C*-dynamical systems from number theory

Part 5: C*-algebras and dynamical systems from algebraic number fields

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June 2010

Algebraic number fields

- An algebraic number field K is a finite algebraic extension of $\mathbb{Q}$.
- $\mathcal{O}$ = ring of algebraic integers.
  = zeros of monic polynomials with integer coefficients.
- $\mathcal{O}^*$ = units of $\mathcal{O}$ = integers whose inverses (in K) are also integers.
- J_K = fractional ideals in K, i.e. nonzero finitely-generated $\mathcal{O}$-submodules $\mathfrak{a}$ of K; J_K is an abelian group under $\mathfrak{a}\mathfrak{b} := \{\sum a_i b_i : a_i \in \mathfrak{a}, b_i \in \mathfrak{b}\}$.
- Unique factorization fails in general for algebraic numbers, but holds for ideals: J_K is free abelian generated by prime ideals.
- Dedekind zeta function of K:

$$\zeta_K(\beta) := \sum_{\mathfrak{a}\subset\mathcal{O}} N_{\mathfrak{a}}^{-\beta} = \prod_{\mathfrak{p}\in\mathcal{P}_K} (1 - N_{\mathfrak{p}}^{-\beta})^{-1},$$

  where $N_{\mathfrak{a}} := |\mathcal{O}/\mathfrak{a}|$ is the absolute norm of the ideal $\mathfrak{a} \subset \mathcal{O}$, and the Euler product is over the prime ideals $\mathcal{P}_K$; it converges for $\Re\beta > 1$ and diverges for $\Re\beta \leq 1$

Abstract class field theory (à la Artin and Tate)

Aim: to describe $\mathcal{G}(K^{ab}/K)$ in terms of the arithmetic properties of K.
Given a number field K and an absolute value v on K, let K_v be the corresponding completion. For nonarchimedean v, let $\mathcal{O}_v \subset K_v$ be the completion of the algebraic integers $\mathcal{O}$ in K, and let K_v^* and $\mathcal{O}_v^*$ be the corresponding groups of invertible elements in those rings. Define the group of ideles to be the restricted product

$$\mathbb{A}_K^* := \prod_{v \text{ infinite}} K_v^* \times \prod_{v \text{ finite}} (K_v^*; \mathcal{O}_v^*)$$

with no restriction at the infinite (or archimedean) places.
There is a group homomorphism (known as the Artin map) of the idele class group $\mathbb{A}_K^*/K^*$ onto $\mathcal{G}(K^{ab}/K)$ with kernel the connected component of the identity.
Example: for $K = \mathbb{Q}$ we have $\mathbb{A}_\mathbb{Q}^* = \mathbb{R}^* \times \prod_p (\mathbb{Q}_p^*; \mathbb{Z}_p^*)$ because the finite absolute values are the p-adic ones $|\cdot|_p$, giving $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, and the only infinite one gives $\mathbb{Q} \hookrightarrow \mathbb{R}$.

Hilbert's 12th problem

For $K = \mathbb{Q}$, the Kronecker-Weber Theorem states that

$$\mathbb{Q}^{\mathrm{ab}} = \mathbb{Q}^{\mathrm{cycl}} := \mathbb{Q}[\{exp(2i\pi r) : r \in \mathbb{Q}/\mathbb{Z}\}]$$

that is, one only has to adjoin the roots of unity to $\mathbb{Q}$ in order to obtain $\mathbb{Q}^{\mathrm{ab}}$. The corresponding Galois group $\mathcal{G}^{\mathrm{ab}}$ is isomorphic to $\operatorname{Aut} \mathbb{Q}/\mathbb{Z} \cong \hat{\mathbb{Z}}^*$.

Problem (H12)

For any algebraic number field K, obtain a generalization of the Kronecker-Weber theorem giving the concrete embeddings

$$K^{ab} \hookrightarrow \mathbb{C}$$

in terms of specific values of trascendental functions that generate K^{ab}.

For $K = \mathbb{Q}[\sqrt{-d}]$ $(d > 0)$ an imaginary quadratic field, the theory of complex multiplication gives the values of modular functions one has to adjoin to K in order to obtain K^{ab}.

