# Hypocoercivity for kinetic equations with linear relaxation terms


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(A JOINT WORK WITH C. MOUHOT AND C. SCHMEISER)


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# Outline

- The goal is to understand the *rate of relaxation* of the solutions of a kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x V \cdot \nabla_v f = \mathsf{L} f$$

towards a global equilibrium when the collision term acts only on the velocity space. Here $f = f(t, x, v)$ is the distribution function. It can be seen as a probability distribution on the phase space, where $x$ is the position and $v$ the velocity. However, since we are in a linear framework, the fact that $f$ has a constant sign plays no role.

- A key feature of our approach [J.D., Mouhot, Schmeiser] is that it distinguishes the mechanisms of relaxation at *microscopic level* (convergence towards a local equilibrium, in velocity space) and *macroscopic level* (convergence of the spatial density to a steady state), where the rate is given by a spectral gap which has to do with the underlying diffusion equation for the spatial density

# A very brief review of the literature

- Non constructive decay results: [Ukai (1974)] [Desvillettes (1990)]
- Explicit $t^{-\infty}$-decay, no spectral gap: [Desvillettes, Villani (2001-05)], [Fellner, Miljanovic, Neumann, Schmeiser (2004)], [Cáceres, Carrillo, Goudon (2003)]
- *hypoelliptic theory*:
  [Hérau, Nier (2004)]: spectral analysis of the Vlasov-Fokker-Planck equation
  [Hérau (2006)]: linear Boltzmann relaxation operator
- Hypoelliptic theory vs. *hypocoercivity* (Gallay) approach and generalized entropies:
  [Mouhot, Neumann (2006)], [Villani (2007, 2008)]
- Other related approaches: non-linear Boltzmann and Landau equations:
  micro-macro decomposition: [Guo]
  hydrodynamic limits (fluid-kinetic decomposition): [Yu]

# A toy problem

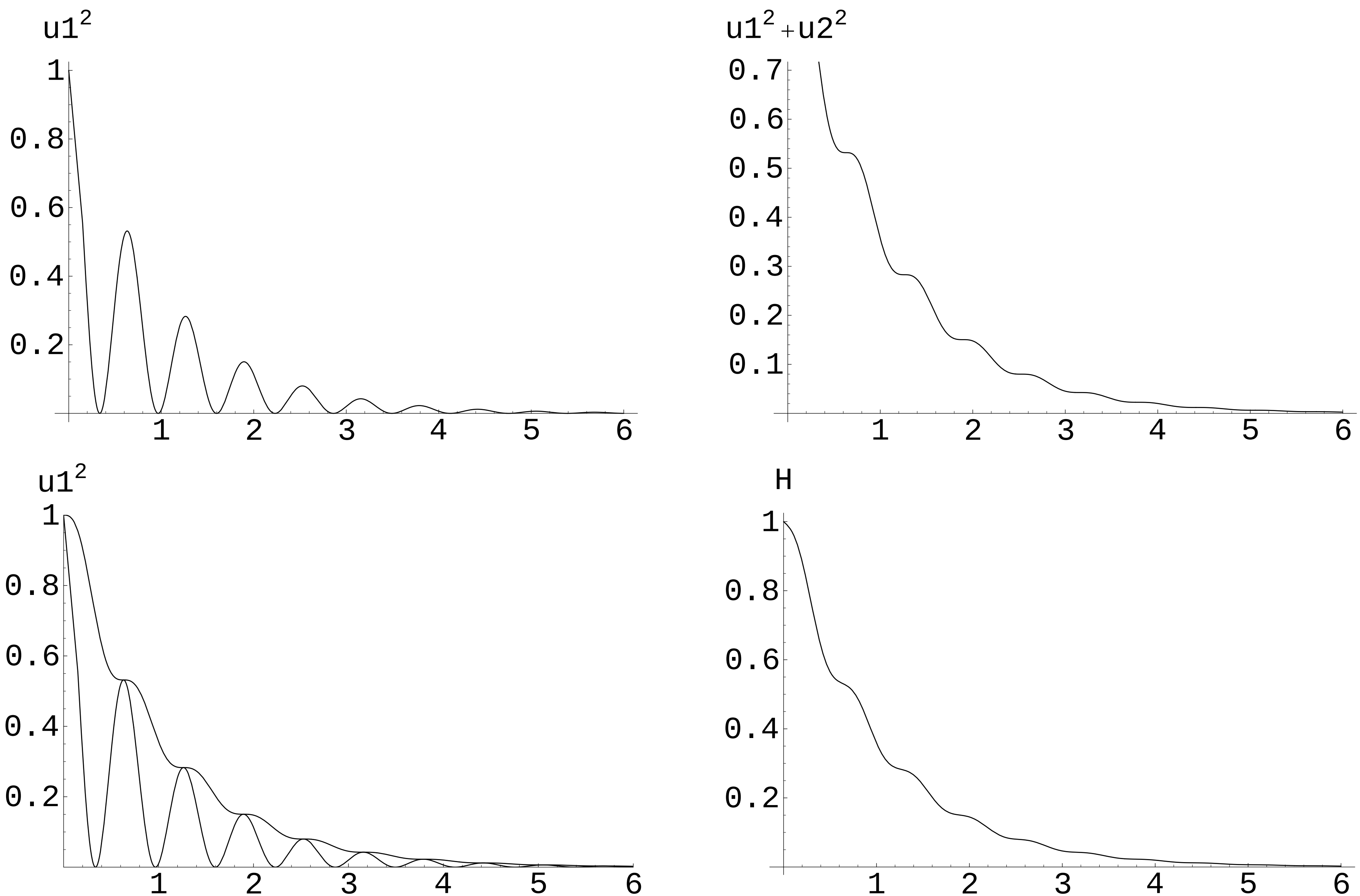
$$\frac{du}{dt} = (L - T)\,u\,, \quad L = \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix}, \quad T = \begin{pmatrix} 0 & -k \\ k & 0 \end{pmatrix}, \quad k^2 \geq \Lambda > 0$$

