Interfaces in random environment

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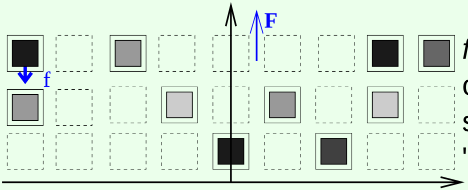
The Random Obstacle Model



$f(x,u,\omega) = c \geq 0$ on $\mathbb{R}^{n+1} \setminus Q_i$,
on "obstacles" Q_i : $f(x,u,\omega) \leq 0$
strength random (exponential r.v.)
"driving" force $F \geq 0$

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Quenched Edwards-Wilkinson Model (QEW) **Questions:**

Pinning/De-pinning: Is it true that

- $0 < F < F_*$: nonnegative stationary solution exists
- $F > F_*$: **no** nonnegative stationary solution exists?

"effective velocity" on scale $\tau = \epsilon^{-1}t$, $y = \epsilon^{-1}x$.

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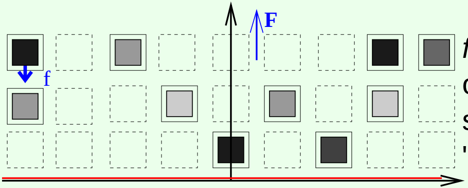

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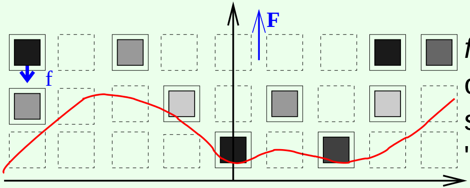

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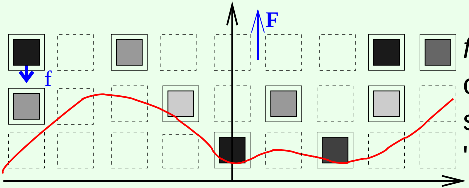

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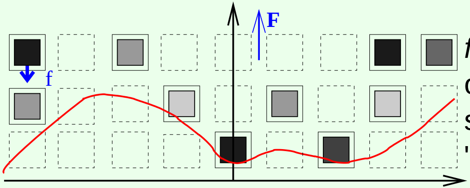

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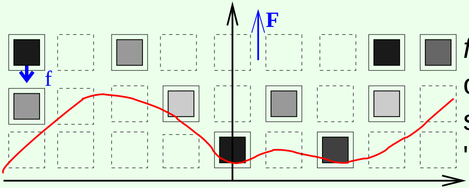

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Forced Mean Curvature Flow as Gradient Flow

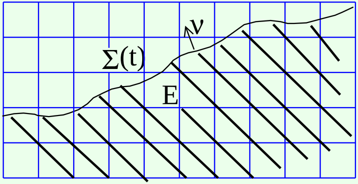
Zoom in on scale of heterogeneities: ($\Sigma = \partial E$.)

Liapunov functional (formal):

$$\mathrm{Area}(\Sigma) + \underbrace{\int_{\mathbb{R}^{n+1}\cap E} f(X)dX}_{\text{“volume''}} \qquad [V(Y)]$$

Gradient flow:

$$V_X = \kappa_X + f(X),\ X \in \Sigma(t) \subset \mathbb{R}^{n+1} \qquad [\dot{Y} = -V'(Y)]$$



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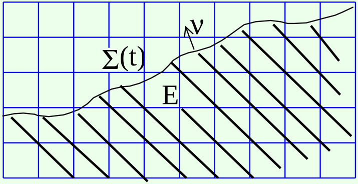

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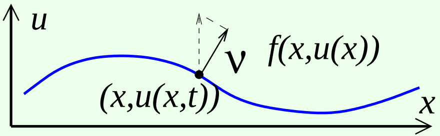
Pinning and De-Pinning: Semilinear Approximation

Forced mean curvature flow: $V_x = \kappa_x + f(x) + F$



If surface is graph $(x, u(x,t))$ then $u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ solves

$$\partial_t u = \sqrt{1 + |\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1 + |\nabla u|^2}}\right) + \sqrt{1 + |\nabla u|^2}\,f(x,u).$$

gradient small, then (heuristic) approximation: **semilinear PDE**

$$u_t = \Delta u + f(x,u) + F, \qquad F \geq 0 : \text{external driving force.}$$

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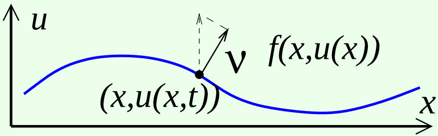