Problem A: (Connes-Marcolli-Ramachandran '05)*

Given an algebraic number field K, construct a C*-dynamical system $(\mathcal{A}, \sigma)$ such that

(i) the partition function is the Dedekind zeta function of K;

(ii) the Galois group $\mathcal{G}(K^{ab}/K)$ acts as symmetries of the system;

(iii) for each inverse temperature $0 < \beta \leq 1$ there is a unique KMS_β-state;

(iv) for each $1 < \beta \leq \infty$ the action of $\mathcal{G}(K^{ab}/K)$ on the extremal KMS_β-states is free and transitive;

(v) there is a K-subalgebra $\mathcal{A}_0$ of $\mathcal{A}$ such that the extremal KMS_∞-states are *fabulous* in the sense that their values on elements of $\mathcal{A}_0$ are algebraic numbers that generate the maximal abelian extension K^{ab} of K; and

(vi) the Galois action of $\mathcal{G}(K^{ab}/K)$ on these values corresponds to the symmetry action induced on the extremal KMS_∞-states.

*(slightly reformulated here)

Ingredients

- Partition function $\operatorname{Tr}\exp(-\beta H)$ with $H =$ Hamiltonian (of a state).
- Dedekind zeta function $\zeta_K(\beta) := \sum_{\mathfrak{a} \in J_K^+} |\mathcal{O}/\mathfrak{a}|^{-\beta}$,

  with $\mathcal{O} = \{$algebraic integers in $K\}$ and $J_K^+ = \{$nonzero ideals in $\mathcal{O}\}$.
- $K^{ab} =$ maximal abelian extension of K (a subfield of $\mathbb{C}$ of infinite degree; the object of class field theory).
- Symmetries = a group of automorphisms of $\mathcal{A}$ that commute with σ.
- KMS (twisted trace) condition at inverse temperature β.
- KMS_∞ states are ground states obtained as limits of KMS_β states as $\beta \to \infty$.
- The extremal KMS_∞ states φ should be equivariant:

$$\varphi(\alpha_{\chi^{-1}}(a)) = \chi(\varphi(a)) \quad a \in \mathcal{A}_0, \quad \chi \in \mathcal{G}(K^{ab}/K)$$

  where the l.h.s. is from the symmetry action on $\mathcal{A}_0$ while the r.h.s. is from the Galois action on K^{ab}.

$K = \mathbb{Q}$ (Bost-Connes, Selecta Math '95)

- C*-algebra = Hecke C*-algebra $\mathcal{C}_{\mathbb{Q}} \cong C(\hat{\mathbb{Z}}) \rtimes \mathbb{N}^{\times}$;
- Dynamics σ = naturally arising from $\mathbb{Q} \rtimes \mathbb{Q}^*_+ \to \mathbb{Q}^*_+ \hookrightarrow \mathbb{R}^*_+$;
- Symmetry group = $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z}) \cong \hat{\mathbb{Z}}^*$ acting on $C^*(\mathbb{Q}/\mathbb{Z}) \cong C(\hat{\mathbb{Z}})$;
- Extremal KMS_∞ states $\leftrightarrow$ unit point masses on $\hat{\mathbb{Z}}^*$;
- Arithmetic $\mathbb{Q}$-subalgebra $\mathcal{A}_0 = \mathbb{Q}[\mathbb{Q}/\mathbb{Z}] \rtimes \mathbb{N}^{\times} \cong$, (viewed as a subalgebra of $\mathcal{C}_{\mathbb{Q}}$);
- $\varphi(\mathcal{A}_0) = \mathbb{Q}$[roots of unity] for every extremal KMS_∞ state φ.

The canonical action of $\mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ on $C^*(\mathbb{Q}/\mathbb{Z})$ (or of $\hat{\mathbb{Z}}^*$ on $C(\hat{\mathbb{Z}})$) as symmetries of the dynamical system corresponds to the action of $\mathcal{G}(\mathbb{Q}^{cycl}/\mathbb{Q}) \cong \mathrm{Aut}(\mathbb{Q}/\mathbb{Z})$ on the values of KMS_∞ states on $\mathcal{A}_0$.