Nonmonotone decay, reminiscent of [Filbet, Mouhot, Pareschi (2006)]

- H-theorem: $\frac{d}{dt}|u|^2 = -2\,u_2^2$
- macroscopic limit: $\frac{du_1}{dt} = -k^2\,u_1$
- generalized entropy: $H(u) = |u|^2 - \frac{\varepsilon\,k}{1+k^2}\,u_1\,u_2$

$$\begin{aligned} \frac{dH}{dt} &= -\left(2 - \frac{\varepsilon\,k^2}{1+k^2}\right) u_2^2 - \frac{\varepsilon\,k^2}{1+k^2}\,u_1^2 + \frac{\varepsilon\,k}{1+k^2}\,u_1\,u_2 \\ &\leq -(2-\varepsilon)\,u_2^2 - \frac{\varepsilon\Lambda}{1+\Lambda}\,u_1^2 + \frac{\varepsilon}{2}\,u_1 u_2 \end{aligned}$$

# Plots for the toy problem



# . . . compared to plots for the Boltzmann equation

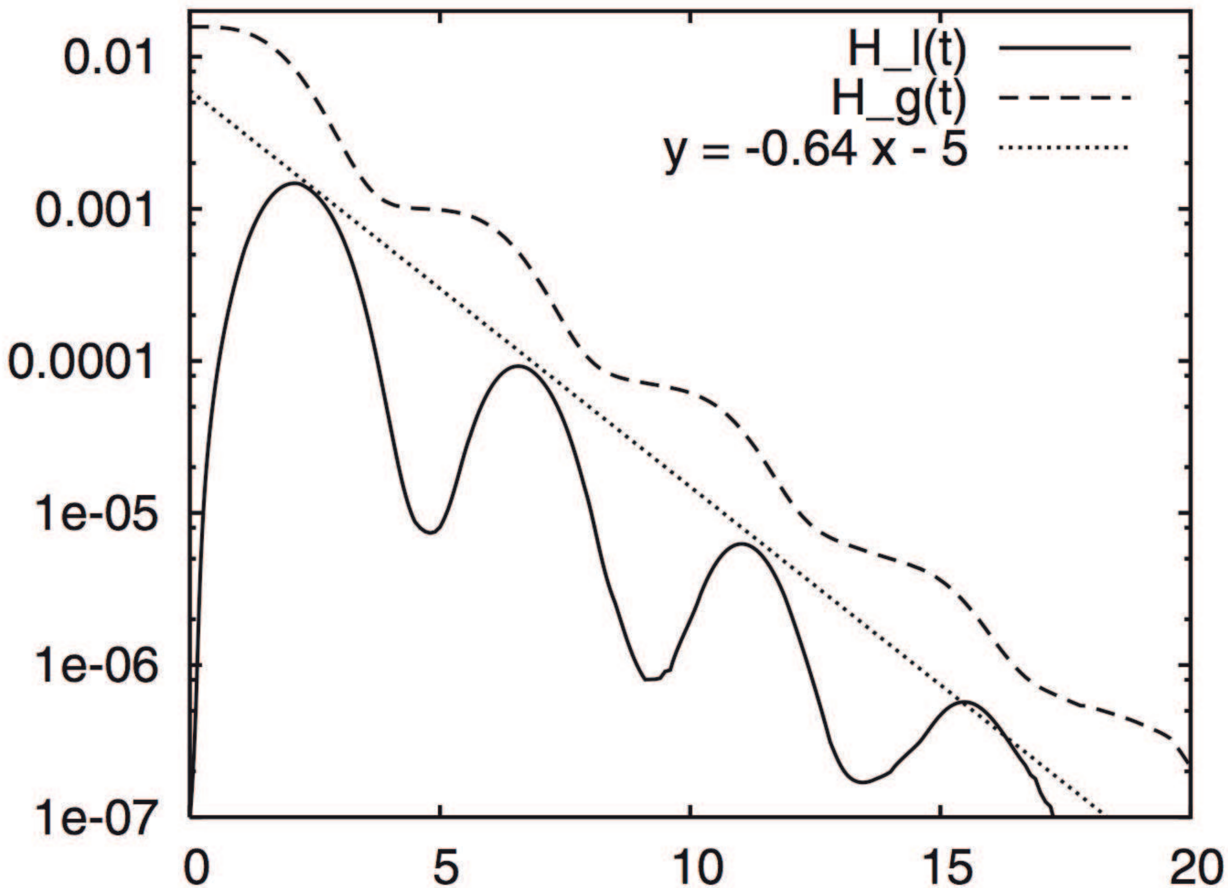



Figure 1: [Filbet, Mouhot, Pareschi (2006)]

# The kinetic equation

$$\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f\,, \quad f = f(t,x,v)\,,\ t>0,\ \ x\in\mathbb{R}^d,\ v\in\mathbb{R}^d \tag{1}$$

- L is a linear collision operator
- $V$ is a given *external potential* on $\mathbb{R}^d$, $d \geq 1$
- $\mathsf{T} := v\cdot\nabla_x - \nabla_x V\cdot\nabla_v$ is a transport operator

There exists a scalar product $\langle\cdot,\cdot\rangle$, such that L is symmetric and T is antisymmetric

$$\frac{d}{dt}\,\|f-F\|^2 = -2\,\|\mathsf{L}\,f\|^2$$

... seems to imply that the decay stops when $f \in \mathcal{N}(L)$
but we expect $f \to F$ as $t \to \infty$ since $F$ generates $\mathcal{N}(L)\cap\mathcal{N}(T)$
Hypocoercivity: prove an H-theorem for a generalized entropy

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}\,f, f\rangle$$

# Examples

- L is a linear relaxation operator L

$$\mathsf{L}\, f = \Pi\, f - f\,, \quad \Pi\, f := \frac{\rho}{\rho_F}\, F(x,v)$$

$$\rho = \rho_f := \int_{\mathbb{R}^d} f\, dv$$

  - Maxwellian case: $F(x,v) := M(v)\, e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2}\, e^{-|v|^2/2} \implies \Pi f = \rho_f\, M(v)$