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Some remarks on periodic forcing

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Pinning and De-Pinning: Semilinear Case

$$V_x = \kappa_x + f(x) + F$$

If surface is graph $(x, u(x,t))$ and **gradient small**, then approximation: $u(x,t) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ periodic in x, solves **semilinear PDE**

$$u_t = \Delta u + f(x,u) + F, \qquad F \geq 0 : \text{external driving force.}$$

$f(\cdot,\cdot)$ is 1-periodic in *both* variables, bounded, smooth, mean zero, “generic,” and f **changes its sign!**

Pinning/De-Pinning

N. Dirr, N.K. Yip, Interfaces and Free Boundaries 8 (2006), 79-109

$$u_t = \Delta u + f(x,u) + F$$

Theorem

$f(\cdot,\cdot)$ is 1-periodic in both *variables, bounded, smooth, mean zero, generic.*

- *There ex. $F_* > 0$ such that for any $0 \le F \le F_*$ there exists a* **periodic stationary solution** *(pinning state) u_F :* $\Delta u_F + f(\cdot, u_F) + F = 0.$
- *For $F > F_*$, there exist* **pulsating wave solutions** *$U_F(x,t)$ with velocity V_F :* $U_F(\cdot, t + 1/V_F) = U_F(\cdot, t) + 1.$

$$V_F = A_f[(F - F_*)_+]^{\frac{1}{2}} + o(|F - F_*|^{\frac{1}{2}})$$

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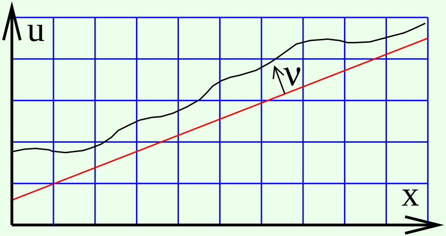
Forced Mean Curvature: Graph

Graph moving over **arbitrary plane** by mean curvature in unbounded domain:

$$\partial_t u = \sqrt{1+|\nabla u|^2}\,\mathrm{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) + \sqrt{1+|\nabla u|^2} f(x,u).$$



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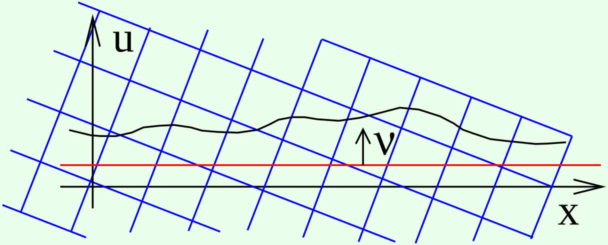
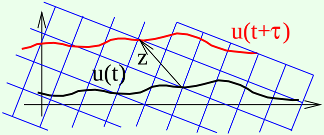


Pulsating Wave



- Bounds in frame moving with velocity $c(\nu)$, continuous in ν
- There ex. c_ν s.t. for any $\mathbf{z} = (x', r) \in \mathbb{R}^{n+1}$ with $\mathcal{O}_\nu(x', r) \in \mathbb{Z}^{n+1}$

$$u(t, x) = u(x - x', t + \tau) - r, \quad r = c_\nu \tau.$$

N. Dirr, **G. Karali, N.K. Yip**, Pulsating Wave for Mean Curvature Flow in Inhomogeneous Medium, EJAM 19 (2008) , 661-699.

Assumption: Forcing small in C^1

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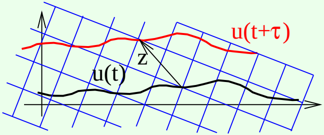

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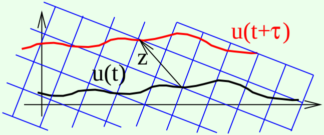

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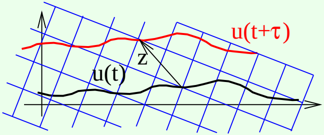

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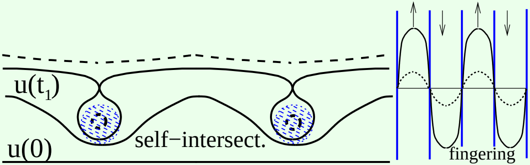

Related Results: MCF in heterogeneous media

- L. Caffarelli, R. De la Llave
- B. Craciun, K. Bhattacharya
- P.L. Lions, P.E. Souganidis
- P. Cardaliaguet, P.L. Lions, P.E. Souganidis
- K. Bhattacharya, P. Dondl

Random Environment

Forcing $f(x,u,\omega)$ (or $(f(X,\omega))$ random with short correlations.

