Equilibrium Pricing in Incomplete Markets
- The Continuous Time Model -

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Outline

• Reminder: The multi-period model.
• The continuous time model.
• Equilibrium dynamics and BSDEs.
• Back to the discrete time model.

The model in discrete time

- We considered a dynamic incomplete market model with:
  - a finite set $\mathbb{A}$ of agents endowed with $H^a$ ($a \in \mathbb{A}$)
  - a finite sample space $(\Omega, \mathcal{F}, \mathbb{P})$
- At time $t = 1, \ldots, T$ the agents maximize preference functionals

$$U_t^a : L(\mathcal{F}_T) \to L(\mathcal{F}_t)$$

that are concave, normalized, monotone, translation invariant,

$$U_t^a(X + Z) = U_t^a(X) + Z \quad \text{for all} \quad Z \in L(\mathcal{F}_t),$$

and time consistent, i.e.,

$$U_t^a(X) = U_t^a(U_{t+1}^a(X)) \quad \text{for all} \quad X \in \mathcal{F}_{t+1}.$$

- The illiquid asset paid dividends so its terminal value was given.

The illiquid asset was priced in equilibrium.
The Random Walk Framework

- The agents’ preferences followed the backward dynamics:

$$U_{t+h}^a - U_h^a = f_t^a(Z_{t+h}^a) + Z_{t+h}^a \cdot \Delta b_{t+h}, \quad U_T^a = H^a.$$  

- From this we concluded that

$$R_{t+h} - R_h = Z_{t+h}^R \cdot \nabla f_t(0) + Z_{t+h}^R \cdot \Delta b_{t+h}, \quad R_T = H$$

$$V_{t+h}^a - V_h^a = g^a(Z_{t+h}^R, Z_{t+h}^a) + Z_{t+h}^a \cdot \Delta b_{t+h}, \quad V_T^a = V^a$$

where

$$Z_{t+h}^R = \pi_t(R_{t+h}) \quad \text{and} \quad Z_{t+h}^a = \pi_t(V_{t+h}^a).$$

What happens if the time between two trading periods tends to zero?
Equilibrium Pricing in Continuous Time

- The agents’ incomes are exposed to financial and non-financial risk:
  \[ W^a = H^a(S_T, R^a_T) \text{ where } R^a_T \sim Q(R_T; \cdot). \]

- Individual risks \( R^a_T \) originate from a common risk factor \( R_T \).

- Preferences are described dynamic translation invariant, ... preference functionals.

- A third party (insurance company) issues a derivative on \( R_T \):
  \[ B_T = H(R_T). \]

- Exchange of risk exposure through trading the derivative.

The derivative in fixed supply and will be priced by an equilibrium approach.
The microeconomic setup

- The agents are exposed to financial and non-financial risk factors:
  - Logarithmic Asset prices follow a diffusion process:
    \[ dS_t = \theta^S(R_t)dt + dW_t^S. \]
  - The external risk process follows a Brownian motion with drift:
    \[ dR_t = \mu dt + dW_t^R. \]

- The agents have bounded random incomes of the form:
  \[
  H^a = h^a(S_T, R_T + W_T^a) + \int_0^T \varphi^a_s(S_s, R_s + W_s^a) ds.
  \]
  where all the Brownian motions \( W^S, W^R, (W^a) \) are independent.

The market is incomplete even after the new asset is introduced.
The optimization problem

- Their risk preferences are described by backward stochastic differential equations (BSDEs):

\[- Y_t^a = H^a - \int_t^T g^a(s, X_s, Z_s^a)\, ds - \int_t^T Z_s^a \cdot dW_s\]

where \( X = (S, R, (W^a)_{a \in A}) \) is the forward process.

- One can show (key!) that all of the entries of the vector

\[ Z_t^a = \left( Z_t^S, Z_t^R, (Z_t^b)_{b \in A} \right) \]

are zero except \( Z^S, Z^R, Z^a \); these refer to the agents’ risk exposure.

- We consider the case of entropic utility functions:

\[ g^a(t, x, z) = \frac{1}{2\gamma_a} \|z\|^2 \quad Y_t^a = \frac{1}{\gamma_a} \log E[-e^{-\gamma_t^a H_a} | \mathcal{F}_t] \]

All BSDEs are assumed to satisfy a comparison principle.
The Derivative

• There is an insurance company or investment bank ...
  – ... that holds a portfolio of climate sensitive securities, and ...
  – ... issues a bond whose payoff $B$ depends on the portfolio risk:

\[
B = h(R_T) + \int_0^T \varphi_s(R_s) ds \quad \text{or} \quad B = H(R_T).
\]

• It is in unit net supply and priced by an equilibrium condition:

\[
B_t = \mathbb{E}_{\mathbb{P}^*}[B|\mathcal{F}_t] \quad \text{w.r.t an endogenous pricing measure } \mathbb{P}^*.
\]

• We state conditions that guarantee that an equilibrium pricing measure exists.

*We focus on risk transfer rather than risk sharing.*
Pricing Schemes

- The derivative is priced in a market environment; hence by a linear pricing scheme

\[ I : L^2(\mathbb{P}) \to \mathbb{R}_+. \]

- Each such scheme can be identified with a measure \( Q \approx P \).

- We assume that the agents have no impact on stock prices so the restriction on \( S \) is given by the price of financial risk \( \theta^S \).

- Notice that the pricing rule is linear for the agents, not for the insurer.

