Coagulation-Fragmentation Models with Diffusion

Klemens Fellner

DAMTP, CMS, University of Cambridge

Faculty of Mathematics, University of Vienna

Jose A. Cañizo, Jose A. Carrillo, Laurent Desvillettes

Coagulation-fragmentation models with diffusion

evolution of a polymer/cluster density $f(t, x, y) \geq 0$

$$\partial_t f - a(y) \triangle_x f = Q_{coag}(f, f) + Q_{frag}(f)$$

time $t \geq 0$, space $x \in \Omega$ normalised with $|\Omega| = 1$

homogeneous Neumann $\nabla_x f(t, x, y) \cdot \nu(x) = 0$ on $\partial\Omega$

non-negative initial data $f_0(x, y)$

size-dependet diffusion coefficients $a(y)$

Continuous and discreet models

discrete in size models $y = i \in \mathbb{N}$, $f(y) = c_i$

$$Q_{coag} = \frac{1}{2} \sum_{j=1}^{i-1} a_{i-j,j}\, c_{i-j}\, c_j - \sum_{j=1}^{\infty} a_{i,j}\, c_i\, c_j,$$

$$Q_{frag} = \sum_{j=1}^{\infty} B_{i+j}\, \beta_{i+j,i}\, c_{i+j} - B_i\, c_i.$$

coagulation-fragmentation coefficients

$$a_{i,j} = a_{j,i} \geq 0, \qquad \beta_{i,j} \geq 0, \qquad (i, j \in \mathbb{N}),$$

$$B_1 = 0, \qquad B_i \geq 0, \qquad (i \in \mathbb{N}),$$

$$i = \sum\nolimits_{j=1}^{i-1} j\, \beta_{i,j}, \qquad (i \in \mathbb{N}, i \geq 2).$$

Continuous and discreet models

continuous in size model $y \in [0, \infty)$

$$Q_{coag}(f,f) = \int_0^y f(y-y')\, f(y')dy' - 2f(y)\int_0^\infty f(y')\, dy'$$

$$Q_{frag}(f) = 2\int_y^\infty f(y')\, dy' - y\, f(y)$$

assume constant coagulation-fragmentation coefficients

[Aizenman, Bak] '79

Overview

Questions:

- existence of solutions (weak, strong)
- conservation of mass or gelation
- degenerate diffusion coefficients
- large-time behaviour
- fast-reaction limit

Main tools:

- duality arguments
- entropy methods (detailed balance)

Discrete coagulation-fragmentation models

Weak formulation, conservation of mass

test-sequence φ_i,

$$\sum_{i=1}^{\infty} \varphi_i \, Q_{coal} = \frac{1}{2} \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} a_{i,j} \, c_i \, c_j \, (\varphi_{i+j} - \varphi_i - \varphi_j),$$

$$\sum_{i=1}^{\infty} \varphi_i \, Q_{frag} = -\sum_{i=2}^{\infty} B_i c_i \left(\varphi_i - \sum_{j=1}^{i-1} \beta_{i,j} \varphi_j \right).$$

conservation of total mass or gelation

$$\|\rho(t,\cdot)\|_{L^1} = \int_\Omega \sum_{i=1}^{\infty} i c_i(t,x) \, dx \leq \int_\Omega \sum_{i=1}^{\infty} i c_i^0(x) \, dx = \|\rho^0\|_{L^1}$$

Existence of global weak solutions in L^1

assumptions on coefficients

$$\lim_{j\to+\infty}\frac{a_{i,j}}{j}=\lim_{j\to+\infty}\frac{B_{i+j}\,\beta_{i+j,i}}{i+j}=0, \qquad \text{(for fixed } i\geq 1),$$

Then, global weak solutions $c_i\in\mathcal{C}([0,T];L^1(\Omega))$, $i\in\mathbb{N}$

$$\sum_{j=1}^{\infty}a_{i,j}c_ic_j\in L^1([0,T]\times\Omega),$$

$$\sup_{t\geq 0}\int_\Omega\left[\sum_{i=1}^{\infty}ic_i(t,x)\right]dx\leq\int_\Omega\left[\sum_{i=1}^{\infty}ic_i^0(x)\right]dx,$$

[Laurençot, Mischler] ’02

L^2 estimates via duality

Assume coefficients like above and $\rho_0 = \sum_{i=1}^{\infty} i c_i^0 \in L^2(\Omega)$