  - Linearized fast diffusion case: $F(x,v) := \omega\, \left(\frac{1}{2}\, |v|^2 + V(x)\right)^{-(k+1)}$
- L is a Fokker-Planck operator
- L is a linear scattering operator (including the case of non-elastic collisions)

# Some conventions. Cauchy problem

- $F$ is a positive probability distribution
- Measure: $d\mu(x,v) = F(x,v)^{-1}\,dx\,dv$ on $\mathbb{R}^d \times \mathbb{R}^d \ni (x,v)$
- Scalar product and norm $\langle f, g\rangle = \iint_{\mathbb{R}^d\times\mathbb{R}^d} f\,g\,d\mu$ and $\|f\|^2 = \langle f, f\rangle$

The equation

$$\partial_t f + \mathsf{T}\,f = \mathsf{L}\,f$$

with initial condition $f(t=0,\cdot,\cdot) = f_0 \in L^2(d\mu)$ such that

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f_0\,dx\,dv = 1$$

has a unique global solution (under additional technical assumptions): [Poupaud], [JD, Markowich,Ölz, Schmeiser]. The solution preserves mass

$$\iint_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,v)\,dx\,dv = 1 \quad \forall\, t \geq 0$$

# Maxwellian case: Assumptions

We assume that $F(x,v) := M(v)\, e^{-V(x)}$ with $M(v) := (2\pi)^{-d/2}\, e^{-|v|^2/2}$ where $V$ satisfies the following assumptions

(H1) *Regularity:* $V \in W^{2,\infty}_{\mathrm{loc}}(\mathbb{R}^d)$

(H2) *Normalization:* $\int_{\mathbb{R}^d} e^{-V}\, dx = 1$

(H3) *Spectral gap condition: there exists a positive constant* $\Lambda$ *such that*

$$\int_{\mathbb{R}^d} |u|^2\, e^{-V}\, dx \leq \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\, e^{-V}\, dx$$

*for any* $u \in H^1(e^{-V}\, dx)$ *such that* $\int_{\mathbb{R}^d} u\, e^{-V}\, dx = 0$

(H4) *Pointwise condition 1: there exists* $c_0 > 0$ *and* $\theta \in (0,1)$ *such that* $\Delta V \leq \frac{\theta}{2}\, |\nabla_x V(x)|^2 + c_0\ \forall\, x \in \mathbb{R}^d$