Since $\mathbb{Q}^{cycl} = \mathbb{Q}^{ab}$ by the Kronecker-Weber theorem this raises the tantalizing possibility of a connection between quantum dynamical systems and explicit class field theory.

A Hecke algebra construction for algebraic number fields

Theorem (L-van Frankenhuijsen '06)

1. *For any number field K, the Hecke C*-algebra $\mathcal{C}_K := C^*(P_K, P_{\mathcal{O}})$ is isomorphic to a semigroup crossed product $C(\hat{\mathcal{O}})^{\overline{\mathcal{O}^*}} \rtimes (\mathcal{O}^\times/\mathcal{O}^*)$, on which the absolute norm $N : \mathcal{O}^\times \to (1, \infty)$ induces a dynamics σ analogous to the Bost-Connes system for $\mathcal{C}_{\mathbb{Q}} = C^*(P_{\mathbb{Q}}^+, P_{\mathbb{Z}}^+)$.*
2. *For totally imaginary fields K of class number $h_K = 1$ the KMS state structure, the symmetries and the partition function of the resulting system $(\mathcal{C}_K, \sigma)$ are as desired (for these fields, the system is* isomorphic *to the one in [L-L-N] and [H-P], and is related to the ones in [Harari-Leichtnam Selecta Math '97]).*
3. *the* 'obvious' *choice of $\mathcal{A}_0$ = Hecke *-algebra over K does not produce enough values $\varphi(a_0)$ with φ extremal KMS$_\infty$ and $a_o \in \mathcal{A}_0$ to generate K^{ab} (only the cyclotomic extension is generated); moreover the symmetry action of $\mathcal{G}(K^{ab}/K) \cong \hat{\mathcal{O}}^*/\overline{\mathcal{O}^*}$ does not match the Galois action on values of KMS$_\infty$ states.*

$K = \mathbb{Q}[\sqrt{-d}]$ (Connes-Marcolli-Ramachandran, Selecta Math '05)

Solution: $\mathcal{A} :=$ groupoid C*-algebra of the commensurability equivalence relation of 1-dimensional K-lattices modulo $\mathbb{C}^*$, with the natural dynamics from absolute norm on K.
To make KMS computations easier, it is convenient to realize $\mathcal{A}$ as the semigroup crossed product $C(Y) \rtimes J_K^+$, with $J_K^+ =$ the semigroup of integral ideals acting by multiplication on

$$Y \cong \mathcal{G}(K^{ab}/K) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}.$$

Arithmetic subalgebra $\mathcal{A}_0$: 1-dimensional K-lattices embed in 2-dimensional $\mathbb{Q}$-lattices of the GL_2 system of Connes-Marcolli. This, together with results about modular functions on elliptic curves is used to extract $\mathcal{A}_0$ from the arithmetic algebra of the GL_2-system.

No similar results are known outside $\mathbb{Q}$ and imaginary quadratic fields.

A system for arbitrary K: the space X

We construct a direct generalization for general number fields of the Connes Marcolli, Ramachandran system for $\mathbb{Q}[\sqrt{-d}]$, which is also isomorphic to a particular case of a very general construction for Shimura varieties from [Ha and Paugam, IMRP 05].

Let $\mathbb{A}_{K,f} = \prod_{v<\infty}(K_v; \mathcal{O}_v)$ be the finite adeles over K. A finite idele $j \in \mathbb{A}_{K,f}{}^*$ acts on

$$\mathcal{G}(K^{ab}/K) \times \mathbb{A}_{K,f},$$

via the Artin map $s\colon \mathbb{A}_K{}^* \to \mathcal{G}(K^{ab}/K)$ on the first component and via multiplication on the second one:

$$j(\gamma, m) = (\gamma s(j)^{-1}, jm)$$

We *balance* this product over the compact group of finite integral ideles $\hat{\mathcal{O}}^* := \prod_{v<\infty} \mathcal{O}_v^*$ (i.e. we take the quotient modulo the sub-action of $\hat{\mathcal{O}}^*$):