Then, for all $T > 0$

$$\|\rho\|_{L^2(\Omega_T)} \leq \left(1 + \frac{\sup_i\{d_i\}}{\inf_i\{d_i\}}\right) T \, \|\rho_0\|_{L^2(\Omega)},$$

or for degenerate diffusion

$$\int_0^T \int_\Omega \left[\sum_{i=1}^{\infty} i \, d_i \, c_i(t,x)\right]\left[\sum_{i=1}^{\infty} i \, c_i(t,x)\right] \leq 4 \, T \sup_{i \in \mathbb{N}^*}\{d_i\} \|\rho_0\|_{L^2(\Omega)}.$$

[Cañizo, Desvillettes, F.] ’09 preprints

Proof of duality bounds

Denoting $A(t,x) = \frac{1}{\rho}\sum_{i=1}^{\infty} i\, d_i\, c_i$, then $\|A\|_\infty \le \sup_{i\in\mathbb{N}^*}\{d_i\}$ and

$$\partial_t \rho - \Delta_x (A\,\rho) = 0.$$

Multiplying with the solution w of the dual problem:

$$-\left(\partial_t w + A\,\Delta_x w\right) = H\sqrt{A},$$

$$\nabla_x w \cdot n(x)|_{\partial\Omega} = 0, \qquad w(T,\cdot) = 0$$

for any smooth function $H := H(t,x) \ge 0$ leads to

$$\int_0^T\!\!\int_\Omega H(t,x)\sqrt{A(t,x)}\,\rho(t,x)\,dxdt = \int_\Omega w(0,x)\,\rho(0,x)\,dx.$$

Proof of duality bounds

Multiplying the dual problem by $-\Delta_x w$ yields

$$\int_0^T \int_\Omega A\,(\Delta_x w)^2\,dxdt \le \int_0^T \int_\Omega H^2\,dxdt.$$

and

$$\int_0^T \int_\Omega \frac{|\partial_t w|^2}{A}\;dxdt \le 4\int_0^T \int_\Omega H^2\,dxdt.$$

Hence,

$$|w(0,x)|^2 \le \left(\int_0^T \sqrt{A}\,\frac{|\partial_t w|}{\sqrt{A}}\,dt\right)^2 \le 4\,T\,\|A\|_{L^\infty(\Omega)}\int_0^T \int_\Omega H^2\,dxdt.$$

Proof of duality bounds

Recalling above

$$\int_0^T \int_\Omega H\,\sqrt{A}\,\rho\,dxdt \le \|\rho(0,\cdot)\|_{L^2(\Omega)}\,\|w(0,\cdot)\|_{L^2(\Omega)}$$
$$\le 2\,\sqrt{T\,\|A\|_{L^\infty(\Omega)}}\,\|H\|_{L^2([0,T]\times\Omega)}\,\|\rho(0,\cdot)\|_{L^2(\Omega)}.$$

for all (nonnegative smooth) functions H, we obtain by duality that

$$\|\sqrt{A}\,\rho\|_{L^2(\Omega)} \le 2\,\sqrt{T\,\|A\|_{L^\infty(\Omega)}}\,\|\rho(0,\cdot)\|_{L^2(\Omega)}.$$

Absence of gelation

Assume $a_{i,j} \leq (i+j)\,\theta(j/i)$ for all $j \geq i \in \mathbb{N}$
for a bounded function $\theta(x) \to 0$ as $x \to +\infty$.
Then, a superlinear moment is bounded on bounded time intervals, i.e. for a test-sequence $\{\psi_i\}_{i\geq 1}$ with $\lim_{i\to\infty} \psi_i \to \infty$:

$$\int_\Omega \sum_{i=1}^{\infty} i\,\psi_i c_i \leq C(T)$$

As a consequence, the mass is conserved

$$\int_\Omega \rho(t,x)\,dx = \int_\Omega \rho_0(x)\,dx \quad \text{for all } t \geq 0.$$