**The market price of external risk will be endogenous.**
Pricing Schemes

- The set of all possible pricing rules is given by

\[ \mathcal{P} = \{ Q \approx \mathbb{P} \text{ and } S \text{ is a } Q\text{-martingale} \} \]

- For each \( Q \in \mathcal{P} \) the density is a uniformly integrable martingale:

\[
Z_t = \exp \left( - \int_0^t \begin{pmatrix} \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \\ \theta^S \\ \theta^R \

\right) \right) - \frac{1}{2} \int_0^t |\theta_s|^2 ds \right)
\]

- The set of all pricing linear rules can be identified with the set of market prices of external risk \( \theta^R = (\theta^R_s) \)

such that \( (Z_t) \) defined by the above equation is an u.i. martingale.

**The set of pricing rules is identified by the market prices of external risk.**
Risk Sharing vs Risk Transfer

• In a model of risk sharing the bond pays no dividends.

• Exchange of risk exposures takes place through a fictitious asset:

\[ d\bar{B}_t^\theta = \theta_t^R dt + dW^R \]

with a given volatility process that can be normalized.

• In a model of risk transfer exchange of risk exposures takes place through market prices:

\[ dB_t^\theta = \kappa_t^R (\theta_t^R dt + dW^R) + ...dt + ...dW_t^S. \]

with an endogenous volatility process that cannot be normalized!

We characterize the equilibrium market price of risk and represent \( \kappa^R \).
Utility Optimization and Market Clearing

- There is no \textit{a-priori} reason that the bond “adds somethings”.

- Given a candidate $\theta^R$ for the market price of external risk:
  - ... let us assume the market is complete (we verify this later), ...
  - ... and introduce a pricing measure $\mathbb{P}^\theta$ along with ...
  - ... the corresponding bond price process ($B_t^\theta$), ...
  - ... and solve the agents’ optimization problems.

- For a given admissible trading strategy $\pi$ in both markets:
  
  $$-dY^a_t(\pi) = g^a(t, X_t, Z_t)dt - Z_t dW_t$$

  with terminal condition
  
  $$Y^a_T(\pi) = -H^a - V^a,\theta_T(\pi).$$

- The agent’s goal is the minimize $Y^a_0(\pi)$.

\textbf{We first solve this problem for a given pricing measure.}
Optimal Trading Strategies and Equilibrium

• Let \((Y^a, Z^a)\) be the unique solution of the agent’s utility BSDE.

• For a given measure \(\mathbb{P}^\theta\) the optimal strategy \(\pi^a\) is of the form:

\[
\pi_t^a = (\pi_t^{a,S}, \pi_t^{a,B}) = G^a(t, Z_t^a, \theta_t^S, \kappa_t^R).
\]

• We need to satisfy the equilibrium condition:

\[
\sum_a \pi_t^{a,B} = 1.
\]

• General equilibrium theory in a complete market environment:

  competitive equilibria \(\leftrightarrow\) representative agent equilibria.

• Due to the the specific structure of idiosyncratic risk exposures:

  Analysis can be reduced to a representative agent economy.

We can describe equilibrium prices by a single BSDE.
The Representative Agent

- Assume that only two agent are active in the market: \( \mathbb{A} = \{a, b\} \).
- The representative agent minimizes aggregate risk: the BSDE is
  \[
  -dY_{t}^{ab} = g^{ab}(t, X_t, Z_t)dt - Z_t dW_t
  \]
  with the terminal condition
  \[
  Y_T^{ab}(\pi) = -H^a - H^b - H - V_T^{ab,\theta}(\pi)
  \]
  where the driver \( g^{a,b} \) is defined by the \textit{inf}-convolution:
  \[
  g^{ab}(t, z) = \inf_{x} \{ g^{a}(t, z - x) + g^{b}(t, x) \} = \frac{1}{2\gamma} \| z \|^2.
  \]
- Under some assumptions the agent’s minimization problem has a solution for a given pricing measure.

\textbf{We choose} \( \theta^R \) \textbf{such that} \( \pi_t^{ab,B} \equiv 0 \).
The Equilibrium Market Price of External Risk

**Theorem:** Assume that the derivative’s payoff is strictly monotone in the external risk process \((R_t)\) and consider the quadratic BSDE

\[
dY_t = -Z_t dW_t + \frac{1}{2} \left[ -(Z_t^R)^2 + (\theta^S)^2 - 2\theta^S Z_t^S - \sum_{a \in A} (Z_t^a)^2 \right] dt
\]

with terminal condition \(Y_T = -H^{ab}\) where

\[
Z = \left( Z^S, Z^R, (Z^a)_{a \in A} \right).
\]

Then \(Z^R\) is an equilibrium market price of external risk. Under additional assumptions each \(Z^a\) is bounded.

**What does all this have to do with discrete time models?**
The Equilibrium Market Price of External Risk

Theorem: If the preceding BSDE has a unique solution such that \( Z^a \) is bounded, then the solution can be approximated by equation in discrete time of the form:

\[
\bar{Y}_t = \mathbb{E}[\bar{Y}_{t+h}|\mathcal{F}_t] + f_t(\bar{Z}_{t+h})
\]

which is just the equilibrium dynamics corresponding to a discrete time model.

Corollary: The equilibrium dynamics of the discrete time model converge to an equilibrium dynamics of a continuous time model.

Notice: The convergence result requires an equilibrium in the continuous time!
Summary and Conclusion

- We considered discrete time model of general equilibrium pricing under translation invariant preferences.
- Equilibria exist if and only if a representative agent exists.
- In a random walk framework equilibria can be characterized in terms of a coupled system of backward equation.
- For a simple benchmark model: convergence of equilibrium dynamics to equilibria of a continuous time model
- Existence and differentiability if quadratic BSDEs.
- Applications of BSDEs to problems of cross hedging.