$$X := \mathcal{G} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}.$$

A system for arbitrary K: C*-algebra and dynamics

The action of the finite ideles drops to an action of the fractional ideals $J_K \cong \mathbb{A}_{K,f}{}^*/\hat{\mathcal{O}}^*$ on the balanced product

$$X := \mathcal{G} \times_{\hat{\mathcal{O}}^*} \mathbb{A}_{K,f}.$$

Finally, we restrict the 2nd component to be a finite **integral** adele:

$$Y := \mathcal{G} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$$

and form the restricted groupoid

$$J_K \boxtimes Y = \{(g,x) : g \in J_K,\ x \in Y,\ gx \in Y\}.$$

This restricted groupoid has a **C*-algebra** $\mathcal{A}$: $C_r^*(J_K \boxtimes Y)$, which turns out to be isomorphic to the semigroup crossed product $C(Y) \rtimes J_K^+$ of $C(Y)$ by an endomorphic action of the integral ideals J_K^+.

The usual norm $N\colon J_K \to \mathbb{Q}_+^* \hookrightarrow \mathbb{R}_+^*$ gives rise to a **dynamics** σ determined by $\sigma_t(Fu_g) = N_g^{it}\, Fu_g$, where $F \in C(Y)$ and $g \in J_K^+$.

Theorem (L-Larsen-Neshveyev, JNT 08)

For any number field K, the system $(\mathcal{A}, \sigma)$ with
$\mathcal{A} = C^*_r(J_K \boxtimes Y) \cong C(Y) \rtimes J_K^+$, $Y := \mathcal{G} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}$, *and* $\sigma_t(Fu_g) = N_g^{it} Fu_g$
for $F \in C(Y)$, $g \in J_K^+$, *and* $t \in \mathbb{R}$, *satisfies:*

1. *the partition function is* $\zeta_K(\beta) = \sum_{\mathfrak{a} \in J_K^+} N(\mathfrak{a})^{-\beta}$, *the Dedekind zeta-function;*
2. $\mathcal{G} := \mathcal{G}(K^{ab}/K)$ *acts as symmetries of* $(\mathcal{A}, \sigma)$ *via*
$$\alpha_\chi(F)(\gamma, m) = F(\chi^{-1}\gamma, m) \qquad \chi \in \mathcal{G};$$
3. *there is a unique* KMS_β*-state for each* $0 < \beta \leq 1$, *and none for* $\beta < 0$;
4. *for each* $1 < \beta \leq \infty$ *the extremal* KMS_β*-states are indexed by* $w \in Y_0 := \mathcal{G} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}^* \cong \mathcal{G}$, *and are given by the Radon measures*
$$\varphi_{\beta,w}(F) = \frac{1}{\zeta_K(\beta)} \sum_{\mathfrak{a} \in J_K^+} N(\mathfrak{a})^{-\beta} F(\mathfrak{a}w) \quad \text{for} \quad F \in C(Y),$$
where $\varphi_{\infty,w}(F) = F(w)$.

Arithmetic subalgebra and fabulous states

The system $(C(\mathcal{G} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}) \rtimes J_K^+, \sigma)$ satisfies the first four properties required in Problem A, essentially because they have been built into it, but the construction does not take into account the last two properties involving the **fabulous KMS$_\infty$ states** and the **arithmetic subalgebra**.

These last two conditions are unlikely to be easily verified because any advance in this direction has potential implications in explicit class field theory, although ultimately this will depend on the explicit formulas, for the fabulous states $\varphi_{\beta,w}$, and for the arithmetic elements $f \in \mathcal{A}_0$.

So it is not surprising that only the systems for $K = \mathbb{Q}$ [B-C, Selecta Math 95] and $K = \mathbb{Q}[\sqrt{-d}]$ [C-M-R Selecta Math '05] have been fully understood.