[Cañizo, Desvillettes, F.] ’09 preprints

Specialised proof for absence of gelation

In particular $a_{i,j} = \sqrt{i\,j}$ and $B_i = 0$, $\sum_{i=0}^{\infty} i\,\log i\, c_i(0,x)\,dx. < \infty$
Then, (using $\log(1+x) \le C\,\sqrt{x}$)

$$\frac{d}{dt}\int_\Omega \sum_{i=1}^{\infty} i\,\log i\, c_i\,dx$$

$$= \int_\Omega \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \sqrt{ij}\,c_i\,c_j\left(i\,\log(1+\frac{j}{i}) + j\,\log(1+\frac{i}{j})\right)dx$$

$$\le 2\int_\Omega \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} i\,j\,c_i\,c_j\,dx \le 2\int_\Omega \rho(t,x)^2\,dx.$$

Discrete coagulation-fragmentation models

Specialised proof for absence of gelation

In particular $a_{i,j} = \sqrt{i\,j}$ and $B_i = 0$, $\sum_{i=0}^{\infty} i\ \log i\, c_i(0,x)\,dx. < \infty$
As a consequence, we have for all $T > 0$

$$\int_\Omega \sum_{i=0}^{\infty} i\ \log i\, c_i(T,x)\,dx \le \int_\Omega \sum_{i=0}^{\infty} i\ \log i\, c_i(0,x)\,dx$$
$$+ 2\int_0^T \int_\Omega \rho(t,x)^2\,dxdt,$$

and the propagation of the moment $\int \sum_{i=0}^{\infty} i\ \log i\, c_i(\cdot,x)dx$ ensures the conservation of the mass.

Degenerate Diffusion: Absence of gelation

Assume

$$d_i \geq C\, i^{-\gamma}, \qquad a_{i,j} \leq C \left(i^{\alpha} \, j^{\beta} + i^{\beta} \, j^{\alpha} \right),$$

with $\alpha + \beta + \gamma \leq 1$, $\alpha, \beta \in [0, 1)$, $\gamma \in [0, 1]$,
Then, the mass is conserved

$$\int_{\Omega} \rho(t, x)\, dx = \int_{\Omega} \rho_0(x)\, dx \qquad \text{for all } t \geq 0.$$

Discrete coagulation-fragmentation models

Existence theory

via duality we have uniform L^2-bound of approximating systems independent of $a_{i,j}$

the assumption $\lim_{j\to+\infty} \frac{a_{i,j}}{j} = 0$ is needed for the limit of

$$Q_{coag}^{-,M} = c_i^M \sum_{j=1}^{\infty} a_{i,j}\, c_j^M .$$

as c_i^M converges to c_i weak-$*$ in $L^\infty(\Omega_T)$,
we need $\sum_{j=1}^{\infty} a_{i,j}\, c_j^M \to \sum_{j=1}^{\infty} a_{i,j}\, c_j$ strongly in $L^1(\Omega_T)$

$$\int_0^T \int_\Omega \left| \sum_j a_{i,j}(c_j^M - c_j) \right| dx dt \le 2 \sup_{j \ge J_0} \left| \frac{a_{i,j}}{j} \right| \rho + \sup_{j \le J_0} \| c_j^M - c_j \|_{L^1(\Omega_T)}$$

Discrete coagulation-fragmentation models

Existence theory

existence of generalised quadratic models

$$\partial_t c_i - d_i \, \Delta_x c_i = \frac{1}{2} \sum_{k+l=i} a_{k,l} \, c_k \, c_l - \sum_{k=1}^{\infty} a_{i,k} \, c_i \, c_k$$
$$+ \frac{1}{2} \sum_{k,l=1}^{\infty} \sum_{i<\max\{k,l\}} b_{k,l} \, c_k \, c_l \, \beta_{i,k,l} - \sum_{k=1}^{\infty} b_{i,k} \, c_i \, c_k$$

global L^1-existence in 1D provided

$$\lim_{l\to\infty} \frac{a_{k,l}}{l} = 0, \; \lim_{l\to\infty} \frac{b_{k,l}}{l} = 0, \; \lim_{l\to\infty} \sup_k \left\{ \frac{b_{k,l}}{kl} \, \beta_{i,k,l} \right\} = 0 \quad k, i \in \mathbb{N}$$