(H5) *Pointwise condition 2: there exists* $c_1 > 0$ *such that* $|\nabla_x^2 V(x)| \leq c_1\, (1 + |\nabla_x V(x)|)\ \forall\, x \in \mathbb{R}^d$

(H6) *Growth condition:* $\int_{\mathbb{R}^d} |\nabla_x V|^2\, e^{-V}\, dx < \infty$

# Maxwellian case: Main result

**Theorem 1.** *If $\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f$, for $\varepsilon > 0$, small enough, there exists an explicit, positive constant $\lambda = \lambda(\varepsilon)$ such that*

$$\|f(t) - F\| \leq (1+\varepsilon)\,\|f_0 - F\|\, e^{-\lambda t} \quad \forall\, t \geq 0$$

- The operator L has no regularization property: hypo-coercivity fundamentally differs from hypo-ellipticity
- Coercivity due to L is only on velocity variables

$$\frac{d}{dt}\|f(t) - F\|^2 = -\|(1-\Pi)f\|^2 = -\iint_{\mathbb{R}^d\times\mathbb{R}^d} |f - \rho_f\, M(v)|^2\, dv\, dx$$

- T and L do not commute: coercivity in $v$ is transferred to the $x$ variable. In the diffusion limit, $\rho$ solves a Fokker-Planck equation

$$\partial_t \rho = \Delta\rho + \nabla\cdot(\rho\,\nabla V) \quad t > 0\,, \quad x \in \mathbb{R}^d$$

the goal of the hypo-coercivity theory is to quantify the interaction of T and L and build a norm which controls $\|\cdot\|$ and decays exponentially

# The operators. A Lyapunov functional

On $L^2(d\mu)$, define

$$\mathsf{b}f := \Pi\,(v\,f)\,,\quad \mathsf{a}f := \mathsf{b}\,(\mathsf{T}\,f)\,,\quad \hat{\mathsf{a}}\,f := -\Pi\,(\nabla_x f)\,,\quad \mathsf{A} := (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1}\,\hat{\mathsf{a}}\cdot\mathsf{b}$$

$$\mathsf{b}\,f = \frac{F}{\rho_F}\int_{\mathbb{R}^d} v\,f\,dv = \frac{F}{\rho_F}\,j_f \quad \text{with} \quad j_f := \int_{\mathbb{R}^d} v\,f\,dv$$

$$\mathsf{a}\,f = \frac{F}{\rho_F}\left(\nabla_x\cdot\int_{\mathbb{R}^d} v\otimes v\,f\,dv + \rho_f\,\nabla_x V\right),\quad \hat{\mathsf{a}}\,f = -\frac{F}{\rho_F}\,\nabla_x\rho_f$$

$$\mathsf{A}\,\mathsf{T} = (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1}\,\hat{\mathsf{a}}\cdot\mathsf{a}$$

Define the Lyapunov functional (generalized entropy)

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}\,f, f\rangle$$

# ...a Lyapunov functional (continued): positivity, equivalence

$$\langle \hat{\mathsf{a}} \cdot \mathsf{a}\, \Pi\, f, f \rangle = \frac{1}{d} \int_{\mathbb{R}^d} \left| \nabla_x \left( \tfrac{\rho_f}{\rho_F} \right) \right|^2 m_F \, dx$$

with $m_F := \int_{\mathbb{R}^d} |v|^2 \, F(\cdot, v) \, dv$. Let $g := \mathsf{A}\, f$, $u = \rho_g / \rho_F$, $j_f := \int_{\mathbb{R}^d} v\, f \, dv$

$$(1 + \hat{\mathsf{a}} \cdot \mathsf{a}\, \Pi)\, g = \hat{\mathsf{a}} \cdot \mathsf{b} f \iff g - \frac{1}{d}\, \nabla_x \left( m_F \, \nabla_x u \right) \frac{F}{\rho_F} = -\frac{F}{\rho_F}\, \nabla_x \, j_f$$

$$\rho_F \, u - \frac{1}{d}\, \nabla_x \left( m_F \, \nabla_x u \right) = -\nabla_x \, j_f$$

$\|\mathsf{A}\, f\|^2 = \int_{\mathbb{R}^d} |u|^2 \, \rho_F \, dx$ and $\|\mathsf{T}\, \mathsf{A}\, f\|^2 = \frac{1}{d} \int_{\mathbb{R}^d} |\nabla_x u|^2 \, m_F \, dx$ are such that

$$2\, \|\mathsf{A}\, f\|^2 + \|\mathsf{T}\, \mathsf{A}\, f\|^2 \le \|(1 - \Pi)\, f\|^2$$