- **Fluctuations**

$$\int_V (f(X) - \mathbb{E}f(X))dX \sim \sqrt{V}N(0,\sigma)$$

$\Rightarrow$ Interfaces not "flat"

- **Rare events** (e.g. large obstacles, "tail of distribution") matter, not just average

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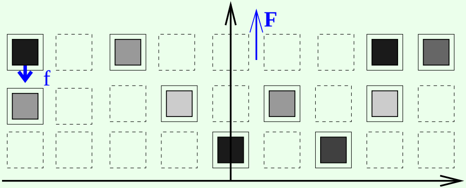
Random Obstacle Model: Precise Setting



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$(\ell_{i,j}(\omega))_{(i,j)\in\mathbb{Z}^n\times(\mathbb{Z}+\frac{1}{2})}$ are a family of independent identically distributed exponential random variables.

There exists $\lambda_0 > 0$ such that $\mathbb{P}\{\ell(i,j)(\omega) > r\} = \lambda_0 e^{-\lambda_0 r}$ for $r \geq 0$

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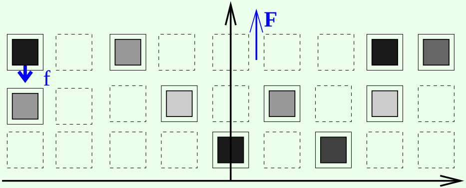

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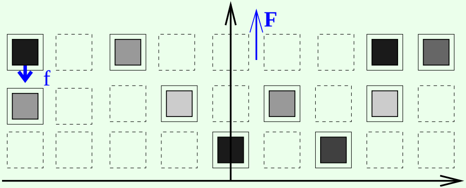

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$(\ell_{i,j}(\omega))_{(i,j)\in\mathbb{Z}^n\times(\mathbb{Z}+\frac{1}{2})}$ are a family of independent identically distributed exponential random variables. There exists $\lambda_0 > 0$ such that $\mathbb{P}\{\ell(i,j)(\omega) > r\} = \lambda_0 e^{-\lambda_0 r}$ for $r \geq 0$

Nonnegative Stationary Solutions for Random Obstacle Model

$$\left.\begin{aligned} 0 &= \Delta u(x,\omega) + f(x,u(x,\omega),\omega) + F \quad \text{on } \mathbb{R}^n \\ u(x) &\geq 0 \end{aligned}\right. \qquad (*)$$

Theorem (N.D., J. Coville, S. Luckhaus)

Let $n = 1$ and u solve $(*)$ on $[-N, N]$ with $u(-N) = u(N) = 0$. Then there exist $F_0 > 0$, C and K such that for $F > F_0$

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Corollary ($n = 1$)

There is almost surely no global nonnegative stationary solution of $(*)$.

Theorem (N.D., P. Dondl, M. Scheutzow)

Let $n = 1, 2$. There ex. $0 < F_1$ such that for $0 < F < F_1$ $(*)$ has almost surely a solution with $\mathbb{E}[u(x,\omega)] = c < \infty$ for all $x \in \mathbb{R}^n$.

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Barrier for/limit of

$$\begin{aligned} \partial_t u(x,t,\omega) &= \Delta u(x,t,\omega) + f(x, u(x,t,\omega), \omega) + F \quad \text{on } \mathbb{R}^n \\ u(x,0) &= 0 \end{aligned}$$

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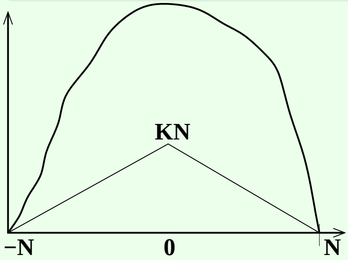

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- Coarse-graining: As $u_{xx} = -F$ between obstacles, path determined by values on $\partial(\mathbb{R} \setminus (\mathbb{Z} + [-\delta, \delta])) \Rightarrow \bar{v}^\delta : \mathbb{Z} \to \delta\mathbb{Z}$
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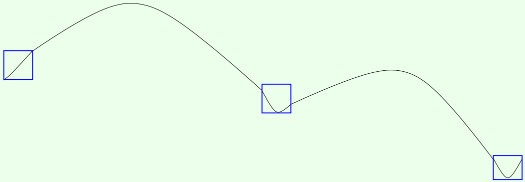