[Cañizo, Desvillettes, F.] '09 preprints

Continuous coagulation-fragmentation models

[Aizenman, Bak]'79 inhomogeneous

Continuous in size density $f(t,x,y)$ with $y \in [0,\infty)$

$$\partial_t f - a(y)\triangle_x f = \int_0^y f(y-y')\,f(y')dy' - 2f(y)\int_0^\infty f(y')\,dy'$$
$$+\,2\int_y^\infty f(y')\,dy' - y\,f(y)$$

homogeneous Neumann, non-negative initial density $f_0(x,y)$

diffusion may degenerate at most linearly for large sizes

$$a(y) \le a^*(\delta)\,,\ \forall y \in [\delta,\delta^{-1}], \qquad 0 < \frac{a_*}{1+y} \le a(y)\,,\ \forall y \in [0,\infty).$$

[Aizenman, Bak]'79 inhomogeneous

Macroscopic densities

amount of monomers N, number density M

$$N = \int_0^\infty y' f(y')\, dy' \,, \quad M = \int_0^\infty f(y')\, dy'$$

conservation of the total mass

$$\partial_t N - \triangle_x \left(\int_0^\infty y' a(y') f(y')\, dy' \right) = 0$$

$$\partial_t M - \triangle_x \left(\int_0^\infty a(y') f(y')\, dy' \right) = N - M^2$$

[Aizenman, Bak]'79 inhomogeneous

Entropy (free energy functional)

Entropy

$$H(f)(t,x) = \int_0^\infty (f \ln f - f)\ dy\,,$$

Entropy dissipation

$$\begin{aligned}
\frac{d}{dt}\int_\Omega H(f)\,dx &= -D_H(f) \\
D_H(f) &= \int_\Omega \int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx \\
&\quad + \int_\Omega \int_0^\infty \int_0^\infty (f'' - f\,f')\ln\left(\frac{f''}{f\,f'}\right) dy\,dy'dx
\end{aligned}$$

Inequality by [Aizenman, Bak]'79

$$\int_0^\infty\int_0^\infty (f(y+y') - f(y)f(y'))\ln\left(\frac{f(y+y')}{f(y)f(y')}\right) dy\,dy' \geq$$

$$M\;H(f|f_{\sqrt{N},N}) + 2(M-\sqrt{N})^2$$

Entropy dissipation

$$\begin{aligned} D_H(f) \;\; &\geq \;\; \int_\Omega\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dy\,dx \\ &\quad + M\;H(f|f_{\sqrt{N},N}) + 2(M-\sqrt{N})^2 \end{aligned}$$

[Aizenman, Bak]'79 inhomogeneous

Local and global equilibria

Intermediate equilibria with the very moments N and $M^2 = N$

$$f_{\sqrt{N},N} = e^{-\frac{1}{\sqrt{N}}y}$$

the global equilibrium

$$f_\infty = e^{-\frac{y}{\sqrt{N_\infty}}}$$

- is constant in x satisfying $M_\infty^2 = N_\infty$
- preserves the initial mass $N_\infty = \int_0^\infty N(x)\,dx$

[Aizenman, Bak]'79 inhomogeneous

relative entropy, additivity

relative entropy

$$H(f|g) = H(f) - H(g)$$

additivity

$$H(f|f_\infty) = H(f|f_{\sqrt{N},N}) + H(f_{\sqrt{N},N}|f_\infty)$$

$f_{\sqrt{N},N}$ and f_∞ do not need to have the same L^1_y-norm, but nevertheless

$$\int_\Omega H(f_{\sqrt{N},N}|f_\infty)\,dx = 2\left(\sqrt{\int_\Omega N\,dx} - \int_\Omega \sqrt{N}\,dx\right) \geq 0$$

[Aizenman, Bak]'79 inhomogeneous

Existence results

[Am, AW] global existence and uniqueness of classical solutions (1D, not [Aizenman, Bak])

[LM] global existence of weak solutions satisfying the entropy dissipation inequality

$$\int_\Omega H(f(t))\,dx + \int_0^t D_H(f(s))\,ds \le \int_\Omega H(f_0)\,dx$$

Diffusivity $a(y) \in L^\infty([1/R, R])$ for all $R > 0$
Without rate: Equilibrium states attract all global weak solutions

[Aizenman, Bak]'79 inhomogeneous

Theorems

Nonnegative initial data $(1 + y + \ln f_0) f_0 \in L^1((0,1) \times (0,\infty))$ with positive initial mass $\int_0^1 N_0(x)\,dx = N_\infty > 0$ on $\Omega = (0,1)$. As above at most linearily degenerate diffusion.