As a consequence

$$(1 - \varepsilon) \|f\|^2 \le 2\, H(f) \le (1 + \varepsilon) \|f\|^2$$

# . . . a Lyapunov functional (continued): decay term

$$H(f) := \frac{1}{2}\,\|f\|^2 + \varepsilon\,\langle \mathsf{A}\,f, f\rangle$$

The operator T is skew-symmetric on $L^2(d\mu)$. If $f$ is a solution, then

$$\frac{d}{dt} H(f - F) = D(f - F)$$

$$D(f) := \ldots$$
$$\ldots \underbrace{\langle f, \mathsf{L}\,f\rangle}_{\text{micro}} - \varepsilon\,\underbrace{\langle \mathsf{A}\,\mathsf{T}\,\Pi\,f, f\rangle}_{\text{macro}} - \varepsilon\,\langle \mathsf{A}\,\mathsf{T}\,(1-\Pi)\,f, f\rangle + \varepsilon\,\langle \mathsf{T}\,\mathsf{A}\,f, f\rangle + \varepsilon\,\langle \mathsf{L}\,f, (\mathsf{A}+\mathsf{A}^*)\,f\rangle$$

**Lemma 2.** *For some $\varepsilon > 0$ small enough, there exists an explicit constant $\lambda > 0$ such that*

$$D(f - F) + \lambda\,H(f - F) \le 0$$

# Preliminary computations (1/2)

- Maxwellian case: $\Pi f = \rho_f \, M(v)$ with $\rho_f := \int_{\mathbb{R}^d} f \, dv$
- Replace $f$ by $f - F$: $0 = \iint_{\mathbb{R}^d \times \mathbb{R}^d} f \, dx \, dv = \langle f, F \rangle$, $\int_{\mathbb{R}^d} (\Pi \, f - f) \, dv = 0$
- $L$ is a linear relaxation operator: $\mathsf{L} \, f = \Pi \, f - f$

$$\langle L f, f \rangle \le -\|(1 - \Pi) \, f\|^2$$

- The other terms: for any $c$,

$$
\begin{aligned}
-\varepsilon \, \langle \mathsf{A} \, \mathsf{T} \, (1 - \Pi) \, f, f \rangle &= -\varepsilon \, \langle \mathsf{A} \, \mathsf{T} \, (1 - \Pi) \, f, \Pi \, f \rangle \\
&\le \frac{c}{2} \, \|\mathsf{A} \, \mathsf{T} \, (1 - \Pi) \, f\|^2 + \frac{\varepsilon^2}{2 \, c} \, \|\Pi \, f\|^2
\end{aligned}
$$

$$\varepsilon \, \langle \mathsf{T} \, \mathsf{A} \, f, f \rangle = \varepsilon \, \langle \mathsf{T} \, \mathsf{A} \, f, \Pi \, f \rangle \le \frac{c}{2} \, \|\mathsf{T} \, A \, f\|^2 + \frac{\varepsilon^2}{2 \, c} \, \|\Pi \, f\|^2$$

$$\varepsilon \, \langle (\mathsf{A} + \mathsf{A}^*) \, \mathsf{L} \, f, f \rangle \le \varepsilon \, \|(1 - \Pi) \, f\|^2 + \varepsilon \, \|\mathsf{A} \, f\|^2$$

# Preliminary computations (1/2)

$\mathsf{A}\,\mathsf{T}\,(1-\Pi)=\Pi\,\mathsf{A}\,\mathsf{T}\,(1-\Pi)$ and $\mathsf{T}\,\mathsf{A}=\Pi\,\mathsf{T}\,\mathsf{A}$. With $m_F:=\int_{\mathbb{R}^d}|v|^2\,F\,dv$

$$\langle \hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi\,f,f\rangle=\frac{1}{d}\int_{\mathbb{R}^d}\left|\nabla_x\big(\tfrac{\rho_f}{\rho_F}\big)\right|^2 m_F\,dx$$

Recall that one ca compute $\mathsf{A}\,f:=(1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1}\,\hat{\mathsf{a}}\cdot\mathsf{b}\,f$ as follows: let

$$g:=\mathsf{A}\,f \qquad u:=\rho_g/\rho_F$$

By definition of A, $\hat{\mathsf{a}}\cdot\mathsf{b}f=(1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)\,g$ means

$$-\nabla_x\int_{\mathbb{R}^d}v\,f\,dv=:-\nabla_x\,j_f=\rho_F\,u-\frac{1}{d}\,\nabla_x\,(m_F\,\nabla_x u)$$

