Equilibrium Pricing in Incomplete Markets - The One Period Model -

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Outline and related literature

- The microeconomic model
- The representative agent.
- Characterization of equilibrium results.
- Existence of equilibrium results.

The presentation is based on the following papers:

- Cheridito, H, Kupper & Pirvu (2008) "Equilibrium in incomplete markets under translation invariant preferences", in preparation.
- Filipovic & Kupper (2007) "Equilibrium prices for monetary utility functions", Working paper.

The model

- We consider a one-period incomplete market model with:
  - a finite set $\mathbb{A}$ of agents endowed with random incomes H^a
  - a finite sample space $(\Omega, \mathscr{F}, \mathbb{P})$
- Each agent $a \in \mathbb{A}$ maximizes a preference functional

$$U^a : L(\mathscr{F}) \to \mathbb{R}$$

that is normalized, monotone, and translation invariant:

$$U^a(X + m) = U^a(X) + m.$$

- The agents can trade a liquid and an illiquid asset:
  - The price process (S_0, S_1) of the liquid asset is exogenous.
  - The price process (R_0, R_1) of the illiquid asset is endogenous.
- The illiquid asset pays a dividend d_1 at time $t = 1$ so that

$$R_1 = d_1.$$

THE ILLIQUID ASSET ("RISK BOND") WILL BE PRICED IN EQUILIBRIUM.

The optimization problem

- Each agent $a \in \mathbb{A}$ is endowed with a (random) payoff H^a.
- The agent is exposed to financial and non-financial risk factors:

$$H^a \in L(\mathscr{F}) \quad \text{but it could be that} \quad H^a \notin \sigma(S_1, R_1).$$

- Each agent a $\in \mathbb{A}$ trades to maximize her utility functional:

$$\max_{\eta^a, \vartheta^a} U^a \left(H^a + \eta^a \Delta S_1 + \vartheta^a \Delta R_1 \right)$$

where ΔS_1 and ΔR_1 denote the price increments of the assets.

The market is incomplete so the agent cannot hedge all of her risk exposure.

Equilibrium pricing in a static model

Definition: A partial (in the bond market) equilibrium is a trading strategy $\{(\hat{\eta}^a, \hat{\vartheta}^a)\}_{a \in \mathbb{A}}$ along with an initial price R_0 such that:

a) Each agent maximizes her utility from trading:

$$U^a(H^a + \hat{\eta}^a \Delta S_1 + \hat{\vartheta}^a \Delta R_1) \geq U^a(H^a + \eta^a \Delta S_1 + \vartheta^a \Delta R_1)$$

b) The bond markets clears:

$$\sum_{a \in \mathbb{A}} \hat{\vartheta}^a = 1.$$

- We do not require market clearing in the financial market.
- The agents' combined demand in the stock is small.

Our goal is to prove the existence of an equilibrium.

Equilibrium pricing in a static model

• In a complete market one proves existence of equilibrium by
- defining a "representative agent" that holds all endowments;
- choose the prices s.t. it is optimal for the agent not to trade.

• The definition of the representative agent depends on the equilibrium to be supported ($\rightsquigarrow$ fixed point!)

• This approach typically fails when markets are incomplete.

• However: when the agents have monetary utility functions:
- the approach carries over to incomplete markets;
- the definition of the representative agent is independent of the equilibrium.

The representative agent is defined in terms of the convolution of the utility functions.

The representative agent

Assumption (A): The aggregate utility can be maximized:

$$\sum_a U^a\left(H^a + \hat{\eta}^a \Delta S_1 + \hat{\vartheta}^a R_1\right) \geq \sum_a U^a\left(H^a + \eta^a \Delta S_1 + \vartheta^a R_1\right)$$

for all strategies that satisfy partial market clearing: $\sum_a \vartheta^a = 1$.

- The convolution $\Phi : L(\mathscr{F}) \to \mathbb{R}$ of the utilities is defined by

$$\Phi(X) = \sup_{\eta^a, \vartheta^a} \left\{ \sum_a U^a\left(\frac{X}{|\mathbb{A}|} + H^a + \eta^a \Delta S_1 + \vartheta^a R_1\right) \; : \; \sum_a \vartheta^a = 1 \right\}.$$

It can be viewed as the representative agent's utility function.

UNDER CONDITION (A) THE SUP IS ATTAINED AT $X = 0$ AND IS FINITE.

The representative agent

- Since the sample space is finite convex analysis results yield:

$$\Phi(X) = \min_{\xi \in \mathscr{D}} \{\mathbb{E}[\xi * X] - \varphi(\xi)\}$$

where $\mathscr{D}$ is the set of all equivalent probability densities and

$$\varphi(\xi) = \sup_{Y \in L(\mathscr{F})} \{\Phi(Y) - \mathbb{E}[\xi * Y]\}$$

- In particular, there exists a super-gradient $\hat{\xi}$ of Φ at zero:

$$\Phi(0) = \varphi(\hat{\xi}).$$

- The super-gradient satisfies the standard condition of

"At a price system $\hat{\xi}$ the allocation $X = 0$ is optimal."

IN PRINCIPLE THE CONSUMPTION SPACE $L(\mathscr{F})$ IS TOO LARGE.

The representative agent

- The fact that Φ is defined on $L(\mathscr{F})$ mimics completeness.

- Our agent's consumption space is given by the linear subset

$$\mathbb{S}_1 := \{\eta S_1 + \vartheta R_1 \ : \ \eta, \vartheta \in \mathbb{R}\}$$

so we consider the restriction $\hat{\varphi}$ of φ to $\mathbb{S}_1$:

$$\hat{\varphi}(\xi) = \sup_{\eta,\vartheta} \{\Phi(\eta S_1 + \vartheta R_1) - \mathbb{E}[\xi(\eta S_1 + \vartheta R_1)]\} .$$

- Since any $\hat{\xi} \in \partial\Phi(0)$ satisfies the condition

$$\Phi(0) = \hat{\varphi}(\hat{\xi});$$

it can be viewed as a super-gradient at 0 of Φ restricted to $\mathbb{S}_1$.

The space $\partial\Phi(0)$ is just fine (we do not consider the problem of uniqueness).

Characterization and existence of equilibrium

Theorem: The process (R_0, R_1) along with the trading strategy $\{(\eta^a, \vartheta^a)\}_{a \in \mathbb{A}}$ is an equilibrium if and only if the following holds:

a) The bond market clears, i.e., $\sum_{a \in \mathbb{A}} \vartheta^a = 1$.

b) The representative agent maximizes her utility:

$$\Phi(0) = \sum_a U^a(H^a + \eta^a \Delta S_1 + \vartheta^a R_1) = \varphi(\hat{\xi})$$

c) Asset prices are martingales under the measure $\frac{d\mathbb{Q}}{d\mathbb{P}} = \hat{\xi}$, i.e.,

$$S_0 = \mathbb{E}[\hat{\xi} * S_1] \quad \text{and} \quad R_0 = \mathbb{E}[\hat{\xi} * R_1].$$

Corollary: Under Condition (A) an equilibrium exists.

How can we generalize these results to a dynamic framework?