Then, for $\beta < 2$ and $t > 0$

$$\|f(t,\cdot,\cdot) - f_\infty\|_{L^1_{x,y}} \leq C_\beta\, e^{-(\ln t)^\beta},$$

and

$$\int_0^\infty (1+y)^q \,\|f(t,\cdot,y) - f_\infty(y)\|_{L^\infty_x}\, dy \leq C_{\beta,q}\, e^{-(\ln t)^\beta}\;,$$

for all $t \geq t_* > 0$.

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 1) Additivity

$$\int_0^1 H(f|f_\infty)\,dx = \int_0^1 H(f_{\sqrt{N},N}|f_\infty)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right)$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 2) "Reacting" Moments N and M

$$\int_0^1 H(f|f_\infty)\,dx = \int_0^1 H(f_{\sqrt{N},N}|f_\infty)\,dx + 2\left(\sqrt{\overline{N}} - \overline{\sqrt{N}}\right)$$

$$\sqrt{\overline{N}} - \overline{\sqrt{N}} \le \frac{2}{\sqrt{N_\infty}}\left[\|M - \sqrt{N}\|^2_{L^2_x} + \|M - \overline{M}\|^2_{L^2_x}\right].$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 2) "Reacting" Moments N **and** $M > \mathcal{M}_* > 0$

$$\int_0^1 H(f|f_\infty)\,dx \le C\left[\int_0^1 M\,H(f|f_{\sqrt{N},N})\,dx + 2\|M-\sqrt{N}\|^2_{L^2_x}\right] + \frac{4}{\sqrt{N_\infty}}\|M-\overline{M}\|^2_{L^2_x}$$

$$\le C\int_0^1\int_0^\infty\int_0^\infty (f''-ff')\ln\left(\frac{f''}{ff'}\right)dy\,dy'\,dx + \frac{4}{\sqrt{N_\infty}}\|M-\overline{M}\|^2_{L^2_x}\,,$$

Entropy Entropy Dissipation Estimate

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Step 3) Diffusion For a cut-off size $A > 0$, denote

$M_A(t,x) := \int_0^A f(t,x,y)\,dy$ and $M_A^c(t,x) := \int_A^\infty f(t,x,y)\,dy$

$$\|M - \overline{M}\|^2_{L^2_x} = \int_\Omega \left(M_A - \overline{M_A} + M_A^c - \overline{M_A^c}\right)^2 dx$$

$$\leq 2\|M_A - \overline{M_A}\|^2_{L^2_x} + \frac{4}{A^{2p}} \int_\Omega \left(\int_0^\infty y^p f(y)\,dy\right)^2 dx$$

$$\leq C(P, a_*)\, A\, \|M\|_{L^\infty_x} \int_\Omega \int_0^\infty a(y) \frac{|\nabla_x f|^2}{f}\, dy dx$$

$$+ \frac{4}{A^{2p}} \|M\|_{L^\infty_x} \mathcal{M}_{2p}$$

for any $p > 1$.

needs $\|M\|_{L^\infty_x}$, $\|M\|_{L^\infty_t(L^1_x)}$, $\mathcal{M}_* > 0$

Entropy Entropy Dissipation Estimate

Let $f := f(x,y) \geq 0$ be measurable with moments
$0 < \mathcal{M}_* \leq M(x) = \int_0^\infty f(x,y)\,dy \leq \|M\|_{L^\infty_x}$,
$0 < N_\infty = \int_\Omega \int_0^\infty y\, f(x,y)\,dydx$, $\int_\Omega \int_0^\infty y^{2p} f(x,y)\,dxdy \leq \mathcal{M}_{2p}$.

Then, for all $A \geq 1$ and $p > 1$

$$D_1(f) \geq \frac{C}{A\,\|M\|_{L^\infty_x}} \int_\Omega H(f|f_\infty)\,dx - C\,\frac{\mathcal{M}_{2p}}{A^{2p+1}}\,, \qquad \text{(-1)}$$

with a constant $C = C(\mathcal{M}_*, N_\infty, a_*, P(\Omega))$ depending only on $\mathcal{M}_*$, N_∞, a_*, and the Poincaré constant $P(\Omega)$.