$\|\mathsf{A}\,f\|^2=\int_{\mathbb{R}^d}|u|^2\,\rho_F\,dx$ and $\|\mathsf{T}\,\mathsf{A}\,f\|^2=\frac{1}{d}\int_{\mathbb{R}^d}|\nabla_x u|^2\,m_F\,dx$ are such that

$$2\,\|\mathsf{A}\,f\|^2+\|\mathsf{T}\,\mathsf{A}\,f\|^2\le\|(1-\Pi)\,f\|^2$$

# Preliminary computations (2/2)

$$D(f) \leq \underbrace{-\left(1-\frac{c}{2}-2\,\varepsilon\right)\|(1-\Pi)\,f\|^2}_{\text{micro: }\leq 0} - \varepsilon \quad \underbrace{\langle \mathsf{A\,T}\,\Pi\,f, f\rangle}_{\text{macro: ``first estimate''}}$$

$$+\ \frac{c}{2}\,\underbrace{\|\mathsf{A\,T}\,(1-\Pi)\,f\|^2}_{\text{``second estimate...''}} + \frac{\varepsilon^2}{c}\,\|\Pi\,f\|^2$$

...$(\mathsf{A\,T}\,(1-\Pi))^* f = (\hat{\mathsf{a}}\cdot\mathsf{a}\,(1-\Pi))^* g$ with $g = (1+\hat{\mathsf{a}}\cdot\mathsf{a}\,\Pi)^{-1} f$ means

$$\rho_f = \rho_F\,u - \frac{1}{d}\,\nabla_x\,(m_F\,\nabla_x u)$$

where $u = \rho_g/\rho_F$. Let $q_F := \int_{\mathbb{R}^d} |v_1|^4\,F\,dv$, $u_{ij} := \partial^2 u/\partial x_i \partial x_j$

$$\|(\mathsf{A\,T}\,(1-\Pi))^* f\|^2 = \sum_{i,\,j=1}^{d} \int_{\mathbb{R}^d} \left[\left(\frac{2\,\delta_{ij}+1}{3}\,q_F - \frac{m_F^2\,\delta_{ij}}{d^2\,\rho_F}\right) u_{ii}\,u_{jj} + \frac{2(1-\delta_{ij})}{3}\,q_F\,u_{ij}^2\right] dx$$

# Maxwellian case

With $\rho_F = e^{-V} = \frac{1}{d}\, m_F = q_F$, $\mathsf{B} = \hat{\mathsf{a}} \cdot \mathsf{a}\, \Pi$, $g = (1+\mathsf{B})^{-1} f$ means

$$\rho_f = u\, e^{-V} - \nabla_x \left(e^{-V}\, \nabla_x u\right) \quad \text{if} \quad u = \frac{\rho_g}{\rho_F}$$

Spectral gap condition: there exists a positive constant $\Lambda$ such that

$$\int_{\mathbb{R}^d} |u|^2\, e^{-V} dx \le \Lambda \int_{\mathbb{R}^d} |\nabla_x u|^2\, e^{-V} dx$$

Since $\mathsf{A}\,\mathsf{T}\,\Pi = (1+\mathsf{B})^{-1}\,\mathsf{B}$, we get the “first estimate"

$$\underbrace{\langle \mathsf{A}\,\mathsf{T}\,\Pi\, f, f\rangle}_{\text{macro}} \ge \frac{\Lambda}{1+\Lambda}\, \|\Pi\, f\|^2$$

Notice that $\mathsf{B} = \hat{\mathsf{a}} \cdot \mathsf{a}\, \Pi = (\mathsf{T}\,\Pi)^*(\mathsf{T}\,\Pi)$ so that $\langle \mathsf{B}\, f, f\rangle = \|\mathsf{T}\,\Pi\, f\|^2$

# Second estimate (1/3)

We have to bound ($H^2$ estimate)

$$\|(\mathsf{A}\,\mathsf{T}\,(1-\Pi))^* f\|^2 \le 2 \sum_{i,\,j=1}^{d} \int_{\mathbb{R}^d} |u_{ij}|^2\, e^{-V} dx$$

Let $\|u\|_0^2 := \int_{\mathbb{R}^d} |u|^2\, e^{-V} dx$. Multiply $\rho_f = u\, e^{-V} - \nabla_x \left(e^{-V}\, \nabla_x u\right)$ by $u\, e^{-V}$ to get

$$\|u\|_0^2 + \|\nabla_x u\|_0^2 \le \|\Pi\, f\|^2$$

By expanding the square in $|\nabla_x (u\, e^{-V/2})|^2$, with $\kappa = (1-\theta)/(2\,(2+\Lambda\, c_0))$, we obtain an improved Poincaré inequality

$$\kappa\, \|W\ u\|_0^2 \le \|\nabla_x u\|_0^2$$

for any $u \in H^1(e^{-V} dx)$ such that $\int_{\mathbb{R}^d} u\, e^{-V} dx = 0$

Here: $W := |\nabla_x V|$

# Second estimate (2/3)

Multiply $\rho_f = u\,e^{-V} - \nabla_x \left(e^{-V}\,\nabla_x u\right)$ by $W^2\,u$ with $W := |\nabla_x V|$ and integrate by parts

$$\|W\,u\|_0^2 + \|W\,\nabla_x u\|_0^2 - 2\,c_1\left(\|\nabla_x u\|_0 + \|W\,\nabla_x u\|_0\right)\cdot\|W\,u\|_0 \le \frac{\kappa}{8}\,\|W^2 u\|_0^2 + \frac{2}{\kappa}\,\|\Pi\,f\|^2$$