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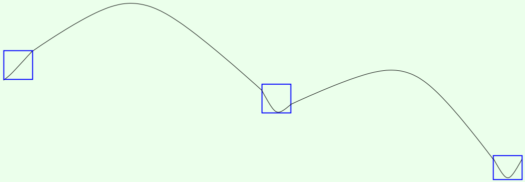

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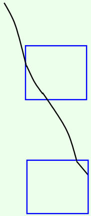

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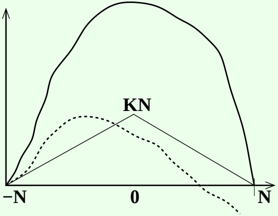

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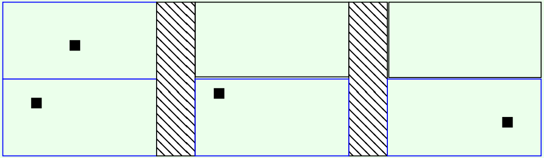

- Suppose: There exists Lipschitz graph $w \geq 1$ which is contained in the open set.
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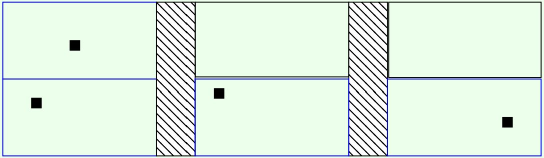

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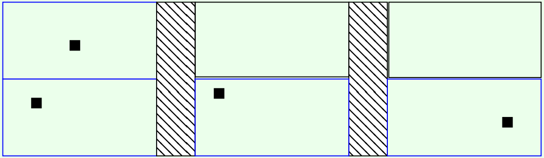

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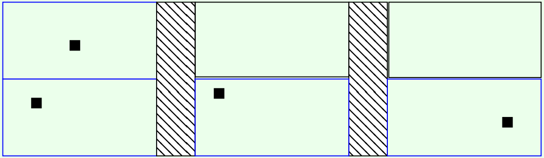

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Lipschitz Graph Percolation

Lemma

There exists a $p_0 > 0$ such that if closed sites are i.i.d. with $\mathbb{P}(z \text{ closed}) < p_0$, then a nonnegative discrete 1-Lipschitz graph $w : \mathbb{Z}^n \to \mathbb{N}$ exists with $(z, w(z))$ closed for all $z \in \mathbb{Z}^n$.

Branching process on z (height of cone): Offspring distribution $\xi(z)$, new height $z + \xi(z)$.

$$\mathbb{E}\Big(\sum_{z \in \mathbb{Z}} e^{\mu z} \xi(z)\Big) < 1 \quad \text{for some } \mu > 0 \ \Rightarrow \ \text{dies out}$$

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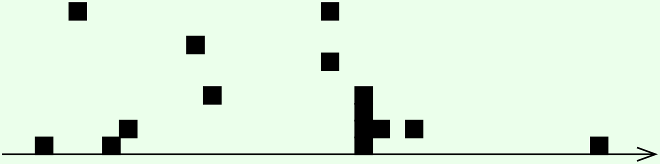

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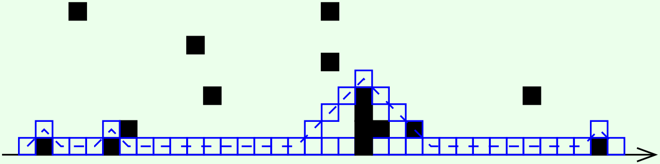

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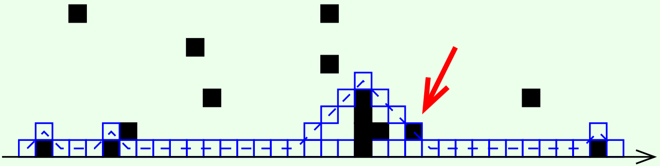

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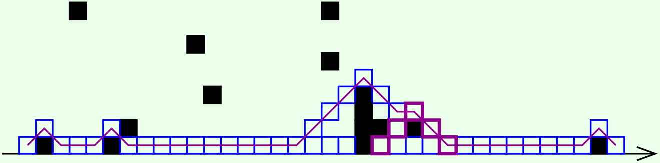

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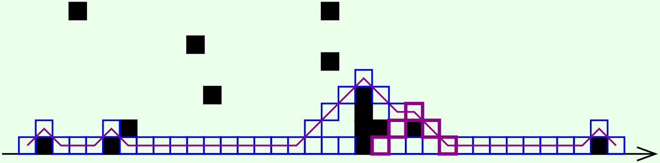

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