A-priori Estimates

$(L^1 \cap L^2) + L^\infty$ bounds in 1D

Lemma

$$\|M(t,\cdot)\|_{L^\infty_x} \le m_\infty + m_2(t).$$

Proof: $f(t,x,y) - f(t,\tilde{x},y) = 2\int_{\tilde{x}}^{x} \sqrt{f}(t,\xi,y)\,\partial_x\sqrt{f}(t,\xi,y)\,d\xi$

$$\int_0^\infty \left| f(t,x,y) - \int_0^1 f(t,\tilde{x},y)\,d\tilde{x} \right| dy$$

$$\le 2\left[\int_0^\infty\!\!\int_0^1 \frac{f(t,x,y)}{a(y)} dxdy\right]^{\frac{1}{2}} \left[\int_0^\infty\!\!\int_0^1 a(y)|\partial_x\sqrt{f}(t,x,y)|^2 dxdy\right]^{\frac{1}{2}}$$

$$M(t,x) \le \int_0^1 M(t,\tilde{x})\,d\tilde{x} + a_*^{-1/2}\,(\mathcal{M}_0^* + N_\infty)^{1/2}\,D(f(t))^{1/2}.$$

A-priori Estimates

Lower $\int_\Omega M(t,x)\,dx$ bound

Lemma: $\int_\Omega M(t,x)\,dx \geq \mathcal{M}_{0*}$

$$\frac{d}{dt}\int_0^1 M(t,x)\,dx \geq \int_0^1 N_0(x)\,dx - (m_\infty + m_1(t))\int_0^1 M(t,x)\,dx$$

Idea:

$$\begin{aligned}\int_0^1 M(t,x)\,dx \geq &\int_0^1 M_0(x)\,dx\, e^{-\int_0^t (m_\infty + m_1(\sigma))\,d\sigma} \\ &+ \int_0^1 N_0(x)\,dx \int_0^t e^{-\int_s^t (m_\infty + m_1(\sigma))\,d\sigma}\,ds \\ \geq & e^{-\mu_1}\left[e^{-m_\infty t}\|M_0\|_{L^1_x} + \frac{1-e^{-m_\infty t}}{m_\infty}\|N_0\|_{L^1_x}\right]\end{aligned}$$

A-priori Estimates

Moments $M_p(f)(t) := \int_0^1 \int_0^\infty y^p\, f\, dy\, dx$

Lemma: For $p > 1$ and for a.a. $t \geq t_* > 0$

$$M_p(f)(t) \leq (2^{2p}\, C)^p =: \mathcal{M}_p^*$$

for $C = C(t_*, f_0)$ depending only on the initial datum and t_*.

Idea: fragmentation produces moments

$$\frac{d}{dt} M_p(f)(t) \leq (2^p - 2)\, M_p(f)(t)\, [m_\infty + m_2(t)] - \frac{p-1}{p+1} M_{p+1}(f)(t).$$

interpolation $-\frac{p-1}{p+1} M_{p+1}(f) \leq \frac{\epsilon^{-p}}{p+1} N_\infty - \frac{p}{p+1} \epsilon^{-1}\, M_p(f)$

use Duhamel's formula, $\int_{t_*}^t m_2\, ds \leq \mu_2 \sqrt{t - t_*}$, Vallée-Poussin

A-priori Estimates

Lower bound $M(t,x) \geq \mathcal{M}_* > 0$

Lemma: Let $t_* > 0$ be given. Then, there is a strictly positive constant $\mathcal{M}_*$ (depending on t_*, a_* and $a^*(\delta)$)

$$M(t,x) \geq \mathcal{M}_*.$$

Idea: linear lower bound for lost terms

$$\partial_t f - a(y)\,\partial_{xx} f = g_1 - y\,f - \|M(t,\cdot)\|_{L^\infty_x}\,f$$

where g_1 is nonnegative, then

$$(\partial_t + a(y)\,\partial_{xx})\left(f\,e^{ty+\int_0^t \|M(s,\cdot)\|_{L^\infty_x}\,ds}\right) = g_2$$

where g_2 is nonnegative.