Improved Poincaré inequality applied to $W\,u - \int_{\mathbb{R}^d} W\,u\,e^{-V}\,dx$ gives

$$\kappa\,\|W^2 u\|_0^2 \le \int_{\mathbb{R}^d} |\nabla_x (W\,u)|^2\,e^{-V}\,dx + 2\,\kappa \int_{\mathbb{R}^d} W\,u\,e^{-V}\,dx \int_{\mathbb{R}^d} W^3\,u\,e^{-V}\,dx$$

(...) $\kappa\,\|W^2\,u\|_0^2 \le 4\,\|W\,\nabla_x u\|_0^2 + 8\,c_1^2\,(\|u\|_0^2 + \|W\,u\|_0^2) + 4\,\kappa\,\|W\|_0^4\,\|u\|_0^2$

$$\|W\,\nabla_x u\|_0 \le c_5\,\|\Pi\,f\|$$

# Second estimate (3/3)

Multiply $\rho_f = u\, e^{-V} - \nabla_x \left(e^{-V}\, \nabla_x u\right)$ by $\Delta u$ and integrating by parts, we get

$$\|\nabla_x^2 u\|_0^2 - \left(\|W\, \nabla_x u\|_0 + \|\Pi\, f\|\right) \|\nabla_x^2 u\|_0 \leq \|W\, u\|_0\, \|\nabla_x u\|_0$$

Altogether (...)

$$\|(\mathsf{A}\,\mathsf{T}\,(1-\Pi))\, f\|^2 \leq c_6\, \|\,(1-\Pi)\, f\|^2.$$

Summarizing, with $\lambda_1 = 1 - \frac{c}{2}\,(1+c_6) - 2\,\varepsilon$ and $\lambda_2 = \frac{\Lambda\,\varepsilon}{1+\Lambda} - \frac{\varepsilon^2}{c}$

$$D(f) \leq -\lambda_1\, \|(1-\Pi)\, f\|^2 - \lambda_2\, \|\Pi\, f\|^2$$

# The Vlasov-Fokker-Planck equation

$$\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f \quad \text{with} \quad \mathsf{L}\, f = \Delta_v f + \nabla_v (v\, f)$$

Under the same assumptions as in the linear BGK model... same result !
A list of changes

- $\underbrace{\langle f, \mathsf{L}\, f\rangle}_{\text{micro}} = -\|\nabla_v f\|^2 \le -\|(1-\Pi)\, f\|^2$ by the Poincaré inequality
- Since $\Pi\, f = \rho_f\, M(v)$, where $M(v)$ is the gaussian function, and $F(x,v) = M(v)\, e^{-V(x)}$

$$\langle \mathsf{A}\, f, \mathsf{L}\, f\rangle = \iint \rho_{\mathsf{A}\, f}\, M(v)\, (\mathsf{L}\, f)\, \frac{dx\, dv}{F} = \iint \rho_{\mathsf{A}\, f}\, (\mathsf{L}\, f)\, e^V\, dx\, dv = 0$$

- $\mathsf{A}\, f = u\, F$ means $\rho_F\, u - \frac{1}{d}\, \nabla_x\, (m_F\, \nabla_x u) = -\nabla_x\, j_f$, $j_f := \int_{\mathbb{R}^d} v\, f\, dv$.
Hence $j_{\mathsf{L}\, f} = -j_f$ gives

$$\langle \mathsf{A}\, \mathsf{L}\, f, f\rangle = -\langle \mathsf{A}\, f, f\rangle$$

# Motivation: nonlinear diffusion as a diffusion limit

$$\varepsilon^2\,\partial_t f + \varepsilon\Big[v\cdot\nabla_x f - \nabla_x V(x)\cdot\nabla_v f\Big] = Q(f)$$

$$\text{with}\quad Q(f) = \gamma\left(\frac{1}{2}\,|v|^2 - \overline{\mu}(\rho_f)\right) - f$$

Local mass conservation determines $\overline{\mu}(\rho)$

Theorem [Dolbeault, Markowich, Oelz, CS, 2007] $\rho_f$ converges as $\varepsilon \to 0$ to a solution of

$$\partial_t\rho = \nabla_x\cdot\left(\nabla_x\nu(\rho) + \rho\nabla_x V\right)$$

with $\nu'(\rho) = \rho\,\overline{\mu}'(\rho)$

$\gamma(s) = (-s)_+^k$, $\nu(\rho) = \rho^m$, $0 < m = m(k) < 5/3$ ($\mathbb{R}^3$ case)

