous section for a set of regularly sampled frequencies (say, 20 frequencies over $[\omega_0 - B, \omega_0 + B]$).

Observe the refocused wave profile, compare with

$$\phi_r^{tr}(x) = \frac{1}{a_{tr}} \exp\Big(-\frac{x^2}{r_{tr}^2}\Big) \exp\Big(-\frac{x^2}{r_a^2}\Big)$$

and observe the statistical stability of the refocused wave (i.e. repeat the experiment with different realizations).