A-priori Estimates

Lower bounds $M(t,x) \geq \mathcal{M}_*$ and $N(t,x) \geq \mathcal{N}_*$

Fourier series and Poisson's formula for $\partial_t h - a\,\partial_{xx} h = G \in L^1$ with homogeneous Neumann boundaries on $(0,1)$

$$h(t,x) = \frac{1}{2\sqrt{\pi}} \int_{-1}^{1} \tilde{h}(0,z) \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a\,t}}\, e^{-\frac{(2k+x-z)^2}{4a\,t}}\, dz$$

$$+ \frac{1}{2\sqrt{\pi}} \int_0^t \int_{-1}^{1} \tilde{G}(s,z) \sum_{k=-\infty}^{\infty} \frac{1}{\sqrt{a\,(t-s)}}\, e^{-\frac{(2k+x-z)^2}{4a\,(t-s)}}\, dz ds$$

$\tilde{h}$ and $\tilde{G}$ "mirrored" around 0

A-priori Estimates

Lower bounds $M(t,x) \geq \mathcal{M}_*$ **and** $N(t,x) \geq \mathcal{N}_*$

$$f(t_1+t,x,y) \geq C \int_0^1 f(t_1,z,y)\, e^{-(2t_* + \frac{1}{a_*\, t_*})\, y}\, dz,$$

Moment bound $\int_0^1 \int_0^\infty y^2\, f(t,x,y)\, dy\, dx \leq \mathcal{M}_2^*$

$$M(t_1+t,x) \geq C\, e^{-(2t_* + \frac{1}{a_*\, t_*})\frac{1}{\delta}} \int_0^1 \int_\delta^{1/\delta} f(t_1,z,y)\, dydz$$
$$\geq C\, e^{-(2t_* + \frac{1}{a_*\, t_*})\frac{1}{\delta}} \left(\mathcal{M}_{0*} - \delta\, N_\infty - K\,\delta - \int H(f)\, dx / \ln K \right).$$

Choosing δ and K, we get that $M(t_1+t,x) \geq \mathcal{M}_*$.

Proof of Theorem

Algebraic rate for all $p > 1$

Algebraic rate for all $p > 1$

We have for any $A > 1$

$$\frac{d}{dt}\int_0^1 H(f|f_\infty)\,dx \le -\frac{C}{\|M\|_{L^\infty_x}}\frac{1}{A}\int_0^1 H(f|f_\infty)\,dx + \frac{C_p\,2^{8p^2}}{A^{2p+1}}\,,$$

where $\|M\|_{L^\infty_x}(t) \le m_\infty + m_2(t)$.
balance r.h.s. by choosing $A = A(t) > 2$ as

$$\frac{1}{A} \le C_3^{-1/2}\left(\frac{C_4\int_0^1 H(f|f_\infty)\,dx}{\|M\|_{L^\infty_x}\,2^{8p^2}}\right)^{\frac{1}{2p}}\,,$$

Thus, Gronwall yields algebraic rate for all $p > 1$.

Proof of Theorem

Algebraic rate for all $p > 1$

Faster than polyniminal rate

Then, by summing w.r.t. $p \in \mathbb{N}$

$$\int_0^1 H(f(t)|f_\infty)\,dx \le L(t - C_7),$$

where (for all $1 < \alpha < 2$)

$$L^{-1}(t) = \sum_{q\ge 1,\ \text{even}} \frac{t^q}{(C_{10}\,q)^q\,2^{2q^2}} = \sum_{q\ge 1,\ \text{even}} t^q\,e^{-2q^2\ln 2 - q\ln(q\,C_{10})}$$
$$\ge C(\alpha)\,e^{[\ln^{2(\alpha-1)}(t)]}$$

for all t large enough and $1 < \alpha < 2$.

Fast-Reaction limit

formal

$$\partial_t f^\varepsilon - a(y)\triangle_x f^\varepsilon = \frac{1}{\varepsilon}\left(Q_{coag}(f^\varepsilon, f^\varepsilon) + Q_{frag}(f^\varepsilon)\right)$$

formal limit: $f^\varepsilon \to e^{-\frac{y}{\sqrt{N^0}}}$

$$\partial_t N^0 - \triangle_x n(N^0) = 0$$

where

$$n(N) := \int_0^\infty a(y) y e^{-\frac{y}{\sqrt{N}}}\, dy$$

$$0 < a_* N \le n(N) \le a^* N\,, \qquad 0 < a_* \le n'(N) \le a^*$$

Fast-Reaction limit

Entropy, entropy dissipation

$$-\varepsilon \frac{d}{dt} \int_\Omega H(f^\varepsilon)\, dx \geq \int_\Omega M^\varepsilon\, H(f^\varepsilon | f_{M^\varepsilon, N^\varepsilon})\, dx + 2 \int_\Omega ((M^\varepsilon) - \sqrt{N^\varepsilon})^2\, dx$$