# Linearized fast diffusion case

Consider a solution of $\partial_t f + \mathsf{T}\, f = \mathsf{L}\, f$ where $\mathsf{L}\, f = \Pi\, f - f$, $\Pi\, f := \frac{\rho}{\rho_F}$

$$F(x,v) := \omega \left( \frac{1}{2}\, |v|^2 + V(x) \right)^{-(k+1)} \,, \quad V(x) = \left(1 + |x|^2\right)^\beta$$

where $\omega$ is a normalization constant chosen such that $\iint_{\mathbb{R}^d \times \mathbb{R}^d} F \, dx \, dv = 1$ and $\rho_F = \omega_0 \, V^{d/2-k-1}$ for some $\omega_0 > 0$

**Theorem 3.** *Let $d \geq 1$, $k > d/2+1$. There exists a constant $\beta_0 > 1$ such that, for any $\beta \in (\min\{1, (d-4)/(2k-d-2)\}, \beta_0)$, there are two positive, explicit constants $C$ and $\lambda$ for which the solution satisfies:*

$$\forall \, t \geq 0 \,, \quad \|f(t) - F\|^2 \leq C \, \|f_0 - F\|^2 \, e^{-\lambda t}.$$

Computations are the same as in the Maxwellian case except for the “first estimate" and the “second estimate"

# Linearized fast diffusion case: first estimate

For $p = 0, 1, 2$, let $w_p^2 := \omega_0 \, V^{p-q}$, where $q = k + 1 - d/2$, $w_0^2 := \rho_F$

$$\|u\|_i^2 = \int_{\mathbb{R}^d} |u|^2 \, w_i^2 \, dx$$

Now $g = (1 + \hat{\mathsf{a}} \cdot \mathsf{a} \, \Pi)^{-1} f$ means

$$\rho_f = w_0^2 \, u - \frac{2}{2k-d} \, \nabla_x \left( w_1^2 \, \nabla_x u \right)$$

Hardy-Poincaré inequality [Blanchet, Bonforte, J.D., Grillo, Vázquez]

$$\|u\|_0^2 \leq \Lambda \, \|\nabla_x u\|_1^2$$

under the condition $\int_{\mathbb{R}^d} u \, w_0^2 \, dx = 0$ if $\beta \geq 1$. As a consequence

$$\langle \mathsf{A} \, \mathsf{T} \, \Pi \, f, f \rangle \geq \frac{\Lambda}{1 + \Lambda} \, \|\Pi \, f\|^2$$

# Linearized fast diffusion case: second estimate

Observe that $\rho_f = w_0^2\, u - \frac{2}{2k-d}\, \nabla_x \left( w_1^2\, \nabla_x u \right)$ multiplied by $u$ gives

$$\|u\|_0^2 + (q-1)^{-1}\, \|\nabla_x u\|_1^2 \leq \|\Pi\, f\|^2$$

By the Hardy-Poincaré inequality (condition $\beta < \beta_0$)

$$\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\, |u|^2\, dx - \frac{\left( \int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\, u\, dx \right)^2}{\int_{\mathbb{R}^d} V^{\alpha+1-q-\frac{1}{\beta}}\, dx} \leq \frac{1}{4\,(\beta_0-1)^2} \int_{\mathbb{R}^d} V^{\alpha+1-q}\, |\nabla_x u|^2\, dx$$

By multiplying $\rho_f = w_0^2\, u - \frac{2}{2k-d}\, \nabla_x \left( w_1^2\, \nabla_x u \right)$ by $V^\alpha\, u$ with $\alpha := 1 - 1/\beta$ or by $V\, \Delta u$ we find

$$\|\nabla_x^2 u\|_2^2 \leq C\, \|\Pi\, f\|^2$$

# Diffusion limits and hypocoercivity

The strategy of the method is to introduce at kinetic level the macroscopic quantities that arise by taking the diffusion limit

| kinetic equation | diffusion equation | functional inequality (macroscopic) |
| --- | --- | --- |
| Vlasov + BGK / Fokker-Planck | Fokker-Planck | Poincaré (gaussian weight) |
| linearized Vlasov-BGK | linearized porous media | Hardy-Poincaré |
| nonlinear Vlasov-BGK | porous media | Gagliardo-Nirenberg |

- from kinetic to diffusive scales: parabolic scaling and diffusion limit
- heuristics: convergence of the macroscopic part at kinetic level is governed by the functional inequality at macroscopic level
- interplay between diffusion limits and hypocoercivity is still work in progress as well as the nonlinear case

# Concluding remarks

- hypo-coercivity vs. hypoellipticity
- diffusion limits, a motivation for the “fast diffusion case”
- other collision kernels: scattering operators
- other functional spaces
- nonlinear kinetic models
- hydrodynamical models

Reference

J. Dolbeault, C. Mouhot, C. Schmeiser, Hypocoercivity for kinetic equations with linear relaxation terms, CRAS 347 (2009), pp. 511–516