Towards K-lattices

To get a system with 'arithmetic content' another approach is needed that does not use $\mathcal{G} = \mathcal{G}(K^{ab}/K)$ explicitly but yields the same

$$\mathcal{A} = C_r^*(J_K \boxtimes (\mathcal{G} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}})) \quad \text{and} \quad \sigma_t(Fu_g) = N_g^{it} Fu_g.$$

A purely arithmetic construction based on K alone is motivated by

Definition (Connes-Marcolli '06)

An *n-dimensional* $\mathbb{Q}$*-lattice* is a pair (L, φ), where $L \subset \mathbb{R}^n$ is a lattice and $\varphi\colon \mathbb{Q}^n/\mathbb{Z}^n \to \mathbb{Q}L/L$ is a homomorphism.

Definition (Connes-Marcolli-Ramachandran '05)

A 1*-dimensional* K*-lattice* is a pair (Λ, ϕ) in which Λ is a f. g. $\mathcal{O}$-submodule of $\mathbb{C}$ such that $\Lambda \otimes_{\mathcal{O}} K \cong K$ and $\phi : K/\mathcal{O} \to K\Lambda/\Lambda$ is a module map.

(in both cases the notion of commensurabilty leads to a convolution product and a C*-algebra.)

K-lattices modulo commensurability

Let K_∞ be the product of all the completions of K at infinite places, so

$$K_\infty := \prod_{v \text{ infinite}} K_v,$$

which is isomorphic to $\mathbb{R}^{[K:\mathbb{Q}]}$ as an additive topological group. By an n-**dimensional** $\mathcal{O}$-**lattice** we mean a lattice L in K_∞^n such that $\mathcal{O}L = L$.

Definition

An n-**dimensional** K-**lattice** is a pair (L, φ), where $L \subset K_\infty^n$ is an n-dimensional $\mathcal{O}$-lattice and $\varphi \colon K^n/\mathcal{O}^n \to KL/L$ is an $\mathcal{O}$-module map.

Definition

Two n-dimensional K-lattices (L_1, φ_1) and (L_2, φ_2) are **commensurable** if the lattices L_1 and L_2 are commensurable and $\varphi_1 = \varphi_2$ modulo $L_1 + L_2$.

Let $\mathcal{R}_{K,n}$ be the equivalence relation of commensurability of n-dimensional K-lattices.

Proposition (L-Larsen-Neshveyev, JNT 08)

The quotient of the equivalence relation $\mathcal{R}_{K,1}$ of commensurability of 1-dimensional K-lattices by the scaling action of $(K_\infty^*)^\circ$, the connected component of the identity in K_∞^*, is a groupoid isomorphic to

$$(\mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*) \boxtimes ((\mathbb{A}_K^*/K^*(K_\infty^*)^\circ) \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}). \quad (1)$$

Recall now that the original construction of our system $(\mathcal{A}, \sigma)$ was based on the groupoid

$$J_K \boxtimes (\mathcal{G} \times_{\hat{\mathcal{O}}^*} \hat{\mathcal{O}}) \quad (2)$$

and that

$$J_K \cong \mathbb{A}_{K,f}^*/\hat{\mathcal{O}}^*$$

However, by class field theory, we know that

$$\mathcal{G} \cong \mathbb{A}_K^*/\overline{K^*(K_\infty^*)^\circ},$$

while the second component in the groupoid from (1) does not involve a closure.

A topological quotient is needed for K-lattices

The groupoids in (1) and (2) are *almost* the same, but there is the nuance that in using $\mathcal{G} := \mathcal{G}(K^{ab}/K)$ for our topological restricted groupoid $J_K \boxtimes Y$ we were effectively taking the **quotient of $\mathbb{A}_K^*$ by the closure of $K^*(K_\infty^*)^\circ$.**

In terms of equivalence of K-lattices this means that given a K-lattice (L, φ) we would have to identify not only all K-lattices $(kL, k\varphi)$ with $k \in (K_\infty^*)^\circ$, but also all K-lattices of the form $(kL, k\psi)$, where ψ is a limit point of the maps $u\varphi$ with $u \in \mathcal{O}^* \cap (K_\infty^*)^\circ$ in the topology of pointwise convergence.

For $K = \mathbb{Q}$ or $K =$ an imaginary quadratic field, $\mathcal{O}^*$ is finite so this nuance does not arise.