Thus

$$\int_0^\infty \int_\Omega M^\varepsilon H(f^\varepsilon | f_{\sqrt{N^\varepsilon}, N^\varepsilon})\, dx\, dt \leq \varepsilon\, C$$

Assumption $M^\varepsilon \geq \mathcal{M}_*$

$$\| f^\varepsilon - e^{-\frac{y}{\sqrt{N^\varepsilon}}} \|^2_{L^2_t(L^1_{x,y})} \leq \varepsilon\, C(\mathcal{M}_*)$$

Fast-Reaction limit

Expansion

Looking for $f_1^\varepsilon \in L^2_{t,x}(L^1_y((1+y)\,dy))$

$$f^\varepsilon = e^{-\frac{y}{\sqrt{N^\varepsilon}}} + \varepsilon^\theta f_1^\varepsilon$$

$\nabla_x f_1^\varepsilon \cdot \nu(x) = 0$ and $\theta > 0$

Then

$$\partial_t N^\varepsilon - \triangle_x n(N^\varepsilon) = \varepsilon^\theta \triangle_x \int_0^\infty a(y) y f_1^\varepsilon\, dy \;:= \varepsilon^\theta\, \triangle_x g^\varepsilon$$

where $g^\varepsilon \in L^2_{t,x}$ with $\nabla_x g^\varepsilon \cdot \nu(x) = 0$

Fast-Reaction limit

Compactness

Let $g^\varepsilon \in L^2_{t,x}$ with $\nabla_x g^\varepsilon \cdot \nu(x) = 0$. Take initial data $N_{in} \in L^2_x$.
Then, the solutions of the nonlinear diffusion equation

$$\partial_t N^\varepsilon - \triangle_x n(N^\varepsilon) = \varepsilon^\theta \triangle_x g^\varepsilon$$

$$\nabla_x N^\varepsilon \cdot \nu(x)|_{\partial\Omega} = 0$$

converge in $L^2_{t,x}$ to the solution N of

$$\partial_t N - \triangle_x n(N) = 0$$

$$\nabla_x N \cdot \nu(x)|_{\partial\Omega} = 0$$

Fast-Reaction limit

Compactness : proof

Duality argument : $w \geq 0$, $w(T) = 0$, $\nabla_x w \cdot \nu(x)|_{\partial\Omega} = 0$

$$-\partial_t w - \frac{n(N^\varepsilon) - n(N)}{N^\varepsilon - N} \triangle_x w = H$$

Holds that $\|\triangle_x w\|_{L^2([0,T]\times\Omega)} \leq C\|H\|_{L^2([0,T]\times\Omega)}$

Then

$$\left|\int_0^T \int_\Omega (N^\varepsilon - N)\, H dx dt\right| \leq \varepsilon^\theta \|g^\varepsilon\|_{L^2_{t,x}} \, \|\triangle_x w\|_{L^2([0,T]\times\Omega)}$$

Since $H \geq 0 \in C_0^\infty([0,T] \times \Omega)$ is arbitrary

$$\|N^\varepsilon - N\|_{L^2_{t,x}} \leq C\,\varepsilon^\theta\,\|g^\varepsilon\|_{L^2_{t,x}} \leq C\,\varepsilon^\theta$$

THANK YOU!

References

[AB] M. Aizenman, T. Bak, “Convergence to Equilibrium in a System of Reacting Polymers”, *Commun. math. Phys.* **65** (1979), 203–230.

[Am] H. Amann, “Coagulation-fragmentation processes”, Arch. Rational Mech. Anal. **151** (2000), 339-366.

[AW] Amann, H., Walker, C., “Local and global strong solutions to continuous coagulation-fragmentation equations with diffusion”, J. Differential Equations **218** (2005), 159-186.

[LM] Ph. Laurençot, S. Mischler, “The continuous coagulation-fragmentation equation with diffusion”, Arch. Rational Mech. Anal. **162** (2002), 45-99.